Endo-classes for p-adic classical groups

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Abstract

For a unitary, symplectic, or special orthogonal group over a non-archimedean local field of odd residual characteristic, we prove that two intertwining cuspidal types are conjugate in the group. This completes work of the third author who showed that every irreducible cuspidal representation of such a classical group is compactly induced from a cuspidal type, now giving a classification of irreducible cuspidal representations of classical groups in terms of cuspidal types. Our approach is to completely understand the intertwining of the so-called self dual semisimple characters, which form *the* fundamental step in the construction. To this aim, we generalise Bushnell–Henniart's theory of endo-class for simple characters of general linear groups to a theory for self dual semisimple characters of classical groups, and introduce (self dual) endo-parameters which parametrise intertwining classes of (self dual) semisimple characters.

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1 Introduction

Let G be a unitary, symplectic or special orthogonal group over a non-archimedean local field F of odd residual characteristic and let G^+ be either G in the unitary and symplectic case or the ambient orthogonal group. In previous works, the third author has developed an approach to the smooth dual of G via Bushnell–Kutzko types. In this article, we accomplish an important part of this programme and complete recent work ([12] and [18]) in refining the third author's construction [23] of all cuspidal representations of G to a classification. Moreover, we develop a natural framework for our results on semisimple characters, *the theory of endo-parameters* for self dual semisimple characters of G^+ (and of G), which contains in its core a generalisation of Bushnell–Henniart's theory of endo-class ([5]).

In [23], the third author explicitly constructs pairs (J, λ) consisting of a compact open subgroup J of G and an irreducible representation λ of J. If these pairs satisfy certain conditions

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then they are called *cuspidal types*. The main result of ibid. says that every irreducible cuspidal representation π of G contains a cuspidal type (J, λ), and that in this case it is compactly induced $\pi \simeq \mathrm{ind}_J^G \lambda$. If π contains two cuspidal types then the cuspidal types necessarily intertwine, and in the first main result of this paper we show:

Theorem A (intertwining implies conjugacy). Cuspidal types intertwine in G if and only if they are conjugate in G.

As conjugate cuspidal types induce isomorphic representations, this completes a classification by types of the irreducible cuspidal representations of G. This result was expected by analogy with other classifications of cuspidal representations via types for other connected reductive groups (such as Bushnell–Kutzko for GL_n [9], the third author and Sécherre for inner forms of GL_n [16], and Hakim–Murnaghan for all connected reductive groups under tame conditions [11]), but our proof has required a substantial amount of work and relies on the main results of a number of papers (recently, [12] and [18]). We expect this theorem to find many applications in arithmetic - and will be useful whenever detailed analysis of cuspidal representations of G is required.

In the second theme of this paper, we generalise Bushnell and Henniart's notions of potential simple character and endo-class, as defined in [5], to self dual potential semisimple characters and endo-class for G. As well as appearing in an essential way in our proof of Theorem A, this theory warrants independent study and allows a parametrisation of intertwining classes of (self dual) semisimple characters of GL_n, G^+ and of G via *endo-parameters* which we introduce at the end of the paper.

Potential applications include a *local Langlands correspondence for endo-parameters* or *Ramification Theorem* for classical groups (see [7, 6.1 Theorem] and, recently, [8] for the state of the art for $\mathrm{GL}_F(V)$), and a decomposition via endo-parameters of the category of all smooth representations of G (see [6, Type Theorem] for simple characters of $\mathrm{GL}_F(V)$). Both of which are the subject of current work of the authors. Let W_F denote the Weil group of F, and P_F the wild inertia subgroup of W_F. We expect the local Langlands correspondence for endo-parameters to take the following form:

Conjectural form of the LLC for endo-parameters. Suppose that the defining algebraic group for G is quasi-split.

(i) There is a unique bijection, compatible with the local Langlands correspondence for G, between the set of endo-parameters of self dual semisimple characters for G (forgetting Witt type data) and the set of W_F-conjugacy classes of representations of P_F which are extendible to a Langlands parameter $W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G(\mathbb{C})$.

(ii) Let $\phi : W_F \to {}^L G(\mathbb{C})$ be a Langlands parameter for an L-packet $\Pi(\phi)$ of G. There is a unique bijection, compatible with the local Langlands correspondence for G, between endo-parameters of representations in $\Pi(\phi)$ and the irreducible representations of $C_{{}^L G}(\mathrm{im}(\phi|_{P_F}))/Z_{{}^L G}(\mathbb{C})$.

For symplectic groups Part (i) of the Conjecture follows from current work of the third author with Blondel and Henniart [2].

Let F/F_0 be a quadratic or trivial extension with $\mathrm{Gal}(F/F_0) = \langle \sigma \rangle$. If V is an F-vector space with a ε-hermitian form h defining a unitary, symplectic or orthogonal group $G^+ = \mathrm{U}(V, h)$, we let Σ denote the cyclic group of order two generated by the inverse of the adjoint anti-involuton of h so that $G^+ = \mathrm{GL}(V)^\Sigma$. We fix a character of the additive group F^+ trivial on F_0. Let $E = F[\beta]$ be a sum of field extensions of F, and k be an integer satisfying certain bounds (see Section 8) - a *semisimple pair.* To a quadruple (V, φ, Λ, r) consisting of an F-vector space V, an embedding $\varphi : E \to \mathrm{End}_F(V)$, a $\varphi(\mathfrak{o}_E)$-lattice sequence in V, and an integer r closely related to k, using work of Bushnell–Kutzko [9] and the third author [22], we can associate a set $\mathcal{C}(\Lambda, r, \varphi(\beta))$ of *semisimple characters* of a compact open subgroup of $\mathrm{GL}(V)$ (whose *group level* is controlled by k). If in this quadruple V is equipped with a hermitian form h and the rest of the data satisfies certain duality properties, Σ acts on this set with fixed points, and the subset of characters fixed by Σ defines by unique restriction to G^+ the set $\mathcal{C}_-(\Lambda, r, \varphi(\beta))$ of *self dual semisimple characters* of G^+. Moreover, there are natural *transfer* maps between the sets of characters defined by different quadruples.

Given a semisimple pair, a *pss-character* (resp. *self dual pss-character*) is a function from the class of all quadruples (V, φ, Λ, r) (resp. $((V, h), \varphi, \Lambda, r)$) to the class of all semisimple characters (resp. class of all self dual semisimple characters) whose values are related by transfer. For two pss-characters (resp. self dual pss-characters), we define *comparison pairs* in Section 8; these are essentially pairs of quadruples with the same underlying space (resp. hermitian space), where we can test if the values of the pss-characters (resp. self dual pss-characters) intertwine. Note that, in the symplectic case, we need a more restrictive notion of comparison pair a *Witt comparison pair* (see Section 8). We call two pss-characters (resp. self dual pss-characters) *endo-equivalent* if we can find a pair of values which intertwine and if both semisimple pairs have the same degree. Our second main theorem is the surprising fact that this definition is equivalent to the assertion that for all suitable comparison pairs the values intertwine:

Theorem B (10.6). Let Θ_- and Θ'_- be two self dual pss-characters and let Θ and Θ' be their lifts. Then, the following assertions are equivalent:

(i) The self dual pss-characters Θ_- and Θ'_- are endo-equivalent;

(ii) The lifts Θ and Θ' are endo-equivalent.

(iii) There is a unique matching ζ between the simple blocks of Θ and the simple blocks of Θ' such that for all Witt ζ-comparison pairs the corresponding realisations of Θ_- and Θ'_- intertwine over G^+.

An important corollary of Theorem B is a transitivity of G^+-intertwining of self dual semisimple characters statement in Corollary 10.7, which we extend to an analogous statement for G-intertwining of semisimple characters in Corollary 11.3. While intertwining of characters is obviously a reflexive and symmetric relation in general, it is clearly not necessarily transitive, thus the transitivity statement we obtain is a reflection of the structure in the collection of all (self dual) semisimple characters. These statements are key to our proof of intertwining implies conjugacy for cuspidal types (Theorem A).

By a deep result of Bushnell–Henniart [6, Intertwining Theorem], simple characters of general linear groups intertwine if and only if they are endo-equivalent. This not only implies that intertwining of simple characters with same group level is transitive, it also shows that endo-classes parametrise intertwining classes of simple characters of GL_n. In the final section we prove a broad generalisation of this result to semisimple and self dual semisimple characters by introducing *endo-parameters.*

A semisimple character is called *full* if it lies in a set of semisimple characters $\mathcal{C}(\Lambda, 0, \beta)$, and an endo-class is called *full* if it contains a pss-characters supported on a semisimple pair (k, β) with $k = 0$. A GL-*endo-parameter* is a function from the set of all full simple endo-classes $\mathcal{E}$ to the natural numbers (including zero) with finite support.

Theorem C (13.6). The set of intertwining classes of full semisimple characters for $\mathrm{GL}_F(V)$ is in bijection with the set of those endo-parameters f which satisfy

$$\sum_{c\in\mathcal{E}} \deg(c)f(c) = \dim_F V.$$

The definition of endo-parameters for classical groups is much more intricate. We associate to a self dual simple character a Witt tower of hermitian spaces over a self dual field extension of F, and show that if two skew semisimple characters intertwine then block-wise their Witt towers *match*. We encode this information into an invariant of a skew semisimple character we call its *Witt type*, and denote the set of all Witt types for (σ,ϵ) by $\mathcal{W}_{\sigma,\epsilon}$. A self dual semisimple character is called elementary if it is skew or its associated indexing set contains two elements. A (σ,ϵ)-*endo-parameter* is a map from the set of full elementary (σ,ϵ)-endo-classes $\mathcal{E}_-$ to $\mathbb{N}\times\mathcal{W}_{\sigma,\epsilon}$ with finite support. Our final main result shows that (σ,ϵ)-*endo-parameters* parametrise intertwining classes of full self dual semisimple characters:

Theorem D (13.11). The set of intertwining classes of full self-dual semisimple characters for G^+ is in bijection with the set of (σ,ϵ)-endo-parameters $f=(f_1,f_2)$ which satisfy

$$\sum_{c\in\mathcal{E}_-} \deg(c)(2f_1(c) + \mathrm{diman}(f_2(c))) = \dim_F V,$$

such that the sum of the Witt tower over F of the $f_2(c)$s is equal to the Witt tower of h. We extend this theorem to parametrise intertwining classes of self dual semisimple characters of special orthogonal groups in Corollary 13.12.

We now describe the structure of the article. In Section 2, we carefully recast definitions and key results relevant to the sequel. In Section 3, we introduce comparison pairs, allowing us in Section 4 to study endo-equivalence for simple characters for unitary and special orthogonal, but not symplectic groups. The symplectic case requires quite a lot more work. We introduce matching Witt towers in Section 5 and more restrictive comparison pairs where we require the corresponding Witt towers to match called *Witt comparison pairs*. The main result of Section 5 shows that matching of Witt towers is inherited along *defining sequences* of simple strata. The additional condition of Witt comparison pairs allows us in Section 7 to study endo-equivalence for simple characters for symplectic groups following a study of intertwining simple characters for symplectic groups in Section 6. In Section 8, we recall definitions of semisimple strata and characters, and define potential semisimple characters. In Section 9, making significant use of recent work of the second and third authors [18], we investigate intertwining semisimple characters. In Section 10, we can finally introduce semisimple endo-equivalence and prove Theorem B. Section 11 investigates the difference between intertwining under orthogonal and special orthogonal groups. We then deduce Theorem A (intertwining implies conjugacy for cuspidal types) in Section 12, using the work of the previous sections to reduce to the *level zero part* where it is known thanks to recent results of the first and third authors [12]. Section 13 introduces endo-parameters and proves Theorems C & D on the parametrisation of intertwining classes of (self dual) semisimple characters by endo-parameters. The appendix generalises results on intertwining and conjugacy of skew characters of the second and third authors [18] to self dual characters.

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2 Preliminaries

2.1 Notation

Let F/F_0 be a trivial or quadratic extension of (locally compact) non-archimedean local fields of odd residual characteristic p, and σ be the generator of $\mathrm{Gal}(F/F_0)$. If E is any non-archimedean local field, we denote by $\mathfrak{o}_E$ the ring of integers of E, $\mathfrak{p}_E$ its maximal ideal, $k_E = \mathfrak{o}_E/\mathfrak{p}_E$ the residue field, and ν_E the additive valuation on E, which we will always normalise to have image $\mathbb{Z} \cup \{\infty\}$. We fix a uniformiser $\varpi_F \in \mathfrak{p}_F$ such that $\sigma(\varpi_F) = -\varpi_F$ if F/F_0 is ramified, and $\sigma(\varpi_F) = \varpi_F$ otherwise. If E/F is an algebraic extension of non-archimedean local fields, we write E_{tr} and E_{ur} for the maximal tamely ramified and maximal unramified subextension of E/F, respectively. Let ψ_0 be a character of F_0, with conductor $\mathfrak{p}_{F_0}$. We put $\psi_F = \psi_0 \circ \mathrm{Tr}_{F/F_0}$.

Let $\epsilon = \pm 1$, and V be a N-dimensional F-vector, space equipped with a non-degenerate ϵ-hermitian form $h : V \times V \to F$, defined with respect to the involution σ. Thus, h is a biadditive form $h : V \times V \to F$, satisfying

$$h(xv, yw) = \sigma(x)\epsilon\sigma(h(w, v))y,$$

for all $v, w \in V$, and $x, y \in F$. We let $A = \mathrm{End}_F(V)$, $\widetilde{G} = \mathrm{Aut}_F(V)$, and σ_h denote the adjoint anti-involution induced on A by h. Let σ denote both the involution $\sigma(g) = \sigma_h(g)^{-1}$, $g \in \widetilde{G}$, on $\widetilde{G}$ and the involution $\sigma(a) = -\sigma_h(a)$, $a \in A$, on A. Let $\Sigma = \{1, \sigma\}$, where 1 acts as the identity on $\widetilde{G}$ and A.

We set

$$G^+ = \widetilde{G}^\Sigma = \{g \in G : h(gv, gw) = h(v, w), \text{ for all } v, w \in V\},$$

the F_0-points of a unitary group (if F/F_0 is quadratic, $\epsilon = \pm 1$), a symplectic group (if $F = F_0$, $\epsilon = -1$), or an orthogonal group (if $F = F_0$, $\epsilon = 1$), defined over F_0. If G^+ is not orthogonal we set $G = G^+$, and in the orthogonal case we let G denote the special orthogonal subgroup of elements of determinant 1. We call G a *classical group*. For $\widetilde{H}$ a σ-stable subgroup of $\widetilde{G}$, we write $H^+ = \widetilde{H} \cap G^+$ and $H = \widetilde{H} \cap G$.

We denote set of σ-skew-symmetric elements of A by A_- and, for any σ-stable subset X of A, we write $X_- = X \cap A_-$ for the skew-symmetric elements and X_+ for the symmetric elements.

For a symmetric or skew-symmetric element a of A we define the *twist* of h by a to be the form $h^a : V \times V \to F$ defined by

$$h^a(v, w) = h(v, aw),$$

for all $v, w \in V$. If a is invertible the adjoint anti-involutions are then conjugate by a

$$\sigma_{h^a}(x) = a^{-1}\sigma_h(x)a,$$

for all $x \in A$.

Let $E = F[\beta]$ be a simple field extension of F. If σ extends to a unique involution σ_E on E, we call E a *self dual* extension and in this case we denote by E_0 the fixed field of σ_E. Let (V, h) be an ε-hermitian (F/F_0)-space. Let E be a self dual field extension of F equipped with an F-embedding $\varphi : E \hookrightarrow \mathrm{End}_F(V)$. We call φ *self dual* if $\sigma_h \circ \varphi = \varphi \circ \sigma_E$.

An $\mathfrak{o}_F$-*lattice sequence* in V is a map $\Lambda : \mathbb{Z} \to \{\mathfrak{o}_F\text{-lattices in } V\}$ which is decreasing (we have $\Lambda(k) \subseteq \Lambda(k-1)$, for all $k \in \mathbb{Z}$) and periodic (there exists $e(\Lambda) \in \mathbb{Z}$ such that $\Lambda(n + e(\Lambda)) =$

$\mathfrak{p}_F(\Lambda(n))$, for all $n \in \mathbb{Z}$). An $\mathfrak{o}_F$-lattice sequence Λ in V defines an $\mathfrak{o}_F$-lattice sequence in A, by setting

$$\mathfrak{a}_n(\Lambda) = \{a \in A : a\Lambda(i) \subseteq \Lambda(i+n),\ i \in \mathbb{Z}\}, \quad \text{for } n \in \mathbb{Z}.$$

The $\mathfrak{o}_F$-lattice $\mathfrak{a}(\Lambda) = \mathfrak{a}_0(\Lambda)$ is a hereditary $\mathfrak{o}_F$-order in A with Jacobson radical $\mathfrak{a}_1(\Lambda)$. If $\dim_{k_F}(\Lambda(i)/\Lambda(i+1))$ is independent of $i \in \mathbb{Z}$ we say that Λ is *regular*; or equivalently, that $\mathfrak{a}(\Lambda)$ is a *principal order*. The normaliser in $\mathrm{GL}_F(V)$ of Λ is a compact mod-centre subgroup

$$\mathfrak{n}(\Lambda) = \{g \in \mathrm{GL}_F(V) : \text{there exists } n \in \mathbb{Z},\ g(\Lambda(k)) = \Lambda(k+n),\ \text{for } k \in \mathbb{Z}\}.$$

We have a group homomorphism $\nu_\Lambda : \mathfrak{n}(\Lambda) \to \mathbb{Z}$ by setting $\nu_\Lambda(g) = n$, if $g(\Lambda(0)) = \Lambda(n)$. The kernel of ν_Λ is a compact open subgroup $P(\Lambda)$ of $\mathrm{GL}_F(V)$ which coincides with the group of units in $\mathfrak{a}(\Lambda)$. This subgroup has a decreasing separable filtration by compact open pro-p subgroups $P_n(\Lambda) = 1 + \mathfrak{a}_n(\Lambda)$, for $n \geqslant 1$.

If L is an $\mathfrak{o}_F$-lattice in V, we put $L^\sharp = \{v \in V : h(v, L) \subseteq \mathfrak{p}_F\}$. An $\mathfrak{o}_F$-lattice sequence Λ in V is called *self dual* (with respect to the ϵ-hermitian form h on V) if there exists $d \in \mathbb{Z}$, such that $\Lambda(d-k) = \Lambda(k)^\sharp$, for all $k \in \mathbb{Z}$. If Λ is self dual, then $P^-(\Lambda) = P(\Lambda) \cap G^+$ is a compact open subgroup of G. Intersecting the decreasing separable filtration $P_n(\Lambda)$ of $P(\Lambda)$ with G^+ gives a decreasing separable filtration $P_n^-(\Lambda)$ of $P^-(\Lambda)$ by compact open pro-p subgroups. The quotient group $M^-(\Lambda) = (P^-(\Lambda) \cap G)/P_1^-(\Lambda)$ is the group of k_{F_0}-points of a reductive group $\mathcal{M}^-$ defined over k_{F_0}. However, the algebraic group $\mathcal{M}^-$ need not be connected. We let $P^\circ(\Lambda)$ denote the inverse image in $P^-(\Lambda) \cap G$ of the k_{F_0}-points of the connected component of $\mathcal{M}^-$, and call $P^\circ(\Lambda)$ a *parahoric subgroup* of G.

2.2 Witt towers, transfer of forms

First we recall the definition of the Witt group over local fields. Let $a \in F$ be such that $a = \epsilon\sigma(a)$. We write $\langle a \rangle$ for the one-dimensional E-vector space with ϵ-hermitian form $h(\alpha, \beta) = \sigma(\alpha)a\beta$, for $\alpha, \beta \in F$. We let $\mathbb{H}$ denote the *hyperbolic plane*, and, for $m \in \mathbb{N}$, let $m\mathbb{H}$ denote the orthogonal sum of m copies of $\mathbb{H}$. We will always choose a basis so that the Gram matrix of $m\mathbb{H}$ is anti-diagonal with ϵ's and 1's on the anti-diagonal. Let V be an ϵ-hermitian F-space, by this we mean an F-vector space equipped with non-degenerate ϵ-hermitian form h with respect to σ. Then there exists $m \in \mathbb{N}^0$ and an anisotropic space V_0 such that V is isometric to $m\mathbb{H} \oplus V_0$; moreover, the integer m and the isometry class of V_0 are unique. We call V_0 the *anisotropic part* of V (or of h) and write $\mathrm{diman}(V) = \dim(V_0)$, the *anisotropic dimension* of V.

For an ϵ-hermitian form h, we call the class of all ϵ-hermitian spaces with anisotropic part isometric to the anisotropic part of h the *Witt tower* of h. The *Witt group* $W_{\sigma,\epsilon}(F)$ is the abelian group formed from the abelian semigroup of Witt towers of ϵ-hermitian forms induced by orthogonal sum of ϵ-hermitian spaces.

(i) Unitary case: If F/F_0 is quadratic and $\epsilon = \pm 1$, $W_{\sigma,\epsilon}(F)$ is of order 4 and is isomorphic to the Klein group if $-1 \in N_{F/F_0}(E^\times)$, and is cyclic if not.

(ii) Orthogonal case: If F/F_0 is trivial and $\epsilon = 1$, $W_{\mathrm{id},1}(F)$ is of order 16.

(iii) Symplectic case: If F/F_0 is trivial and $\epsilon = -1$, $W_{\mathrm{id},-1}(F)$ is trivial.

For ϵ-hermitian spaces (V, h) and (V', h'), we will write $h \equiv h'$ if they define the same element of the Witt group, i.e. are in the same Witt tower.

Let $E = F[\beta]/F$ be a self dual field extension with the induced involution σ_E, in particular β is zero or $E \neq E_0$, since $\sigma_E(\beta) = -\beta$, and $\lambda : E \to F$ be a non-zero F-linear form on E which is (σ_E, σ)-equivariant. Such forms exist, indeed, we can choose such a linear form λ by setting $\lambda(1) = 1$ and $\lambda(\beta) = \lambda(\beta^2) = \cdots = \lambda(\beta^{n-1}) = 0$. We will denote this particular choice by λ_β. Let (V, h) be an ϵ-hermitian E-space. Then $(V, \lambda \circ h)$ is a non-degenerate ϵ-hermitian F-space, called the *transfer* of (V, h). It is easy to see that the transfer preserves orthogonal sums, isometries, and hyperbolic spaces. Hence it induces a group homomorphism

$$\lambda^* : W_{\sigma_E,\epsilon}(E) \to W_{\sigma,\epsilon}(F).$$

In general, it is easy to see that this homomorphism is not bijective (for example, by taking E/F of even dimension). However, we have the following rather surprising theorem:

Theorem 2.1. Assume that if F/F_0 is trivial then $\epsilon = 1$ (the non-symplectic case). Then restriction of the transfer homomorphism $\lambda^* : W_{\sigma_E,\epsilon}(E) \to W_{\sigma,\epsilon}(F)$ to spaces of the same parity of dimension is injective.

Note that, the assertion of the theorem would not be true without the assumption that $E = F[\beta]$ is a self dual field extension.

Proof. By twisting, and composing with a transfer $W_{\sigma,1}(F) \to W_{\mathrm{id},1}(F_0)$ if necessary, it is sufficient to prove it in the orthogonal case $F = F_0$. For separable extensions the statement with $\lambda = \mathrm{Tr}_{E/F}$ is proved in [13]. In ibid. it is explained that the statement also holds for inseparable extensions (cf. [13, Remark 1.4]), while a different proof for inseparable extensions can be found in the proof of [18, Theorem 4.4] where particular linear forms are chosen. However, the following simple observation shows that if the result holds for one (σ_E, σ)-equivariant linear form then it holds for all such forms:

Lemma 2.2. Let λ, λ' be non-zero (σ_E, σ)-equivariant F-linear forms on E, then there exists a unique $\gamma \in E_0$ such that $\lambda(x\gamma) = \lambda'(x)$ for all $x \in E$. Moreover, the image $\lambda^*(W_{\sigma_E,\epsilon}(E))$ of λ^* in $W_{\sigma,\epsilon}(F)$ is independent of the choice of λ.

Proof. From the bijection of E onto its dual space, there exists a unique $\gamma \in E$ such that $\lambda(x\gamma) = \lambda'(x)$ for all $x \in E$. As λ, λ' are self dual, one finds $\lambda(\sigma_E(x)\sigma_E(\gamma)) = \lambda(\sigma_E(x)\gamma)$, which implies that $\gamma = \sigma_E(\gamma)$, i.e. $\gamma \in E_0$. (Note that, similarly, for any $\gamma \in E_0$, the form $\gamma \cdot \lambda(x) = \lambda(x\gamma)$ is always self dual.) The second part follows as multiplication by $E(\gamma)$ (given by the tensor product) defines an automorphism of the Witt ring $W_{\sigma_E,\epsilon}(E)$. □

This completes the proof of the theorem. □

Proposition 2.3 ([18, Theorem 4.4])**.** The map λ^* maps the Witt tower of maximal anisotropic dimension to the Witt tower of maximal anisotropic dimension.

We will need more precise information on the image of the transfer homomorphism in particular instances.

Proposition 2.4. Suppose $\epsilon = 1$. Let $E = F[\beta]$ be a self dual extension of F of degree n, (V, h_E) be an ϵ-hermitian E-space, and λ be any (σ_E, σ)-equivariant F-linear form on E.

(i) Granted $p \neq 2$, we have

$$\det(\lambda^*(V)) = \det(\lambda^*\langle 1\rangle)^{\dim_E(V)} N_{E/F}(\det(V)).$$

(ii) In $W_{\sigma,\epsilon}(F)$, we have

$$\lambda_\beta^*(\langle 1\rangle) \equiv \begin{cases} \langle 1\rangle & \text{if } n \text{ is odd;} \\ \langle 1\rangle \oplus \langle (-1)^{\frac{n}{2}+1} N_{E/F}(\beta)\rangle & \text{otherwise.} \end{cases}$$

Proof. The analogue of these statements for transfer of quadratic forms are proved by Scharlau in [15, Lemma 5.8, Theorem 5.12]. The hermitian case follows *mutatis. mutandis.* being careful of the extra signs which appear, and for this reason we sketch the proof of Part (ii). Suppose β has minimal polynomial $X^n + \beta_{n-1}X^{n-1} + \cdots + \beta_1 X + \beta_0$. Then we can easily calculate the Gram matrices of the form $\lambda_\beta^*(\langle 1\rangle)$:

$$J(\lambda_\beta^*(\langle 1\rangle)) = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +\beta_0 \\ \vdots & 0 & 0 & 0 & -\beta_0 & \star \\ \vdots & 0 & 0 & \iddots & \star & \star \\ \vdots & 0 & (-1)^{n-1}\beta_0 & \star & \star & \star \\ 0 & (-1)^n\beta_0 & \star & \star & \star & \star \end{pmatrix},$$

In particular, if $n = [E:F]$ is odd, then $\lambda_\beta^*(\langle 1\rangle) \simeq \frac{n-1}{2}\mathbb{H} \oplus \langle 1\rangle$, as it is an orthogonal sum of $\langle 1\rangle$ with the subspace X generated by $\beta, \beta^2, \ldots, \beta^n$; however, X has a totally isotropic subspace of half its dimension generated by $\beta, \beta^2, \ldots \beta^{\frac{n-1}{2}}$, hence is hyperbolic. Similarly, if $n = [E:F]$ is even, then $\lambda_\beta^*(\langle 1\rangle) \simeq \frac{n-2}{2}\mathbb{H} \oplus \langle 1\rangle \oplus \langle \delta\rangle$, with $\delta = (-1)^{\frac{n}{2}+1} N_{E/F}(\beta)$. □

We will need two properties of the norm map:

Lemma 2.5. Let $F \neq F_0$ and E/F be a finite field extension and σ_E be a non-trivial involution on E extending σ and $\alpha \in E_0$.

(i) Then $N_{E/F}(\alpha) = N_{E_0/F_0}(\alpha)$.

(ii) We have $\alpha \in N_{E/E_0}(E^\times)$ if and only if $N_{E_0/F_0}(\alpha) \in N_{F/F_0}(F^\times)$.

Proof. Consider all field extensions in a given Galois closure $\mathfrak{F}$ of F_0. Let $m = [E:F]_{\mathrm{sep}}$ be the degree of the separable part of the extension and $d = [E:F]_{\mathrm{rad}}$ be the degree of the inseparable part of the extension so that $[E:F] = dm$. Similarly, define d_0 and m_0 for the extension E_0/F_0. Then, as $p \neq 2$, we have $d = d_0$ and $m = m_0$. Let $\tau_1, \ldots, \tau_m$ denote the m distinct F-homomorphisms $E \hookrightarrow \mathfrak{F}$. For each i, the homomorphism $\tau_i|_{E_0} : E_0 \hookrightarrow \mathfrak{F}$ is an F_0-homomorphism, and for $i \neq j$, we have $\tau_i|_{E_0} \neq \tau_j|_{E_0}$ as $E = FE_0$ and they are distinct as F-homomorphisms $E \hookrightarrow \mathfrak{F}$. The first assertion now follows by definition of the norms. The second assertion for purely inseparable extensions E/F follows directly by considering valuations, as in this case $[E:F]$ is necessarily odd. Thus, by transitivity of the norm, we can suppose that E/F and hence E_0/F_0 is separable. By local class field theory, for any finite abelian extension K/k (contained in a given separable closure) of local fields (so, for example as $p \neq 2$, E/E_0 or F/F_0), we have an isomorphism

$$\mathrm{Art}_{K/k} : k^\times/N_{K/k}(K^\times) \simeq \mathrm{Gal}(K/k).$$

Moreover, the base change property of class field theory applied to the field extensions above, gives on $E_0^\times/N_{E/E_0}(E^\times)$

$$\mathrm{Res}^E_F \circ \mathrm{Art}_{E/E_0} = \mathrm{Art}_{F/F_0} \circ N_{E_0/F_0}.$$

We have $\mathrm{Gal}(E/E_0) = \{1, \sigma_E\}$ and $\mathrm{Gal}(F/F_0) = \{1, \sigma\}$, with $\mathrm{Res}^E_F(\sigma_E) = \sigma$, and the Artin reciprocity maps are isomorphisms. Hence Art_{E/E_0} is trivial on the class of α if and only if $\mathrm{Art}_{F/F_0} \circ N_{E_0/F_0}$ is also trivial on this class, and the second assertion follows. □

Remark 2.6. In the quadratic unitary case:

(i) As dimension and determinant modulo the norm group $N_{F/F_0}(F^\times)$ form a complete set of invariants, i.e. characterise ϵ-hermitian spaces up to isometry completely, Lemma 2.5 and Proposition 2.4 combine to characterise the standard transfer λ^*_β of the self dual field extension $E = F[\beta]$ completely.

(ii) Let x be a non-zero element of F_0:

  (a) If -1 is a square in F, then x is a norm with respect to N_{F/F_0} if and only if x is a square in F.

  (b) If -1 is not a square in F, then x is a norm with respect to N_{F/F_0} if and only if either $\nu_{F_0}(x)$ is even and x is a square in F or $\nu_{F_0}(x)$ is odd and x is not square in F.

2.3 Self dual embeddings and transfer

Let $\varphi : E \hookrightarrow A$ be a self dual embedding. This gives V the structure of an E-vector space and we write V_φ when we want to make it clear that we are considering V as an E-vector space via φ. The F-linear map

$$\mathrm{Hom}_E(V_\varphi, E) \to \mathrm{Hom}_F(V, F)$$
$$\psi \mapsto \lambda \circ \psi$$

is an isomorphism of F-vector spaces, and the (F/F_0)-form h defines an isomorphism $V \to \mathrm{Hom}_F(V, F)$ by $v \mapsto h(v, -)$. We let $\psi_v \in \mathrm{Hom}_E(V_\varphi, E)$ be the unique E-linear map such that $h(v, -) = \lambda \circ \psi_v$, then define $h_\varphi : V_\varphi \times V_\varphi \to E$ by $h_\varphi(v, w) = \psi_v(w)$.

Lemma 2.7 ([4, Lemma 5.3])**.** The map $h_\varphi : V_\varphi \times V_\varphi \to E$ is a nondegenerate hermitian (E/E_0)-form. Moreover, it is the unique hermitian (E/E_0)-form on V_φ such that $h(v, w) = \lambda(h_E(v, w))$ for all $v, w \in V$.

Sometimes, it will be more convenient to take an element $\beta \in A$ generating a field, and consider the embedding to be the inclusion of the subfield of A. We denote the form given by the lemma for this embedding by h_β.

Let $\varphi, \varphi' : E \hookrightarrow A$ be self dual F-embeddings and (V_φ, h_φ) and $(V_{\varphi'}, h_{\varphi'})$ be the hermitian (E/E_0)-spaces defined by (V, h) and a fixed (σ_E, σ)-invariant F-linear form as in Lemma 2.7. An observation of the second author in [17], we use later without reference, is the following useful corollary of Lemma 2.7.

Corollary 2.8 ([17, Proposition 1.3])**.** The hermitian (E/E_0)-spaces (V_φ, h_φ) and $(V_{\varphi'}, h_{\varphi'})$ are isometric if and only if the embeddings φ and φ' are conjugate in $\mathrm{U}(V, h)$.

2.4 Potential simple characters

For the remainder, we fix our quadratic or trivial extension F/F_0, σ the generator of $\mathrm{Gal}(F/F_0)$, and a choice of sign ϵ.

A *stratum* in A is a 4-tuple $[\Lambda, n, r, \beta]$ where Λ is an $\mathfrak{o}_F$-lattice sequence, $n, r \in \mathbb{Z}$, with $n \geqslant r \geqslant 0$, and $\beta \in \mathfrak{a}_{-n}(\Lambda)$. The fraction $\frac{n}{e(\Lambda)}$ is called the *level* of the stratum. We call the stratum $[\Lambda, r, r, 0]$ a *zero-stratum*. A stratum $[\Lambda, n, r, \beta]$ is called *pure* if it is zero or $E = F[\beta]$ is a field, Λ is an $\mathfrak{o}_E$-lattice sequence, and $\nu_\Lambda(\beta) = -n$. Let $\mathfrak{n}_k(\beta, \Lambda) = \{x \in \mathfrak{a}(\Lambda) : \beta x - x\beta \in \mathfrak{a}_k(\Lambda)\}$ and define the *critical exponent* $k_0(\beta, \Lambda)$ by

$$k_0(\beta, \Lambda) = \max\{\nu_\Lambda(\beta), \sup\{k \in \mathbb{Z} : \mathfrak{n}_k(\beta, \Lambda) \nsubseteq \mathfrak{a}(\Lambda_E) + \mathfrak{a}_1(\Lambda)\}\},$$

for non-zero β and $k_0(0, \Lambda) = -\infty$. A pure stratum $[\Lambda, n, r, \beta]$ is called *simple* if $k_0(\beta, \Lambda) < -r$, in particular if $n = r$ then the stratum has to be zero. A stratum $[\Lambda, n, r, \beta]$ is called *self dual* if Λ is a self dual $\mathfrak{o}_F$-lattice sequence and $\beta \in A_-$.

For a pure stratum we call $[E : F]$ the *degree* of the stratum, and for all objects which are defined using a field extension $E|F$ we call $[E : F]$ the *degree* of this object, e.g. of a simple pair, a ps-character, a character, etc, see below.

Let $[\Lambda, n, r, \beta]$ be a *simple stratum* in A with non-zero β. Associated to $[\Lambda, n, r, \beta]$ and our initial choice of ψ_F, in the work of Bushnell and Kutzko [9] - extended to non-strict lattice sequences by the third author in [22], are a compact open subgroup $H^{r+1}(\beta, \Lambda)$ of $P(\Lambda)$ and a set of characters $\mathcal{C}(\Lambda, r, \beta)$ of $H^{r+1}(\beta, \Lambda)$ called *simple characters*. For the stratum $[\Lambda, r, r, 0]$ we define $\mathcal{C}(\Lambda, r, 0)$ to be the singleton consisting only of the trivial character on $H^{r+1}(0, \Lambda)$ which is defined as $P_{r+1}(\Lambda)$. This trivial character is also called a simple character.

Let $[\Lambda, n, r, \beta]$ be a self dual simple stratum in A. Then $H_-^{r+1}(\beta, \Lambda) = H^{r+1}(\beta, \Lambda) \cap G$ is a compact open subgroup of $P^-(\Lambda)$, and we can define a set of *self dual simple characters* $\mathcal{C}_-(\Lambda, r, \beta)$ of $H_-^{r+1}(\beta, \Lambda)$ by restriction from $\mathcal{C}(\Lambda, r, \beta)$. This restriction of characters coincides with the Glauberman correspondence by [20, §2]. A character $\theta \in \mathcal{C}(\Lambda, r, \beta)$ is called *σ-invariant* if it is fixed by the involution $x \mapsto \sigma(x) = \sigma_h(x)^{-1}$, for all $x \in \widetilde{G}$, and we write $\mathcal{C}(\Lambda, r, \beta)^{\sigma_h}$ for the subset of σ-invariant characters. By the Glauberman correspondence, if $\theta_- \in C_-(\Lambda, r, \beta)$ is a simple character, then there is a unique $\theta \in \mathcal{C}(\Lambda, r, \beta)^{\sigma_h}$ whose restriction to $H_-^{r+1}(\beta, \Lambda)$ is θ_-. We call θ the *lift* of θ_-.

Let $E = F[\beta]$ be a field extension and denote $n_F(\beta) = -\nu_E(\beta)$. Let $\Lambda(E)$ be the $\mathfrak{o}_F$-lattice sequence $i \mapsto \mathfrak{p}_E^i$, $i \in \mathbb{Z}$. This is the unique (up to translation) $\mathfrak{o}_E$-lattice chain in E, and $\nu_{\Lambda(E)}(\beta) = \nu_E(\beta)$. For any integer $0 \leqslant k \leqslant n_F(\beta) - 1$, the stratum $[\Lambda(E), n_F(\beta), k, \beta]$ in $\mathrm{End}_F(E)$ is pure; and we set

$$k_F(\beta) = \frac{k_0(\beta, \Lambda(E))}{e(E/F)}.$$

Remark 2.9. Note that our definition of $k_F(\beta)$ differs by a normalisation from the standard choice of Bushnell–Henniart [5]; we make this change as with this normalisation $k_F(\beta)$ generalizes to the semisimple case.

A *simple pair* over F is a pair $[k, \beta]$ where $E = F[\beta]$ is a finite field extension of F and k is an integer satisfying $0 \leqslant k < -k_F(\beta)e(E/F)$. A simple pair $[k, \beta]$ over F is called *self dual* if $E = F[\beta]$ is a self dual field extension. Given a simple pair $[k, \beta]$, we consider quadruples (V, φ, Λ, r), consisting of: a finite dimensional F-vector space V; an embedding $\varphi : E \hookrightarrow A$; a $\varphi(\mathfrak{o}_E)$-lattice sequence Λ in V (hence we have $\varphi(E^\times) \subseteq \mathfrak{K}(\Lambda)$); and an integer r such that the *group level* $\left\lfloor \frac{r}{e(\Lambda)} \right\rfloor$ is equal to k. Given such a quadruple (V, φ, Λ, r), we obtain a simple stratum $[\Lambda, n, r, \varphi(\beta)]$ in A, with $n = -\nu_E(\beta)e(\Lambda_E)$, which we call a *realisation* of the simple pair $[k, \beta]$. (It is simple as $k_F(\beta) = \frac{1}{e(\Lambda_E)e(E/F)}k_0(\beta, \Lambda)$ by [9, 1.4.13]). Let $[k, \beta]$ be a simple pair and $\mathcal{Q}(k, \beta)$ denote the class of all such quadruples (V, φ, Λ, r). Suppose that $[k, \beta]$ is self dual. Let $\mathcal{Q}_-(k, \beta)$ denote the class of all quadruples $((V, h), \varphi, \Lambda, r)$ where (V, h) is an ϵ-hermitian F-space; $(V, \varphi, \Lambda, r) \in$

$\mathcal{Q}(k,\beta)$, and φ, Λ are self dual with respect to h. We note that the set $\mathcal{Q}_-(k,\beta)$ just depends on our initial choice of σ, ϵ. If $((V,h),\varphi,\Lambda,r) \in \mathcal{Q}_-(k,\beta)$, then $[\Lambda, n, r, \varphi(\beta)]$ is a self dual simple stratum which we call a (self dual) *realisation* of the self dual simple pair $[k,\beta]$.

For realisations $[\Lambda, n, r, \varphi(\beta)]$ and $[\Lambda', n', r', \varphi'(\beta)]$ of a simple pair $[k,\beta]$ there is a canonical bijection

$$\tau_{\Lambda,\Lambda',\beta} : \mathcal{C}(\Lambda, r, \varphi(\beta)) \to \mathcal{C}(\Lambda', r', \varphi'(\beta)),$$

by [9, 3.6.14], where despite the dependence of $\tau_{\Lambda,\Lambda',\beta}$ on $(\varphi, \varphi', r, r')$ we do not include it in our notation. Let $[k,\beta]$ be a self dual simple pair. By [22, Proposition 2.12], if $[\Lambda, n, r, \varphi(\beta)]$ and $[\Lambda', n', r', \varphi'(\beta)]$ are self dual realisations of $[k,\beta]$, then $\tau_{\Lambda,\Lambda',\beta}$ commutes with the involutions defined on $\mathcal{C}(\varphi(\beta), r, \Lambda)$ and $\mathcal{C}(\varphi'(\beta), r', \Lambda')$ and restricts to give a bijection

$$\tau_{\Lambda,\Lambda',\beta} : \mathcal{C}_-(\Lambda, r, \varphi(\beta)) \to \mathcal{C}_-(\Lambda', r', \varphi'(\beta)).$$

We let $\mathcal{C}(k,\beta)$ denote the class of all simple characters defined by a realisation of a simple pair $[k,\beta]$ and $\mathcal{C}_-(k,\beta)$ denote the class of all self dual simple characters defined by a realisation of a self dual simple pair $[k,\beta]$, i.e.

$$\mathcal{C}(k,\beta) = \bigcup_{\substack{(V,\varphi,\Lambda,r)\\ \in \mathcal{Q}(k,\beta)}} \mathcal{C}(\Lambda, r, \varphi(\beta)), \quad \mathcal{C}_-(k,\beta) = \bigcup_{\substack{(V,\varphi,\Lambda,r)\\ \in \mathcal{Q}_-(k,\beta)}} \mathcal{C}_-(\Lambda, r, \varphi(\beta)). \tag{2.10}$$

Note that, as introduced by Bushnell and Henniart in [5], the standard notation for $\mathcal{C}(k,\beta)$ is $\mathfrak{R}(k,\beta)$.

A *potential simple character*, or *ps-character*, *supported on the simple pair* $[k,\beta]$ is a function,

$$\Theta : \mathcal{Q}(k,\beta) \to \mathcal{C}(k,\beta)$$

such that $\Theta(V,\varphi,\Lambda,r) \in \mathcal{C}(\Lambda, r, \varphi(\beta))$, for $(V,\varphi,\Lambda,r) \in \mathcal{Q}(k,\beta)$, and

$$\Theta(V', \varphi', \Lambda', r') = \tau_{\Lambda,\Lambda',\beta}(\Theta(V,\varphi,\Lambda,r)),$$

for $(V,\varphi,\Lambda,r), (V',\varphi',\Lambda',r') \in \mathcal{Q}(k,\beta)$. For $(V,\varphi,\Lambda,r) \in \mathcal{Q}(k,\beta)$, we call $\Theta(V,\varphi,\Lambda,r)$ a *realisation* of Θ. Thus, a ps-character is determined by any one of its realisations. Let Θ be a ps-character supported on the simple pair $[k,\beta]$ and Θ' be a ps-character supported on the simple pair $[k',\beta']$. We say that Θ and Θ' are *endo-equivalent*, denoted $\Theta \approx \Theta'$, if both have the same degree, $k = k'$ and there exists a finite dimensional vector space V, which is part of a quadruple $(V,\varphi,\Lambda,r) \in \mathcal{Q}(k,\beta)$ and part of a quadruple $(V,\varphi',\Lambda',r') \in \mathcal{Q}(k',\beta')$, such that $\Theta(V,\varphi,\Lambda,r)$ and $\Theta'(V,\varphi',\Lambda',r')$ intertwine in $\mathrm{GL}_F(V)$, i.e. there exist realisations on a common vector space which intertwine. We have the following key results for ps-characters

Theorem 2.11 ([3, Theorem 1.11] and [5, Corollary 8.10])**.** Suppose we are given two endo-equivalent ps-characters Θ and Θ' supported on $[k,\beta]$ and $[k,\beta']$, respectively, and let θ and θ' be realisations of Θ and Θ', respectively, on the same vector space V. Then, θ and θ' intertwine over $\tilde{G}$.

Theorem 2.12 ([3, Corollary 8.3])**.** This defines an equivalence relation on the class of ps-characters.

The equivalence classes are called *simple* GL*-endo-classes*.

Theorem 2.13. Let θ, θ' and θ'' be simple characters with the same group level and the same degree. If θ intertwines with θ' and θ' intertwines with θ'' over $\tilde{G}$, then θ intertwines with θ'' over $\tilde{G}$.

The Theorem is proven for the case of strict lattice sequences in [6, Corollary 2].

Proof. The ps-characters Θ and Θ'' defined by θ and θ'' are endo-equivalent to each other by Theorem 2.13 and thus θ and θ'' intertwine by some element of $\tilde{G}$ by Theorem 2.11. □

The equality of the degrees of simple characters is a result of intertwining if one assumes that both characters have the same period and $r = r'$:

Theorem 2.14. Suppose two simple characters $\theta \in \mathcal{C}(\Lambda, r, \beta)$ and $\theta' \in \mathcal{C}(\Lambda', r, \beta')$ of the same period intertwine. Then $e(E|F) = e(E'|F)$, $f(E|F) = f(E'|F)$ and $k_0(\beta, \Lambda) = k_0(\beta', \Lambda')$.

Proof. Using the †-construction from [12, §3.1] we can assume that Λ and Λ' are lattice chains of the same period. Then by Theorem [9, 3.5.11] these characters are conjugate and thus [9, 3.5.1] provides the desired equalities. □

A ps-character supported on a self dual simple pair $[k, \beta]$ is called *σ-invariant* if for any (or equivalently one) self dual realisation of $[k, \beta]$ the value is σ-invariant. A *self dual ps-character* is a function, $\Theta_- : \mathcal{Q}_-(k, \beta) \to \mathcal{C}_-(k, \beta)$ such that $\Theta_-((V, h), \varphi, \Lambda, r) \in \mathcal{C}_-(\Lambda, r, \varphi(\beta))$, for $((V, h), \varphi, \Lambda, r) \in \mathcal{Q}_-(k, \beta)$, and

$$\Theta_-((V', h'), \varphi', \Lambda', r') = \tau_{\Lambda,\Lambda',\beta}(\Theta_-((V, h), \varphi, \Lambda, r)),$$

for $((V, h), \varphi, \Lambda, r), ((V', h'), \varphi', \Lambda', r') \in \mathcal{Q}_-(k, \beta)$. Thus, again, a self dual ps-character is determined by any one of its realisations. By the Glauberman correspondence, every self dual ps-character comes uniquely from the restriction of a σ-invariant ps-character. We call the values of Θ_- *realisations* and the σ-invariant ps-character $\Theta : \mathcal{Q}(k, \beta) \to \mathcal{C}(k, \beta)$ which restricts to Θ_- the *lift* of Θ_-.

We will need a proposition on minimal elements in tamely ramified field extensions:

Proposition 2.15. Suppose E_1/F and E_2/F are two finite field extensions with the first one being tamely ramified. Further, let b_i be an element of $E_i^\times$, $i = 1, 2$, such that

(i) $F[b_1] = E_1$ and $\nu_{E_1}(b_1)$ is prime to $e(E_1/F)$ which we denote by e,

(ii) $\nu_{E_1}(b_1)/e$ is equal to $\nu_{E_2}(b_2)/e(E_2/F)$, and

(iii) $b_1^e \varpi_F^{-\nu_{E_1}(b_1)} + \mathfrak{p}_{E_1}$ and $b_2^e \varpi_F^{-\nu_{E_1}(b_1)} + \mathfrak{p}_{E_2}$ have the same minimal polynomial over k_F, and the first element generates the residue field k_{E_1} over k_F.

Then, there is an F-embedding $\phi : E_1 \to E_2$ such that

$$\nu_{E_2}(\phi(b_1) - b_2) > \nu_{E_2}(b_2).$$

Proof. Let $\overline{P}$ be the minimal polynomial of $\overline{b_1^e \varpi_F^{-\nu_{E_1}(b_1)}}$ over k_F, and choose a lift $P \in \mathfrak{o}_F[X]$ of $\overline{P}$. Hensel's lemma guaranties the existence of zeroes $\gamma_1 \in E_{1,ur}$ and $\gamma_2 \in E_{2,ur}$ of P congruent to $b_i^e \varpi_F^{-\nu_{E_1}(b_1)}$ mod $\mathfrak{p}_{E_i}$, respectively. There is an F-monomorphism from E_1 into an algebraic closure of E_2 which maps γ_1 to γ_2, and thus we can assume that $F = E_{1,ur}$ and $\gamma_1 = \gamma_2$. By Bezout's lemma it is enough to restrict to the case where b_1 is a uniformiser. Let $\tilde{\varpi} \in \mathfrak{p}_k \setminus \mathfrak{p}_k^2$ be an eth power in E_1. Then $b_1^e \tilde{\varpi}^{-1}$ and $\overline{b_1^e \tilde{\varpi}^{-1}}$ are e-th powers. The latter is equal to $\overline{b_2^e \tilde{\varpi}^{-1}}$ and Hensel's lemma provides e-th roots λ_i of $b_i^e \tilde{\varpi}^{-1}$, respectively, such that $\overline{\lambda}_1$ is equal to $\overline{\lambda}_2$. The F-monomorphism ϕ which maps $b_1 \lambda_1^{-1}$ to $b_2 \lambda_2^{-1}$ satisfies that $\phi(b_1) b_2^{-1}$ is an element of $1 + \mathfrak{p}_{E_2}$. □

Remark 2.16. An element b_1 as in Proposition 2.15 is called a *minimal element* for E_1/F, see [9, 1.4.14].

3 Comparison pairs

Let $[k,\beta]$ be a self dual simple pair. Let $[\Lambda,n,r,\beta]$ be a self dual simple stratum for $[k,\beta]$ (in particular the group level is k) in an ϵ-hermitian (F/F_0)-space V, and let $\theta_-\in C_-(\Lambda,r,\beta)$. This gives V the structure of a ϵ-hermitian (E/E_0)-space defining a class in $W_{\sigma,\epsilon}(E)$, and we will say that the Witt group $W_{\sigma,\epsilon}(E)$ is the *Witt group of the stratum, of θ_-, or of β, with respect ϵ.*

Definition 3.1. Let Θ_- and Θ'_- be two self dual ps-characters supported on possibly different simple pairs $[k,\beta]$ and $[k,\beta']$ for the same k. We call two elements $((V,h),\varphi,\Lambda,r)$ and $((V',h'),\varphi',\Lambda',r')$ in the domain of Θ_- and Θ'_-, respectively, a *comparison pair* if

$$V=V',\ h=h',\ e(\Lambda)=e(\Lambda'),\ r=r'.$$

A comparison pair $(((V,h),\varphi,\Lambda,r),((V,h),\varphi',\Lambda',r'))$ is called *strong* if $\Lambda=\Lambda'$.

We will formulate several statements like "realisations of Θ'_- and Θ'_- with respect to a given comparison pair intertwine over G^+". Here, we mean that G^+ is the classical group defined by the comparison pair, i.e. it is $\mathrm{U}(V,h)$ if the comparison pair is $((V,h),\Lambda,\varphi,r),((V,h),\Lambda',\varphi',r'))$.

We are mainly interested in the case of Witt groups of two given quadratic extensions E/E_0 and E'/E'_0 which have *similar descriptions* in terms of units and uniformisers, i.e either $W_{\sigma,\epsilon}(E)$ and $W_{\sigma',\epsilon}(E')$ are isomorphic to a cyclic group of order 4 (i.e. when -1 is neither a square in E nor E') or both are Kleinian four groups and there are generators $\langle a_1\rangle_\equiv,\langle a_2\rangle_\equiv$ for $W_{\sigma,\epsilon}(E)$ and $\langle a'_1\rangle_\equiv,\langle a'_2\rangle_\equiv$ for $W_{\sigma,\epsilon}(E)$ such that $\frac{\nu_E(a_i)}{e(E|F)}$ is equal to $\frac{\nu_{E'}(a'_i)}{e(E'|F)}$. the next lemma ensures that this is the case for intertwining positive level self dual simple characters.

Lemma 3.2. Suppose that two simple characters $\theta\in\mathcal{C}(\Lambda,r,\beta)^{\sigma_h}$ and $\theta'\in\mathcal{C}(\Lambda',r,\beta')^{\sigma_h}$ intertwine over G^+ and both lattice sequences have the same period. Then, E/E_0 is ramified if and only if E'/E'_0 is ramified.

Proof. Let e be the period of Λ, hence the period of Λ' by assumption. Adding $2e$ shifts of Λ, we obtain the self dual principal lattice chain

$$\Lambda^\dagger=(\Lambda-e+1)\oplus\ldots\oplus(\Lambda-0)\oplus(\Lambda-0)^\sharp\oplus\cdots\oplus(\Lambda-e+1)^\sharp,$$

with respect to ϵ-hermitian form $h^\dagger=\begin{pmatrix} & & h\\ & \iddots & \\ h & & \end{pmatrix}$ ($2e$-times). Put $\beta^\dagger=\bigoplus\beta$ ($2e$-times). Similarly, define, $\Lambda'^\dagger$, $\beta'^\dagger$. Then $\Lambda'^\dagger$ is also a self dual principal lattice chain in $(V^\dagger,h^\dagger)$, with the same jumps in the filtration as $\Lambda^\dagger$; hence $\Lambda^\dagger$ and $\Lambda'^\dagger$ are conjugate in $\mathrm{U}(V^\dagger,h^\dagger)$. The characters $\theta^\dagger=\tau_{\Lambda,\Lambda^\dagger,\beta,\beta^\dagger}(\theta)$ and $\theta'^\dagger=\tau_{\Lambda',\Lambda'^\dagger,\beta',\beta'^\dagger}(\theta')$ are elements of $\mathcal{C}(\Lambda^\dagger,r,\beta^\dagger)^{\sigma_{h^\dagger}}$ and $\mathcal{C}(\Lambda'^\dagger,r,\beta'^\dagger)^{\sigma_{h^\dagger}}$ respectively, which intertwine over $\mathrm{U}(V^\dagger,h^\dagger)$ (cf. [12, §3] for more details on constructions like this). Thus we can restrict to the case that both lattice sequences coincide by [17, Proposition 5.2] and that the characters $\theta^\dagger$ and $\theta'^\dagger$ coincide by intertwining implies conjugacy B.1. By [10, 5.2 (i)] the residue fields of $F[\beta]$ and $F[\beta']$ coincide in $\mathfrak{a}/\mathfrak{a}_1$ and thus the induced action of σ_h on the residue fields also coincide, which finishes the proof. □

4 The non-symplectic case

Here we assume that we are in non-symplectic case, i.e. if $\epsilon = -1$ then $F \neq F_0$.

Proposition 4.1. If two pure self dual strata $[\Lambda, n, n-1, \beta]$ and $[\Lambda', n, n-1, \beta']$ of the same level intertwine over $\widetilde{G}$ then they intertwine over G^+.

Proof. As they intertwine, by [18, Proposition 7.1] the stratum $[\Lambda \oplus \Lambda', n, n-1, \beta \oplus \beta']$ is equivalent to a simple stratum, and thus by Proposition [20, 1.10] the stratum $[\Lambda \oplus \Lambda', n, n-1, \beta \oplus \beta']$ is equivalent to a self dual simple stratum $[\Lambda \oplus \Lambda', n, n-1, \gamma]$ with $\gamma \in \widetilde{G} \times \widetilde{G}$. We can therefore replace β and β' by skew-symmetric elements with the same minimal polynomial over F, implying that they are conjugate by an element of G^+ by [18, Corollary 5.1] (here we use that h is not symplectic). Thus the strata in the statement intertwine over G^+. □

Proposition 4.2. Let $\theta_- \in C_-(\Lambda, r, \beta)$ and $\theta'_- \in C_-(\Lambda, r, \beta')$ be two simple characters of G^+. Then, their lifts (and hence θ_- and θ'_-) are conjugate over G^+ if and only if their lifts intertwine over $\widetilde{G}$.

Proof. Instead of θ_- and θ'_- we consider lifts $\theta \in \mathcal{C}(\Lambda, r, \beta)^{\sigma_h}$ and $\theta' \in \mathcal{C}(\Lambda, r, \beta')^{\sigma_h}$. The proof is by induction on r. The minimal case is given by Proposition 4.1. By the induction hypothesis, Theorem B.1 and the translation principle Theorem A.1 we can assume that there is a self dual simple stratum $[\Lambda, n, r+1, \gamma]$ equivalent to $[\Lambda, n, r+1, \beta]$ and $[\Lambda, n, r+1, \beta']$ such that θ and θ' coincide on $H^{r+1}(\gamma, \Lambda)$. But then there is skew-symmetric element c of $\mathfrak{a}_{-(r+1)}$ such that $\theta = \theta_0 \psi_{\beta - \gamma + c}$ and $\theta' = \theta_0 \psi_{\beta' - \gamma}$ for some $\theta_0 \in \mathcal{C}(\Lambda, r, \gamma)^{\sigma_h}$ and [18, Proposition 9.17 (i)], Proposition 4.1 and [18, Proposition 9.29 (ii)] imply that the θ and θ' intertwine over G^+. The result now follows from Theorem B.1. □

We give a corollary which will be useful in the next section.

Corollary 4.3. Suppose that two simple characters $\theta \in \mathcal{C}(\Lambda, r, \beta)^{\sigma_h}$ and $\theta' \in \mathcal{C}(\Lambda', r, \beta')^{\sigma_h}$ intertwine over $\widetilde{G}$ and both lattice sequences have the same period. Then, E/E_0 is ramified if and only if E'/E'_0 is ramified.

Proof. As in the proof of Lemma 3.2, we can reduce to the case where Λ and Λ' coincide. Proposition 4.2 implies that the characters are conjugate over G^+, and hence intertwine over G^+. Hence we can conclude by Lemma 3.2. □

We now define an equivalence relation for self dual ps-characters in the non-symplectic case:

Definition 4.4. Two self dual ps-characters Θ_- and Θ'_- for possibly different simple pairs, are called *endo-equivalent* if for all strong comparison pairs their respective realisations intertwine over G^+.

Theorem 4.5. Two self dual ps-characters Θ_- and Θ'_- are endo-equivalent if and only if there exists a strong comparison pair such that the realisations of Θ_- and Θ'_- on this comparison pair intertwine over G^+.

Proof. This follows from Proposition 4.2 and Theorem 2.11. □

5 Matching Witt towers

To incorporate the symplectic case we need a new idea because, in contrast to the non-symplectic case, two self dual embeddings of a field extension E/F into (A, σ_h) are not necessarily conjugate by an element of G^+. In fact, we have two conjugacy classes if β is non-zero, i.e. $E \neq E_0$. Thus we need more restrictive comparison pairs in the symplectic case.

Given two self dual field extensions $(E = F[\beta]/F, \sigma_E)$ and $(E' = F[\beta']/F, \sigma_{E'})$ with non-zero skew-symmetric elements β and β' there is a unique bijection

$$w_{\beta,\beta'} : \; W_{\sigma_E,-1}(E) \to W_{\sigma_{E'},-1}(E')$$

which sends the Witt tower of $\langle\beta\rangle$ to the Witt tower of $\langle\beta'\rangle$ and preserves the anisotropic dimension. Similarly, there is a unique bijection

$$w_{\beta^2,\beta'^2} : \; W_{\sigma_E,1}(E) \to W_{\sigma_{E'},1}(E')$$

which sends the Witt tower of $\langle\beta^2\rangle$ to the Witt tower of $\langle\beta'^2\rangle$ and preserves the anisotropic dimension. We first notice exactly when w_{β^2,β'^2} coincides with $w_{1,1}$.

Lemma 5.1. We have $w_{1,1} = w_{\beta^2,\beta'^2}$ if and only if $-1 \in E^{(2)} \cap E'^{(2)}$ or $-1 \notin E^{(2)} \cup E'^{(2)}$ granting β and β' to be non-zero.

Proof. If $-1 \in E^{(2)} \cap E'^{(2)}$ then β^2 and β'^2 are norms and thus $w_{1,1} = w_{\beta^2,\beta'^2}$. If $-1 \notin E^{(2)} \cup E'^{(2)}$ then E/E_0 and E'/E'_0 are both ramified and β and β' have odd valuation. Thus $\nu_E(\beta^2)$ and $\nu_{E'}(\beta'^2)$ are even, but not divisible by 4. As $-1 \notin E^{(2)} \cup E'^{(2)}$ it follows that these squares are not norms and thus $w_{1,1} = w_{\beta^2,\beta'^2}$. Analogously, if $-1 \in E^{(2)}$ and not $-1 \in E'^{(2)}$ then $w_{1,1} \neq w_{\beta^2,\beta'^2}$ □

Let

$$w = \begin{cases} w_{\beta,\beta'} & \text{if } \epsilon = -1; \\ w_{\beta^2,\beta'^2} & \text{if } \epsilon = 1. \end{cases}$$

Let φ, φ' be self dual embeddings of E, E' into A. Let h_φ (resp. $h_{\varphi'}$) be the nondegenerate ϵ-hermitian (E/E_0)-form (resp. ϵ-hermitian (E'/E'_0)-form) defined as in Section 2.3 from these embeddings using the standard equivariant linear forms $\lambda_\beta, \lambda_{\beta'}$ defined by β, β'. For the case of $\beta = \beta' = 0$ we define $h_0 := h$ and w to be the identity of $W_{\sigma,-1}(F) \cup W_{\sigma,1}(F)$.

Definition 5.2. Let $F[\beta]$ and $F[\beta']$ be two self-dual field extension such that β and β' are simultaneously zero or non-zero. We say that the Witt towers of h_φ and $h_{\varphi'}$ *match* if $w((h_\varphi)_\equiv) = (h_{\varphi'})_\equiv$, and similarly we call a comparison pair $(((V, h), \varphi, \Lambda, r,), ((V, h), \varphi', \Lambda', r',))$ *Witt* whenever the Witt towers of h_φ and $h_{\varphi'}$ match.

If we are given a self dual simple stratum $[\Lambda, n, r, \beta]$ then we fix in this section a defining sequence $[\Lambda, n, r + i, \gamma^{(i)}]$ for $i = 0, \ldots, n - r$ of simple strata, as in [9, 2.4.2]. And analogously, for a self dual simple stratum $[\Lambda', n, r, \beta']$ we fix a defining sequence $[\Lambda', n, r+i, \gamma'^{(i)}]$. The main result of this section is that matching of Witt towers is inherited along defining sequences.

Proposition 5.3. Let $\theta \in \mathcal{C}(\Lambda, r, \beta)^{\sigma_h}$ and $\theta' \in \mathcal{C}(\Lambda, r, \beta')^{\sigma_h}$ be simple characters. Suppose that θ and θ' intertwine over $\widetilde{G}$, $e(\Lambda) = e(\Lambda')$, and if G^+ is symplectic that the Witt towers of h_β and $h_{\beta'}$ match. Then the Witt towers of $h_{\gamma^{(i)}}$ and $h_{\gamma'^{(i)}}$ match for all i.

Let β and γ be two non-zero skew-symmetric elements of A generating field extensions E and K of F.

Lemma 5.4. Assume that we are in the non-symplectic case, i.e. $\epsilon = 1$ or $F \neq F_0$. Suppose that $\dim_E V$ and $\dim_K V$ have the same parity and either this is even or $w_{1,1} = w_{\beta^2,\gamma^2}$ and $\dim_F V$ is odd. Then the Witt towers of h_β and h_γ match.

Proof. By twisting with a skew element of F in the unitary case, if necessary, we can always assume that $\epsilon = 1$. If $\dim_E V$ is even and if $\lambda_\beta^*(h_\beta)$ is not hyperbolic then by injectivity of λ_β^* on the set of classes of even anisotropic dimension and as hyperbolic spaces transfer to hyperbolic spaces, the form h is not hyperbolic and thus $\lambda_\gamma^*(h_\gamma)$ is not hyperbolic. Hence

$$w_{\beta^2,\gamma^2}((h_\beta)_\equiv) = (h_\gamma)_\equiv,$$

Similarly, if $\lambda_\gamma^*(h_\beta)$ is hyperbolic we find that $w_{\beta^2,\gamma^2}((h_\beta)_\equiv) = (h_\gamma)_\equiv$.

Next, if $\dim_F V$ is odd, by assumption we have $w_{1,1} = w_{\beta^2,\gamma^2}$. Thus we have to show that $h_\beta \equiv \langle 1\rangle$ in $W_{\sigma_E,1}(E)$ if and only if $h_\gamma \equiv \langle 1\rangle$ in $W_{\sigma_K,1}(K)$. However, as $\dim_F V$ is odd $[E:F]$ and $[K:F]$ are odd and, we have

$$\lambda_\beta^*(\langle 1\rangle) \equiv \lambda_\gamma^*(\langle 1\rangle) \equiv \langle 1\rangle,$$

by Proposition 2.4. Finally, we conclude by injectivity of both transfers on their sets of classes of odd anisotropic dimension. □

Lemma 5.5. Suppose $\epsilon = 1$, $F = F_0$, both $\dim_E V$ and $\dim_K V$ are odd, $w_{1,1} = w_{\beta^2,\gamma^2}$ and there exists a tamely ramified σ_h-invariant extension $\widetilde{F}/F$ in $E \cap K$ not fixed by σ_h. Then $\lambda_\beta^*(\langle 1\rangle) = \lambda_\gamma^*(\langle 1\rangle)$, and in particular the Witt towers of h_β and h_γ match.

Proof. As $\widetilde{F}$ is not fixed by σ_h, the F-dimension $\dim_F V$ is even. By Proposition 2.4, we have

$$a :\equiv \lambda_\beta^*(\langle 1\rangle) \equiv \langle 1\rangle + \langle \pm N_{E/F}(\beta)\rangle, \qquad b :\equiv \lambda_\gamma^*(\langle 1\rangle) \equiv \langle 1\rangle + \langle \pm N_{K/F}(\gamma)\rangle,$$

where the signs are determined by Proposition 2.4, however the signs are not important for this proof. If $[E:\widetilde{F}]$ is even then the $\widetilde{F}$-dimension of V is even, so $[K:\widetilde{F}]$ is even. Hence the images of the maps $\lambda_\beta^* : W_{\sigma_E,1}(E) \to W_{\mathrm{id},1}(F)$ and $\lambda_\gamma^* : W_{\sigma_K,1}(K) \to W_{\mathrm{id},1}(F)$ coincide, consisting of only two elements, the class of hyperbolic F-spaces and the class of the space of maximal anisotropic dimension. If $[E:\widetilde{F}]$ is odd, then the transfer $\lambda_\beta^* : W_{\sigma_E,1}(E) \to W_{\sigma_{\widetilde{F}},1}(\widetilde{F})$ is a bijection, hence the image of $\lambda_\beta^* : W_{\sigma_E,1}(E) \to W_{\mathrm{id},1}(F)$ is equal to the image of the trace transfer $\mathrm{Tr}_{\widetilde{F}/F}^* : W_{\sigma_h,1}(\widetilde{F}) \to W_{\mathrm{id},1}(F)$ (or any other transfer), by Lemma 2.2. Similarly, the same holds for λ_γ^*. Thus, the transfers λ_β^* and λ_γ^* have the same image. Denote by M the common image of λ_β^* and λ_γ^* in $W_{\mathrm{id},1}(F)$.

Now, assume for a contradiction that a and b are different. Then $a-b$ is a non-zero element of M of anisotropic dimension 2. In particular M must have 4 elements, because it contains also the classes of the hyperbolic and the maximal anisotropic space, and hence M has order 4. As λ_β^* and λ_γ^* are injective and the image is order 4, the classes a and b must be non-zero and $a-b$ is either a or b, as in our image we only have two elements of anisotropic dimension not 0 or maximal. Therefore $a = 2b$, as b is non-zero. Hence b has order 4 and M is cyclic. In particular, M has only one element of order 2. But this is impossible, because the class of the space of maximal anisotropic dimension and a both have order 2. □

Lemma 5.6. Let $[\Lambda, n, n-1, \beta]$ and $[\Lambda', n, n-1, \beta']$ be two G^+-intertwining self dual pure strata. Then there is an element $u \in G^+$ such that $E \cap uE'u^{-1}$ contains a tamely ramified σ_h-invariant extension $\widetilde{F}$ of F not fixed by σ_h. Moreover, the field $\widetilde{F}$ can be chosen to be $P^-(\Lambda)$-conjugate to the maximal tamely ramified subextension of $F[\gamma^{(n-1-r)}]/F$, i.e. of a minimal stratum equivalent to $[\Lambda, n, n-1, \beta]$, and if $\Lambda = \Lambda'$ then u can be chosen in $P(\Lambda)$.

Proof. Let $\widetilde{F}$ denote the maximal tamely ramified subextension of $F[\gamma^{(n-1-r)}]/F$ and analogously $\widetilde{F}'$ denote the maximal tamely ramified subextension $F[\gamma'^{(n-1-r)}]/F$. By Proposition 2.15 and [18, Theorem 5.2] we can assume that $\widetilde{F}$ is a subfield of E and that $\widetilde{F}'$ is a subfield of E'. By Proposition 2.15 there is an σ_h-equivariant isomorphism ϕ between $\widetilde{F}/F$ and $\widetilde{F}'/F$, and they are conjugate by an element of G^+ by [18, Theorem 5.2]. Now assume $\Lambda = \Lambda'$, then it follows from [17, Theorem 1.2] that $\widetilde{F}$ and $\widetilde{F}'$ are conjugate by an element of $P^-(\Lambda)$. □

Lemma 5.7. Let $[\Lambda, n, n-1, \beta]$ and $[\Lambda', n, n-1, \gamma]$ be $\widetilde{G}$-intertwining self dual pure strata. Suppose that $F \neq F_0$, $\dim_E V$ and $\dim_K V$ are odd, $\dim_F V$ is even, $e(E/F) = e(K/F)$, and $e(E/E_0) = e(K/K_0)$. Then,

(i) $w_{1,1} = w_{\beta^2,\gamma^2}$;

(ii) $\lambda^*_\beta(\langle 1\rangle) = \lambda^*_\gamma(\langle 1\rangle)$;

(iii) the Witt towers of h_β and h_γ match.

Proof. The conditions imply that (-1) is a square in E if and only if it is a square in K, and thus $w_{1,1} = w_{\beta^2,\gamma^2}$ by Lemma 5.1.

The strata intertwine over G^+ by Proposition 4.1, so after conjugation with the element which intertwines them, we can assume that they intertwine by the identity. By twisting with a skew-symmetric element of F if necessary we can also assume that $\epsilon = 1$. With the same argument as the end of the proof of Lemma 5.4, the first two assertions imply the third. Thus it remains to prove the second assertion.

Let π be a skew-symmetric uniformiser of F. Then we claim that $\pi^{-1}\beta$ is a norm with respect to E/E_0 if and only if $\pi^{-1}\gamma$ is a norm with respect to K/K_0. Indeed, the strata intertwine by 1, so

$$\pi^{-\nu_E(\beta)}\beta^{e(E/F)} \equiv \pi^{-\nu_K(\gamma)}\gamma^{e(K/F)}\ (\mathfrak{a}_1 + \mathfrak{a}'_1).$$

Thus $\pi^{-\nu_E(\beta)}\beta^{e(E/F)} + \mathfrak{p}_E$ is a square in k_E if and only if $\pi^{-\nu_K(\gamma)}\gamma^{e(K/F)} + \mathfrak{p}_K$ is a square k_K. If $e(E/E_0) = 2$ then $e(E/F)$ is odd as there is a skew-symmetric uniformiser of F which is a power of a skew-symmetric uniformiser of E, so the exponent has to be odd. Further, and $\nu_E(\beta)$ is odd as β is skew symmetric. Hence $\pi^{-1}\beta$ is a square if and only $\pi^{-1}\gamma$ is a square. Hence, $\pi^{-1}\beta$ is a norm if and only $\pi^{-1}\gamma$ is a norm by Remark (ii)(b). If $e(E/E_0) = 1$ then $\pi^{-1}\beta$ is a norm with respect to E/E_0 if and only if it has even valuation in E. Now, by our assumptions on the strata and the ramification indices, the integers $\nu_E(\pi^{-1}\beta)$ and $\nu_K(\pi^{-1}\gamma)$ agree, and thus $\pi^{-1}\beta$ is a norm with respect to E/E_0 if and only if $\pi^{-1}\gamma$ is a norm with respect to K/K_0. This completes the proof of the claim.

By Lemma 2.5, $N_{E/F}(\pi^{-1}\beta) = \pi^{-[E:F]}N_{E/F}(\beta)$ is a norm with respect to F/F_0 if and only if $\pi^{-[K:F]}N_{K/F}(\gamma)$ is a norm with respect to F/F_0. But the conditions imply that $[E:F]$ and $[K:F]$ are even, hence $-N_{E/F}(\beta)$ is a norm with respect to F/F_0 if and only if $-N_{K/F}(\gamma)$ is also a norm with respect to F/F_0. Thus, by Proposition 2.4, $\lambda^*_\beta(\langle 1\rangle)$ and $\lambda^*_\gamma(\langle 1\rangle)$ coincide. □

Proposition 5.8. Let $\theta \in \mathcal{C}(\Lambda, r, \beta)^{\sigma_h}$ and $\theta' \in \mathcal{C}(\Lambda', r, \beta')^{\sigma_h}$ be simple characters which intertwine over $\widetilde{G}$. Suppose that $e(\Lambda) = e(\Lambda')$, and if G is symplectic suppose further that θ and θ' intertwine over G^+. Then, the Witt towers of h_β and $h_{\beta'}$ match.

During the proof we use that twisting in the unitary case is a bijection and the following two elementary properties of twisting:

$$(h_\beta)^\beta = (h^\beta)_\beta, \quad (\ )^{\beta'} \circ w_{\beta,\beta'} \circ (\ )^{\beta^{-1}} = w_{\beta^2,\beta'^2}.$$

In particular, thanks to the first property, we can write h_β^β with no ambiguity.

Proof. From the intertwining of the two characters we find

(i) $[\Lambda, n, n-1, \beta]$ and $[\Lambda', n, n-1, \beta']$ intertwine over G^+ by Proposition 4.1;

(ii) $e(E/F) = e(E'/F)$ and $f(E/F) = f(E'/F)$;

(iii) $e(E/E_0) = e(E'/E'_0)$, by Lemma 3.2 and Corollary 4.3.

Now in the non-symplectic case the Lemmas 5.4, 5.5 and 5.7 imply that the Witt towers of h_β and $h_{\beta'}$ match. We are thus left with the symplectic case. The characters intertwine over G, so by conjugating if necessary we can assume that they intertwine by the identity. Thus, $[\Lambda, n, n-1, \beta]$ and $[\Lambda', n, n-1, \beta']$ intertwine by the identity. In particular, if we twist by β and β', then the orthogonal forms h^β and $h^{\beta'}$ are isomorphic by an element of $P^1(\Lambda')P^1(\Lambda)$ by [18, Corollary 3.2]. Hence, $w_{\beta^2,\beta'^2}(h^\beta_{\beta_\equiv})$ is equal to $h^{\beta'}_{\beta'_\equiv}$ by the previous cases and thus $w_{\beta,\beta'}(h_{\beta_\equiv}) = h_{\beta'_\equiv}$. □

Definition 5.9 (Witt comparison pair)**.** A comparison pair for self dual ps-characters, see Definition 3.1, is called *Witt*, if the corresponding Witt towers match.

Remark 5.10. In the non-symplectic case by Proposition 5.8, for two endo-equivalent self dual ps-characters every comparison pair is Witt.

We need two further results for the symplectic case to prove Proposition 5.3.

Lemma 5.11. Let $[\Lambda, n, n-1, \beta]$ and $[\Lambda', n, n-1, \gamma]$ be self dual pure strata which intertwine over G^+. Suppose that $\epsilon = -1$, $F = F_0$, $\dim_E V - \dim_K V$ is even, and either $\dim_E V$ is even, or $w_{1,1} = w_{\beta^2,\gamma^2}$. Then, the Witt towers of h_β and h_γ match.

Proof. We can assume that the strata intertwine by 1, and hence we have an isomorphism from h^β to h^γ given by an element of $P^1(\Lambda')P^1(\Lambda)$ by [18, Corollary 3.2]. Thus, we can restrict to the orthogonal case, and conclude by Lemmas 5.4 and 5.5. □

Lemma 5.12. Let $[\Lambda, n, n-1, \beta]$ and $[\Lambda, n, n-1, \gamma]$ be self dual pure strata which intertwine over G^+. Suppose $\epsilon = -1$, $F = F_0$, $\dim_E V$ is odd, $\dim_K V$ is even and γ is minimal. Then

(i) If $h_\beta \equiv \langle\beta\rangle$, then h_γ is hyperbolic if and only if -1 is a square in E.

(ii) If $h_\beta \not\equiv \langle\beta\rangle$, then h_γ is hyperbolic if and only if -1 is not a square in E.

Proof. By intertwining implies conjugacy for simple strata with same lattice sequence, by conjugating if necessary, we can assume that the strata are equivalent. We show that if $h_\beta \equiv \langle\beta\rangle$ and -1 is a square in E then h_γ is hyperbolic. The other cases can be proved in a similar fashion. As $h_\beta \equiv \langle\beta\rangle$, by twisting we have $h^\beta_\beta = \langle\beta^2\rangle$ and $\langle\beta^2\rangle = \langle 1\rangle$ because -1 is a square in E. However, h^β is isomorphic to h^γ by an element u of $P^1(\Lambda)$ and $[\Lambda, n, n-1, u^{-1}\gamma u]$ is a minimal self dual stratum with respect to h^β equivalent to $[\Lambda, n, n-1, \beta]$. By Lemma 5.6 we can choose u such that $E \cap u^{-1}Ku$ contains a σ_h-invariant, but non-fixed, tamely ramified extension $\widetilde{F}/F$. The conditions on the dimensions imply that $[E : \widetilde{F}]$ is even and thus $\lambda^*_\beta(\langle 1\rangle) = 0$ because of Proposition 2.4 and Proposition 2.3, i.e. h^β and h^γ are hyperbolic. Then, h^γ_γ is hyperbolic, by injectivity of λ^*_γ on classes of even anisotropic dimension, and thus h_γ is hyperbolic. □

We now can finish the proof of Proposition 5.3.

Proof of Proposition 5.3. The non-symplectic statement is given by Proposition 5.8 and the symplectic case follows from Lemmas 5.11 and 5.12 applied to the defining sequences together with the equivalence relation property of matching Witt towers. □

6 The symplectic case

In this section we consider only the symplectic case, that is we fix $\sigma = 1$ and $\epsilon = -1$. We now define the equivalence relation for the symplectic case using Witt comparison pairs.

Definition 6.1. Two ps-characters Θ_- and Θ'_- for possibly different simple pairs are called *endo-equivalent* if for all strong Witt comparison pairs for Θ_- and Θ'_- the corresponding realisations intertwine over G.

We now formulate an analogue of Theorem 4.5 for the symplectic case:

Theorem 6.2. Two self dual ps-characters Θ_- and Θ'_- are endo-equivalent if and only if there exists a strong Witt comparison pair with realisations that intertwine over G.

To prove Theorem 6.2 we need an analogue of Proposition 4.1:

Proposition 6.3. If two non-zero minimal self dual strata $[\Lambda, n, n-1, \beta]$ and $[\Lambda', n, n-1, \beta']$ of the same level intertwine over $\widetilde{G}$ and their Witt towers match then they intertwine over G.

Proof. As in the proof of Proposition 4.1, we replace β and β' with elements with the same minimal polynomial, and Proposition 5.8 ensures that the Witt towers still match. Hence, without loss of generality, we may assume β and β' have the same minimal polynomial. As the Witt towers match, under the isomorphism $\phi : E \to E'$, which maps β to β', h_β and $h_{\beta'}$ match. Thus β is conjugate to β' by an element of G which finishes the proof. □

Proposition 6.4. Let $\theta_- \in C_-(\Lambda, r, \beta)$ and $\theta'_- \in C_-(\Lambda, r, \beta')$ be two simple characters of G. Then, θ_- and θ'_- intertwine over G if and only if their lifts are conjugate over $\widetilde{G}$ and their Witt towers match.

Proof. If θ_- and θ'_- intertwine over G, then their lifts intertwine over $\widetilde{G}$ by the Glauberman correspondence, and hence are conjugate by intertwining implies conjugacy for characters of $\widetilde{G}$.

Moreover, by Proposition 5.8, their Witt towers match. For the converse, it follows from Proposition 5.3 that the Witt towers of the i-th members of the defining sequences of $[\Lambda, n, r, \beta]$ and $[\Lambda, n, r, \beta']$ match, and the result follows *mutatis mutandis* the proof of Proposition 4.2. Note that, in the argument of [18] we use in the proof of Proposition 4.2 there is a step to derived characters, which is here fine because G_γ is unitary, i.e. we do not need matching Witt towers for the derived characters (in fact, we know that they still match because of Proposition 5.3). □

This proposition gives an analogue to Corollary 4.3 for the symplectic case with a similar proof.

Corollary 6.5. Suppose that two simple characters $\theta \in \mathcal{C}(\Lambda, r, \beta)^{\sigma_h}$ and $\theta' \in \mathcal{C}(\Lambda', r, \beta')^{\sigma_h}$ intertwine over $\widetilde{G}$, both lattice sequences have the same period, and the Witt towers of h_β and $h_{\beta'}$ match. Then E/E_0 is ramified if and only if E'/E'_0 is ramified.

Proof. Similar to the proof of Corollary 4.3 using Proposition 6.4. □

7 Simple endo-classes

After Sections 5 and 6, we can improve Proposition 4.2 to allow for non-conjugate lattice sequences, and prove the analogue in the symplectic case using matching Witt towers:

Proposition 7.1. Let $\theta_- \in \mathcal{C}_-(\Lambda, r, \beta)$ and $\theta'_- \in \mathcal{C}_-(\Lambda', r, \beta')$ be self dual simple characters of G^+, and suppose that $e(\Lambda) = e(\Lambda')$.

(i) In the non-symplectic case, the self dual simple characters θ_- and θ'_- intertwine over G^+ if and only if their lifts intertwine over $\widetilde{G}$.

(ii) In the symplectic case, the self dual simple characters θ_- and θ'_- intertwine over G^+ if and only if their lifts intertwine over $\widetilde{G}$ and their Witt towers match.

Proof. Let $\theta \in \mathcal{C}(\Lambda, r, \beta)^{\sigma_h}$ and $\theta' \in \mathcal{C}(\Lambda', r, \beta')^{\sigma_h}$ be the unique lifts of the given self dual simple characters.

Suppose we are in the non-symplectic case. From the $\widetilde{G}$-intertwining we obtain by Corollary 4.3 and Proposition 5.8 that under a Witt basis the anisotropic part of h_β has the same dimension and the same number or diagonal elements of odd valuation as the anisotropic part of $h_{\beta'}$. Thus, there is a self dual $\mathfrak{o}_{E'}$-lattice sequence Λ'' which is G^+-conjugate to Λ. Let $\theta'' = \tau_{\Lambda',\Lambda'',\beta}(\theta') \in \mathcal{C}(\Lambda'', r, \beta')^{\sigma_h}$. The characters θ and θ'' intertwine by some element of $\tilde{G}$ by Theorem 2.13, and thus are conjugate by some element of G^+ by Proposition 4.2. Whence θ and θ' intertwine over G^+. The converse is clear.

The second statement in the symplectic case is completely analogous using Corollary 6.5 and Proposition 6.4 to show that $\widetilde{G}$-intertwining and Witt towers matching implies G^+-intertwining, and Proposition 5.8 for the converse. □

Using the proposition, we can now prove our main result on endo-equivalence of ps-characters:

Theorem 7.2. Let Θ_- and Θ'_- be two self dual ps-characters and Θ and Θ' their lifts. Then, the following assertions are equivalent:

(i) Θ_- and Θ'_- are endo-equivalent;

(ii) Θ and Θ' are endo-equivalent;

(iii) There is a comparison pair such that the realisations of Θ_- and Θ'_- intertwine over G^+;

(iv) For all Witt comparison pairs, the realisations of Θ_- and Θ'_- intertwine over G^+.

In the non-symplectic case these four assertions are equivalent to:

(v) For all comparison pairs, the realisations of Θ_- and Θ'_- intertwine over G^+.

Proof. If Θ and Θ' are endo-equivalent, it follows from Theorem 2.11 and Proposition 7.1 that Θ_- and Θ'_- are endo-equivalent. By definition, this implies there is a (strong) comparison pair such that the respective realisations of Θ_- and Θ'_- intertwine over G^+. And, if there is any comparison pair such that the respective realisations of Θ_- and Θ'_- intertwine over G^+, then by the Glauberman correspondence, this implies Θ and Θ' are endo-equivalent. Hence the first three properties are equivalent. Property (iv) implies Property (iii) and follows from Property (ii) by Proposition 7.1. In the non-symplectic case Property (v) follows from the equivalent properties (ii) and (iv) from Remark 5.10. Furthermore, Property (v) implies Property (iv). □

Remark 7.3. As endo-equivalence of ps-characters for general linear groups is an equivalence relation, see Theorem 2.13 (cf. [5, Corollary 8.10]), from Theorem 7.2 we deduce that endo-equivalence of self dual ps-characters for classical groups is an equivalence relation, and we call the equivalence classes of self dual ps-characters for this relation *simple* (σ, ϵ)*-endo-classes.* or *simple classical endo-classes* with respect to (σ, ϵ).

As already known for $\widetilde{G}$-intertwining of simple characters, see Theorem 2.13 (cf. [6, Corollary 2]), we also obtain that G^+-intertwining of self dual simple characters is a transitive relation, hence an equivalence relation:

Corollary 7.4. Let θ_-, θ'_- and θ''_- be self dual simple characters with the same group level and the same degree. If θ_- intertwines with θ'_- and θ'_- intertwines with θ''_- over G^+, then θ_- intertwines with θ''_- over G^+.

Proof. The corresponding self dual ps-characters Θ_- and Θ'_- and Θ''_- and Θ'_- are endo-equivalent by Theorem 7.2. Hence Θ_- and Θ''_- are endo-equivalent as endo-equivalence is an equivalence relation, by Remark 7.3, and thus θ_- and θ''_- intertwine over G^+, by Proposition 5.8 and again Theorem 7.2. □

8 Semisimple pairs

We recall definitions of the third author of semisimple strata and characters before defining semisimple pairs and potential semisimple characters.

8.1 Semisimple strata and characters

Let $[\Lambda, n, r, \beta]$ be a stratum in A. Suppose $V = \bigoplus_{i\in I} V^i$ is a decomposition of V into F-subspaces. We let $\Lambda^i = \Lambda \cap V^i$ and we let $\beta_i = \mathbf{e}^i\beta\mathbf{e}^i$, where $\mathbf{e}^i : V \to V^i$ is the projection with kernel $\bigoplus_{j\neq i} V^j$. The decomposition $V = \bigoplus_{i\in I} V^i$ of V is called a *splitting* of $[\Lambda, n, r, \beta]$ if $\beta = \sum_{i\in I} \beta_i$ and $\Lambda(k) = \bigoplus_{i\in I} \Lambda^i(k)$, for all $k \in \mathbb{Z}$.

Definition 8.1. A stratum $[\Lambda, n, r, \beta]$ in A is called *semisimple* if it is a zero-stratum or if $\nu_\Lambda(\beta) = -n$ and there exists a splitting $\bigoplus_{i\in I} V^i$ for $[\Lambda, n, r, \beta]$ such that

(i) for $i \in I$, the stratum $[\Lambda^i, n_i, r, \beta_i]$ in $\mathrm{End}_F(V^i)$ is simple, where $n_i = -\nu_{\Lambda^i}(\beta_i)$;

(ii) for $i, j \in I$ with $i \neq j$, the stratum $[\Lambda^i \oplus \Lambda^j, \max\{n_i, n_j\}, r, \beta_i + \beta_j]$ is not equivalent to a simple stratum in $\mathrm{End}_F(V^i \oplus V^j)$.

We write $E = F[\beta]$ and $E_i = F[\beta_i]$, hence $E = \bigoplus_{i\in I} E_i$ is a sum of fields, and we write $B_\beta = C_A(\beta)$. If $[\Lambda, n, r, \beta]$ is self dual with associated splitting $V = \bigoplus_{i\in I} V^i$ then, for each $i \in I$, there exists a unique $\sigma(i) = j \in I$ such that $\overline{\beta_i} = -\beta_j$. We set $I_0 = \{i \in I : \sigma(i) = i\}$ and choose a set of representatives I^+ for the orbits of σ in $I\backslash I_0$. Then we let $I_- = \sigma(I_+)$ so that we have a disjoint union $I = I_+ \cup I_0 \cup I_-$. A semisimple stratum $[\Lambda, n, r, \beta]$ is called *skew* if it is self-dual and the associated splitting $\bigoplus_{i\in I} V^i$ is orthogonal with respect to the ϵ-hermitian form h, i.e. $I = I_0$ in the notation above. By abuse of notation, we call a sum of $\mathfrak{o}_{E_i}$-lattice sequences in V (such as Λ above), an $\mathfrak{o}_E$-lattice sequence.

We call $\dim_F E$ the *degree* of a semisimple stratum and all objects defined using $E|F$ have the same notion of degree, e.g. characters and pss-characters, see below.

Let $[\Lambda, n, r, \beta]$ be a semisimple stratum in A. Associated to $[\Lambda, n, r, \beta]$ (and our initial choice of ψ_F) in [22, Section 3.2] are a compact open subgroup $H^{r+1}(\beta, \Lambda)$ of $P(\Lambda)$, and a set of characters $\mathcal{C}(\Lambda, r, \beta)$ called *semisimple characters*. For each $i \in I$, there is a natural embedding $H^{r+1}(\beta_i, \Lambda^i) \hookrightarrow H^{r+1}(\beta, \Lambda)$ and hence a map $\mathcal{C}(\Lambda, r, \beta) \to \mathcal{C}(\Lambda^i, r, \beta_i)$ which we write $\theta \mapsto \theta_i$. We call the θ_i the *simple block restrictions* of θ.

Suppose that $[\Lambda, n, r, \beta]$ is self dual. Then $H^{r+1}(\beta, \Lambda)$ is invariant under σ, and we define the group $H^{r+1}_-(\beta, \Lambda)$ as the intersection of $H^{r+1}(\beta, \Lambda)$ with G. We call the restrictions of semisimple characters in $\mathcal{C}(\Lambda, r, \beta)$ to $H^{r+1}_-(\beta, \Lambda)$ *self dual semisimple characters*, and denote the set of all such characters by $\mathcal{C}_-(\Lambda, r, \beta)$; this restriction coincides with the Glauberman correspondence (cf. [22, Section 3.6]). As in the simple setting, we write $\mathcal{C}(\Lambda, r, \beta)^{\sigma_h}$ for the subset of σ-invariant semisimple characters. If $\theta_- \in C_-(\Lambda, r, \beta)$ is a self dual semisimple character, then there is a unique $\theta \in \mathcal{C}(\Lambda, r, \beta)^{\sigma_h}$ whose restriction to $H^{r+1}_-(\beta, \Lambda)$ is θ_- by the Glauberman correspondence; we call θ the *lift* of θ_-.

Definition 8.2. Let $[\Lambda, n, 0, \beta]$ be a non-zero semisimple stratum. We let

$$k_0(\beta, \Lambda) = -\min\{r \in \mathbb{Z} : [\Lambda, n, r, \beta] \text{ is not semisimple}\}$$

denote the *critical exponent* of $[\Lambda, n, 0, \beta]$ and $k_F(\beta) := \frac{1}{e(\Lambda)} k_0(\beta, \Lambda)$; by [22, §3.1], this is independent of Λ. For the zero-stratum we keep $k(0, \Lambda) = k_F(0) = -\infty$.

Remark 8.3. It is possible to generalise the critical exponent to all pairs (β, Λ) where β generates a product of fields, and Λ decomposes into $\mathfrak{o}_{E_i}$-lattice sequences, in the following way: Assume β is non-zero, then there is a negative integer l such that $[\Lambda, -\nu_\Lambda(\beta) - le(\Lambda), 0, \varpi_F^l \beta]$ is a semisimple stratum. We can define $k_0(\beta, \Lambda)$ as $k_0(\varpi_F^l \beta, \Lambda) - le(\Lambda)$. This is not used in the sequel.

8.2 Semisimple pairs and potential semisimple characters

Definition 8.4. A *semisimple pair* is a pair $[k, \beta]$ where $E = F[\beta]$ is a sum of pairwise non-isomorphic fields $E = \bigoplus_{i\in I} E_i$, and k is an integer satisfying

$$0 \leqslant \frac{k}{e} < -k_F(\beta),$$

where $e = \mathrm{lcm}_{i\in I}(e(E_i/F))$ is the lowest common multiple of the ramification indices. A semisimple pair $[k, \beta]$ is called self dual if we can extend σ to an involution σ_E on E which maps β to β.

Let $[k, \beta]$ be a self dual semisimple pair, and write the minimal polynomial of β as $\Psi(X) = \prod_{i\in I} \Psi_i(X)$ with $\Psi_i(X)$ irreducible, so $E_i \simeq F[X]/(\Psi_i(X))$. The action of σ_E on the primitive idempotents defines an action on I. We let $I_0 = \{i \in I : \sigma(i) = i\}$, and choose a set of representatives I^+ for the orbits of σ in $I\backslash I_0$. Then we let $I_- = \sigma(I_+)$ so that we have a disjoint union $I = I_+ \cup I_0 \cup I_-$. A self dual semisimple pair $[k, \beta]$ is called *skew* if $I = I_0$ in the notation above.

Given a semisimple pair $[k, \beta]$ with non-zero β, let $\mathcal{Q}(k, \beta)$ denote the class of quadruples (V, φ, Λ, r) consisting of: a finite dimensional F-vector space V; an embedding $\varphi : E \hookrightarrow A$; a $\varphi(\mathfrak{o}_E)$-lattice sequence Λ in V; and an integer r such that the *group level* $\left\lfloor \frac{r}{e(\Lambda_E)} \right\rfloor$ is equal to k, where $e(\Lambda_E)$ is defined to be the greatest common divisor of all $e((\Lambda^i)_{E_i})$. Given such a quadruple (V, φ, Λ, r), we obtain a semisimple stratum $[\Lambda, -\nu_\Lambda(\varphi(\beta)), r, \varphi(\beta)]$ with splitting $V = \bigoplus_{i\in I} V^i$ where $V^i = \ker(\Psi_i(\beta))$; we call this semisimple stratum a *realisation* of the semisimple pair $[k, \beta]$. For the semisimple pair $[k, 0]$ we take the notation of section 2.4.

Given a self dual semisimple pair $[k, \beta]$ with non-zero β, let $\mathcal{Q}_-(k, \beta)$ denote the class consisting of all quadruples $((V, h), \varphi, \Lambda, r)$ where (V, φ, Λ, r) is an element of $\mathcal{Q}(k, \beta)$ and V is equipped with an ϵ-hermitian F-form; and φ, Λ are self dual with respect to this form. Given such a quadruple $((V, h), \varphi, \Lambda, r)$, we obtain a self dual semisimple stratum $[\Lambda, -\nu_\Lambda(\varphi(\beta)), r, \varphi(\beta)]$ with splitting $V = \bigoplus_{i\in I} V^i$ where $V^i = \ker(\Psi_i(\beta))$.

For realisations $[\Lambda, n, r, \varphi(\beta)]$ and $[\Lambda', n', r', \varphi'(\beta)]$ of a semisimple pair $[k, \beta]$ there is a canonical bijection
$$\tau_{\Lambda,\Lambda',\beta} : \mathcal{C}(\Lambda, r, \varphi(\beta)) \to \mathcal{C}(\Lambda', r', \varphi'(\beta)),$$
by [22, Proposition 3.26]. By [22, Proposition 3.32], if $[\Lambda, n, r, \varphi(\beta)]$ and $[\Lambda', n', r', \varphi'(\beta)]$ are self dual realisations of a self dual $[k, \beta]$, then $\tau_{\Lambda,\Lambda',\beta}$ commutes with the involutions defined on $\mathcal{C}(\varphi(\beta), r, \Lambda)$ and $\mathcal{C}(\varphi'(\beta), r', \Lambda')$ and restricts to give a bijection
$$\tau_{\Lambda,\Lambda',\beta} : \mathcal{C}_-(\Lambda, r, \varphi(\beta)) \to \mathcal{C}_-(\Lambda', r', \varphi'(\beta)).$$

Thanks to this result and with the definition of semisimple pairs, we can now define (self dual) potential semisimple characters. At first we define $\mathcal{C}(k, \beta)$ and $\mathcal{C}_-(k, \beta)$ as in (2.10).

Definition 8.5. Let $[k, \beta]$ be a semisimple pair.

(i) A *potential semisimple character*, or *pss-character*, supported on $[k, \beta]$ is a function
$$\Theta : \mathcal{Q}(k, \beta) \to \mathcal{C}(k, \beta)$$
such that $\Theta(V, \varphi, \Lambda, r) \in \mathcal{C}(\Lambda, r, \varphi(\beta))$, for all $(V, \varphi, \Lambda, r) \in \mathcal{Q}(k, \beta)$, and
$$\Theta(V', \varphi', \Lambda', r') = \tau_{\Lambda,\Lambda',\beta}(\Theta(V, \varphi, \Lambda, r)),$$
for all $(V, \varphi, \Lambda, r), (V', \varphi', \Lambda', r') \in \mathcal{Q}(k, \beta)$.

(ii) Suppose that $[k, \beta]$ is self dual. A pss-character supported on a self dual simple pair $[k, \beta]$ is called *σ-invariant* if for any (or equivalently one) self dual realisation of $[k, \beta]$ the value is σ-invariant.

(iii) Suppose that $[k,\beta]$ is self dual. A *self dual potential semisimple character*, or *self dual pss-character*, supported on the self dual semisimple pair $[k,\beta]$ is a function

$$\Theta_- : \mathcal{Q}_-(k,\beta) \to \mathcal{C}_-(k,\beta)$$

such that $\Theta_-((V,h),\varphi,\Lambda,r) \in \mathcal{C}_-(\Lambda,r,\varphi(\beta))$, for all $((V,h),\varphi,\Lambda,r) \in \mathcal{Q}_-(k,\beta)$, and

$$\Theta_-((V',h'),\varphi',\Lambda',r') = \tau_{\Lambda,\Lambda',\beta}(\Theta_-((V,h),\varphi,\Lambda,r)),$$

for all $((V,h),\varphi,\Lambda,r),((V',h'),\varphi',\Lambda',r') \in \mathcal{Q}(k,\beta)$.

As in the simple setting, every self dual pss-character comes uniquely from the restriction of a σ-invariant pss-character, we call the values of a pss-character *realisations*, and we call the unique σ-invariant pss-character restricting to a self dual pss-character its *lift*.

Let $[k,\beta]$ be a self dual semisimple pair, whose indexing set splits as a disjoint union $I = I_+ \cup I_0 \cup I_-$ as above, and $((V,h),\varphi,\Lambda,r) \in \mathcal{Q}_-(k,\beta)$. Then we can write

$$V = \bigoplus_{i\in I_+} (V^i \oplus V^{\sigma(i)}) \oplus \bigoplus_{i\in I_0} V^i.$$

Moreover, writing M for the Levi subgroup of G^+ stabilising this decomposition, as in [14, §3.11] we have

$$H_-^{r+1}(\beta,\Lambda) \cap M \simeq \prod_{i\in I_+} H^{r+1}(\beta_i,\Lambda_i) \times \prod_{i\in I_0} H_-^{r+1}(\beta_i,\Lambda^i).$$

Let Θ_- be a self dual pss-character supported on $[k,\beta]$. Then, after identifying $H_-^{r+1}(\beta,\Lambda)\cap M$ with the decomposition above, we have

$$\Theta_-((V,h),\varphi,\Lambda,r)|_{H^{r+1}(\beta,\Lambda)\cap M} = \bigotimes_{i\in I_+} \Theta_i(V^i,\varphi_i,\Lambda_i,r)^2 \otimes \bigotimes_{i\in I_0} \Theta_{i,-}((V^i,h_i),\varphi_i,\Lambda_i,r)$$

where

(i) For $i \in I_+$, Θ_i is a ps-character supported on the simple pair $[k,\beta_i]$.

(ii) For $i \in I_0$, $\Theta_{-,i}$ is a self dual ps-character supported on the self dual simple pair $[k,\beta_i]$.

Moreover, as in [1, 4.3 Lemma 1],

$$H^{r+1}(\beta_i,\Lambda_i) = H^{r+1}(2\beta_i,\Lambda_i), \quad \Theta_i(V^i,\varphi_i,\Lambda_i,r)^2 \in \mathcal{C}(\Lambda_i,r,2\beta_i), \quad \text{for } i \in I_+.$$

We record these observations, along with the decomposition of a pss-character into ps-characters in the following lemma:

Lemma 8.6. Let $[k,\beta]$ be a semisimple pair with index set I and $\beta = \bigoplus_{i\in I}\beta_i$, and let Θ be a pss-character supported on $[k,\beta]$.

(i) If β_i is non-zero, the pair $[k,\beta_i]$ is a simple pair. Moreover, the map Θ_i, defined by restricting Θ to its i-th component, is well defined and is a ps-character supported on $[k,\beta_i]$.

(ii) Suppose $[k,\beta]$ is self dual with $I = I_+ \cup I_0 \cup I_-$, and Θ_- is a self dual pss-character supported on $[k,\beta]$ with lift Θ.

  (a) For $i \in I_+$, the pair $[k,\beta_i]$ is a simple pair, and the map Θ_i^2, defined by restriction of Θ_- to its $(i \cup \sigma(i))$-th component, is well defined and when the domain is identified with the domain of the i-th component this restriction is the square of a ps-character Θ_i relative to $[k,\beta_i]$, and is a ps-character relative to $[k,2\beta_i]$.

  (b) For $i \in I_0$, the pair $[k,\beta_i]$ is a self dual simple pair, and the map $\Theta_{-,i}$, defined by restriction of Θ_- to its i-th component, is well defined and is a self dual ps-character.

9 Matchings for intertwining semisimple characters

In this section we consider two semisimple strata $[\Lambda, n, r, \beta]$ and $[\Lambda', n, r, \beta']$, both lattice sequences having the same period and the associated splittings $V = \bigoplus_{i\in I} V^i$ and $V' = \bigoplus_{i\in I'} V'^i$, respectively. The starting point for this section is the following result of the second and third authors:

Theorem 9.1 ([18, Theorem 10.1])**.** Let $\theta \in \mathcal{C}(\Lambda, r, \beta)$ and $\theta' \in \mathcal{C}(\Lambda', r, \beta')$ be intertwining semisimple characters. Then there is a unique bijection $\zeta : I \to I'$ and an element g of $\widetilde{G}$ such that, for all $i \in I$,

(i) we have $gV^i = V'^{\zeta(i)}$;

(ii) the characters ${}^g\theta_i$ and $\theta'_{\zeta(i)}$ intertwine.

Moreover, any element which satisfies the first property also satisfies the second property. In particular, $e(E_i|F) = e(E'_{\zeta(i)}|F)$, $f(E_i|F) = f(E'_{\zeta(i)}|F)$, $k_0(\beta_i, \Lambda^i) = k_0(\beta'_{\zeta(i)}, \Lambda'^{\zeta(i)})$ by Theorem 2.14.

A map between splittings of two semisimple strata which satisfies Properties (i), (ii) of the theorem is called a *matching*. Let $\theta \in \mathcal{C}(\Lambda, r, \beta)$ and $\theta' \in \mathcal{C}(\Lambda', r, \beta')$ be semisimple characters. We denote by $I(\theta, \theta')$ the set of elements of $\widetilde{G}$ which intertwine θ with θ', i.e. those $g \in \widetilde{G}$ such that

$$\mathrm{Hom}_{{}^gH^{r+1}(\beta,\Lambda)\cap H^{r+1}(\beta',\Lambda')}({}^g\theta, \theta') \neq 0.$$

We put $I_G(\theta, \theta') = I(\theta, \theta') \cap G$. Given a semisimple character $\theta \in \mathcal{C}(\Lambda, r, \beta)$ and a subset J of the index set I we denote by θ_J the restriction of θ to the intersection of its domain with $\mathrm{Aut}_F(\bigoplus_{i\in J} V^i)$.

Corollary 9.2. Under the assumptions of Theorem 9.1, we have

$$I(\theta, \theta') = S_r(\beta', \Lambda') \left(\prod_{i\in I} I(\theta_i, \theta'_{\zeta(i)}) \right) S_r(\beta, \Lambda)$$

where $S_r(\beta, \Lambda)$ is a subset of $P_1(\Lambda)$ and of the normaliser of each element of $\mathcal{C}(\Lambda, r, \beta)$ which only depends on Λ, β and r (cf. [9, (3.5.1)]). Assume further that for $i \in I_0$, V^i and $V'^{\zeta(i)}$ are isomorphic ϵ-hermitian spaces and both characters are self dual. Then, we have

$$I_G(\theta, \theta') = (S_r(\beta', \Lambda') \cap G^+) \left(\left(\prod_{i\in I_+\cup I_0} I(\theta_{\{i,\sigma(i)\}}, \theta'_{\{\zeta(i),\sigma(\zeta(i))\}}) \right) \cap G^+ \right) (S_r(\beta, \Lambda) \cap G^+).$$

For the proof of Corollary 9.2 we need a generalisation of [22, 3.22], i.e. the description of the intertwining of transfers.

Proposition 9.3. (i) Let $\theta' \in \mathcal{C}(\Lambda', r, \beta)$ be the transfer of $\theta \in \mathcal{C}(\Lambda, r, \beta)$. Then,

$$I(\theta, \theta') = S_r(\beta, \Lambda') B_\beta^\times S_r(\beta, \Lambda).$$

(ii) Let $\theta'_- \in \mathcal{C}_-(\Lambda', r, \beta)$ be the transfer of $\theta_- \in \mathcal{C}_-(\Lambda, r, \beta)$. Then,

$$I(\theta, \theta') \cap G^+ = (S_r(\beta, \Lambda') \cap G^+)(\mathrm{B}_\beta^\times \cap G^+)(S_r(\beta, \Lambda) \cap G^+).$$

See [22, Theorem 3.22] for the case $\Lambda = \Lambda'$.

Proof. This proof is anagolous to the proof of [12, Theorems 3.9 & 3.10].

(i) Let us at first assume that both latices sequences are block-wise principal lattice chains of the same block size. There is an element g in $\mathrm{B}_\beta^\times$ such that $g\Lambda$ is equal to Λ' and the conjugation with g realises the transfer from $\mathcal{C}(\Lambda, r, \beta)$ to $\mathcal{C}(\Lambda', r, \beta)$. Thus we can reduce to the case where θ is equal to θ' which follows from [22, Theorem 3.22].

We now consider the general case. Applying the †-construction of [12, §3.1], we obtain semisimple characters $\theta^\dagger \in \mathcal{C}(\Lambda^\dagger, r, \beta)$, $\theta'^\dagger \in \mathcal{C}(\Lambda'^\dagger, r, \beta)$, where $\Lambda^\dagger$ and $\Lambda'^\dagger$ are principal lattice chains of the same block size in a direct sum of e-copies of V (where $e = e(\Lambda) = e(\Lambda')$). Hence, as above we have the formula (i) for $I(\theta^\dagger, \theta'^\dagger)$. As in [19, Corollary 4.14] we deduce that this formula behaves well under intersection with the Levi group M attached to the † construction, i.e.

$$I(\theta^\dagger, \theta'^\dagger) \cap M = (S_r(\beta^\dagger, \Lambda'^\dagger) \cap M)(B_{\beta^\dagger}^\times \cap M)(S_r(\beta^\dagger, \Lambda^\dagger) \cap M),$$

by [12, Theorem 2.7 (ii) (b)], using the group $\Gamma = \{\pm 1\}^e$, if we have the intersection property

$$S_r(\beta^\dagger, \Lambda'^\dagger) x S_r(\beta^\dagger, \Lambda^\dagger) \cap B_{\beta^\dagger}^\times = (S_r(\beta^\dagger, \Lambda'^\dagger) \cap B_{\beta^\dagger}^\times) x (S_r(\beta^\dagger, \Lambda^\dagger) \cap B_{\beta^\dagger}^\times),$$

for all $x \in B_{\beta^\dagger}^\times$. The proof of the intersection property follows mutatis mutandis to the proof of [12, Lemma 3.6]. We restrict to the first block of M to obtain the desired description of $I(\theta, \theta')$.

(ii) This follows from (i) and a cohomology argument [12, Theorem 2.7 (ii) (b)]; see [12, Corollary 3.7].

□

Proof of Corollary 9.2. By [9, 3.5.1], using the †-construction on the i-th component of Lemma 3.2, we obtain for every index i that E_i/F and $E'_{\zeta(i)}/F$ have the same ramification index and inertia degree. Thus there is an $\mathfrak{o}_E$ -lattice sequence Λ'' which is $\tilde{G}$ conjugate to Λ' by an element which maps Λ''^i to $\Lambda'^{\zeta(i)}$. Let $\theta'' = \tau_{\Lambda'', \Lambda, \beta}(\theta)$ be the transfer of θ to $\mathcal{C}(\Lambda'', r, \beta)$ which, by intertwining implies conjugacy [18, Theorem 10.2], is conjugate to θ' by an element which maps Λ''^i to $\Lambda'^{\zeta(i)}$. Thus we can assume that θ' and θ'' are equal. We can further assume that β and β' have the same associated splitting and that the matching ζ is the identity, see Theorem 9.1. By Proposition 9.3,

$$\begin{aligned} I(\theta, \theta') &= S_r(\beta, \Lambda'') B_\beta^\times S_r(\beta, \Lambda) \\ &= S_r(\beta, \Lambda'') \mathfrak{n}(\Lambda''_E) B_\beta^\times S_r(\beta, \Lambda) \\ &= S_r(\beta', \Lambda') \mathfrak{n}(\Lambda'_{E'}) B_\beta^\times S_r(\beta, \Lambda), \end{aligned}$$

the last equality as $S_r(\beta, \Lambda'')\mathfrak{n}(\Lambda''_E)$ and also $S_r(\beta', \Lambda')\mathfrak{n}(\Lambda'_{E'})$ is the normaliser of every element of $\mathcal{C}(\Lambda'', r, \beta)$. The second assertion follows from a cohomology argument as in [22, 4.14] (cf. also [12, 2.4]). □

We want to prove a conjecture of the second and third authors, cf. [18, Conjecture 10.4]. For this we need a Lemma which reduces the case of intertwining skew-simple characters to the case of transfers.

Corollary 9.4. Suppose $[\Lambda, n, r, \beta]$ and $[\Lambda', n, r, \beta']$ are two skew-simple strata of the same period. And suppose there are characters $\theta \in \mathcal{C}(\Lambda, r, \beta)$ and $\theta' \in \mathcal{C}(\Lambda', r, \beta')$ which intertwine by an element of G^+. Then there is a stratum $[\Lambda', n, r, \beta'']$ such that β'' is conjugate to β by an element of G^+ and $\mathcal{C}(\Lambda', r, \beta')$ is equal to $\mathcal{C}(\Lambda', r, \beta'')$ and there is a lattice sequence Λ'' normalised by $E^\times$ and an element g of G^+ such that $g\Lambda'$ is equal to Λ''.

Proof. As all of the arithmetical invariants are the same, we can build a lattice sequence Λ'' relative to β' such that there exists $g \in G^+$ with $g\Lambda = \Lambda''$ as follows: We can find an E- and an E'-Witt basis (v_j) and (v'_j) for h^β and $h^{\beta'}$ such that Gram matrices M and M' have for every entry the equality of valuations

$$\nu_E(m_{l,k}) = \nu_{E'}(m_{l,k}),$$

and such that (v'_j) splits Λ'. The latter splitting does mean that for every integer t we have a decomposition

$$\Lambda'_t = \bigoplus \mathfrak{p}_{E'}^{\mu_{j,t}} v_j.$$

We now define Λ'' via (just by replacing E' by E and v' by v)

$$\Lambda''_t = \bigoplus \mathfrak{p}_{E}^{\mu_{j,t}} v_j.$$

By [17, Proposition 5.2], there is a $g \in G^+$ such that $g\Lambda' = \Lambda''$. We conjugate θ' to the transfer of θ to Λ'' by an element g' of G^+ by Theorem B.1 (Here we used the equivalence of intertwining 7.4). Then, $\mathcal{C}(\Lambda', r, \beta')$ is equal to $\mathcal{C}(\Lambda', r, g'^{-1}\beta g')$. We define $\beta'' := g'^{-1}\beta g$. □

Theorem 9.5. Let $\theta_- \in C_-(\Lambda, r, \beta)$ and $\theta'_- \in C_-(\Lambda, r, \beta')$ be intertwining self dual semisimple characters. Then, for $i \in I_0$, the spaces V^i and $V'^{\zeta(i)}$ are isometric and the characters $\theta_{-,i}$ and $\theta'_{-,\zeta(i)}$ intertwine by an isomorphism from (V^i, h_i) to $(V'^{\zeta(i)}, h_{\zeta(i)})$, where $h_i, h_{\zeta(i)}$ denote the restrictions of h to $V^i, V^{\zeta(i)}$, respectively. Moreover, the Witt towers of h_{i,β_i} and $h_{\zeta(i),\beta'_{\zeta(i)}}$ match (via the isometry).

Proof. We work with the lifts of θ_- and θ'_-. By Corollary 9.2, it is enough to show the first assertion, and we proceed by induction. For minimal strata this result is known by [18, Proposition 7.10], and having done the proof for $r-1$ then using the Corollary 9.4 and the translation principle (see A.1) we can assume that both strata share the same γ in the first member of a defining sequence, and that $\theta|_{H^{r+2}(\gamma,\Lambda)}$ and $\theta'|_{H^{r+2}(\gamma,\Lambda)}$ intertwine by 1, i.e. they are transfers. Now a modification of [18, Proposition 9.29] for different lattice sequences shows that there is a $c \in \mathfrak{a}_{-(r+1)} \cap \prod_i A^{ii}$ such that $\psi_{s(\beta-\gamma+c)}$ intertwines with $\psi_{s(\beta'-\gamma)}$ via 1, and thus applying the result for strata of [18, Proposition 7.10] block-wise gives the result. The final assertion follows from Proposition 7.1. □

Corollary 9.6. Under the assumptions of Theorem 9.1, suppose that β and β' have the same characteristic polynomial and that θ' is the transfer of θ, then β_i and $\beta_{\zeta(i)}$ have the same characteristic polynomial for all $i \in I$.

Proof. By conjugation we can assume that β and β' coincide. Then being transfers of each other is equivalent to $1 \in I(\theta, \theta')$. The identity satisfies 9.1(i) and 9.1(ii), and the uniqueness statement in Theorem 9.1 implies that ζ is the trivial permutation of the index set, which finishes the proof. □

Remark 9.7. The conclusion of Corollary 9.6 would be false if we skip the transfer condition, as the following example shows. Suppose the characteristic of F is p. Take an element β_1 which is minimal over F and a principal lattice chain Λ normalised by β_1 such that $\nu_\Lambda(\beta_1) = -n$. In fact one can take a non-zero element in F. Take an element $\lambda \in (F \cap \mathfrak{a}_{-r})\backslash\mathfrak{a}_{-r+1}$ where we choose $r = \lfloor \frac{n}{2} \rfloor$. Then by [9, 3.5] the sets $\mathcal{C}(\Lambda, r-1, \beta_1 + l\lambda)$, $l = 0, \ldots, p-1$ coincide. The multiplication by ψ_λ is thus a bijection of $\mathcal{C}(\Lambda, r-1, \beta_1)$. Take an element θ_1 of $\mathcal{C}(\Lambda, r-1, \beta)$ and define $\theta = \bigotimes_{l=0}^{p-1} \theta_1\psi_{l\lambda}$ and $\theta' = \bigotimes_{l=1}^{p} \theta_1\psi_{l\lambda}$ as elements of $\mathcal{C}(\bigoplus_i \Lambda, r-1, \bigoplus_l(\beta_1 + l\lambda))$. Then, the matching from θ to θ' is a cyclic permutation but not the identity.

Finally, we can deduce a semisimple *Skolem–Noether* result:

Corollary 9.8. Let $\theta \in \mathcal{C}(\Lambda, r, \beta)^{\sigma_h}$ and $\theta' \in \mathcal{C}(\Lambda', r, \beta')^{\sigma_h}$ be two G^+-intertwining semisimple characters and suppose that β_i and $\beta'_{\zeta(i)}$ have the same characteristic polynomial for all indexes i. Then β and β' are conjugate in G^+.

Proof. By Theorem 9.5, we can assume that the matching ζ is the identity of I. Now, the characters θ_i and θ'_i intertwine over G^+. Hence, by [18, Theorem 5.2], β_i and β'_i are conjugate by an element of $\mathrm{U}(V^i, h_i)$. Thus β and β' are conjugate in G^+. □

10 Semisimple Endo-classes

We continue with the notation of Section 8.

Definition 10.1. Let $[k, \beta]$ and $[k' = k, \beta']$ be two semisimple pairs of the same degree and Θ and Θ' pss-characters with respect to them, respectively. We call Θ and Θ' *endo-equivalent* if there are realisations θ and θ' of them, respectively, on the same vector space, which intertwine. If the semisimple pairs are self dual then two self dual pss-characters Θ_- and Θ'_- are called *endo-equivalent* if there are realisations θ_- and θ'_- of them, respectively, on the same ϵ-hermitian space (V, h), which intertwine over an element of $\mathrm{U}(V, h)$.

We attach to two endo-equivalent pss-characters a bijection between the index sets defined by the matching of intertwining realisations. The following lemma states that this is well-defined.

Lemma 10.2. Let Θ and Θ' be two endo-equivalent pss-characters with intertwining realizations θ and θ' with matching ζ, and let $\tilde{\theta}, \tilde{\theta}'$ be a second pair of intertwining realisations with matching $\tilde{\zeta}$. Then the matchings coincide.

Proof. By conjugation, see Theorem 9.1, we can assume that the strata for θ an θ' have the same associated splitting and that $V^i = V^{\zeta(i)}$. Similar for the second pair of realisations. Adding up shifts of lattice sequences and cutting down parts of blocks reduces the problem to the case where V^i and $\tilde{V}^i$ have the same dimension and Λ^i and $\tilde{\Lambda}^i$ are principal lattice sequences for all indexes $i \in I$. The characters θ and $\tilde{\theta}$ are transfers and are therefore conjugate by [18, Theorem 10.2] taking the last sentence into account. Making a †-construction again we get as well that θ is conjugate to θ' and $\tilde{\theta}$ is to $\tilde{\theta}'$. We can take conjugations which respect the vector space decompositions, by Corollary 9.2, and thus $\tilde{\theta}'_{\tilde{\zeta}(i)}$ is conjugate to $\tilde{\theta}_i$ which is conjugate to θ_i which is conjugate to $\theta'_{\zeta(i)}$. Thus, the matching from $\tilde{\theta}'$ to θ' is given by $\zeta \circ \tilde{\zeta}^{-1}$. On the other hand, θ' and $\tilde{\theta}'$ are transfers of each other and the matching between them must be therefore the identity of I', by Corollary 9.6. Thus ζ and $\tilde{\zeta}$ coincide. □

We can now define the generalisation of comparison pairs defined in the simple case (see Definition 3.1).

Definition 10.3. Let $[k,\beta]$ and $[k,\beta']$ be two semisimple pairs and $\zeta : I \to I'$ be a bijection from I to I'.

(i) We call a pair
$$((V,\varphi,\Lambda,r),(V',\varphi',\Lambda',r'))$$
of quadruples $(V,\varphi,\Lambda,r)\in\mathcal{Q}(k,\beta)$ and $(V',\Lambda',r',\varphi')\in\mathcal{Q}(k,\beta')$ a ζ- *comparison pair* if
$$V = V',\ r = r',\ e(\Lambda) = e(\Lambda'),\ \dim_F V^i = \dim_F V'^{\zeta(i)}$$
such that
$$k = \left\lfloor \frac{r}{e(\Lambda_E)} \right\rfloor.$$

(ii) A ζ-comparison pair is called *strong* if $\Lambda = \Lambda'$ as $\mathfrak{o}_F$-lattice sequences and, for all $i \in I$ and all integers j, we have
$$\Lambda^i_j/\Lambda^i_{j+1} \cong \Lambda'^{\zeta(i)}_j/\Lambda'^{\zeta(i)}_{j+1}.$$

(iii) Suppose that the semisimple pairs and ζ are self dual and that ζ is self dual, i.e. it commutes with their involutions. A *ζ-comparison pair* for self dual pss-characters is an element of $\mathcal{Q}_-(k,\beta)\times\mathcal{Q}_-(k,\beta')$ which projects onto $\mathcal{Q}(k,\beta)\times\mathcal{Q}(k,\beta')$ to a ζ-comparison pair for pss-characters, with $h = h'$ satisfying that (V^i,h_i) is isometric to $(V^{\zeta(i)},h_{\zeta(i)})$, for all $i \in I_0$.

(iv) A ζ-comparison pair for self dual pss-characters is called *Witt* if it is block-wise Witt on I_0, see Lemma 8.6.

Proposition 10.4. Two pss-characters Θ and Θ' supported on semisimple pairs, say $[k,\beta]$ and $[k,\beta']$, respectively, are endo-equivalent if and only if there is a bijection $\zeta : I \to I'$ such that the ps-characters Θ_i and $\Theta'_{\zeta(i)}$, defined by component-wise restriction, are endo-equivalent, for all $i \in I$.

The map ζ in Proposition 10.4 is uniquely determined by Lemma 10.2 and we call it the *matching* from Θ to Θ'. Given two endo-equivalent self dual pss-characters Θ_- to Θ'_- with lifts Θ to Θ', respectively, then the matching from Θ to Θ' is also denoted as the matching from Θ_- to Θ'_-. This matching is self dual because it is also the matching between intertwining realisations of Θ_- to Θ'_-.

Proof. Suppose we find a bijection $\zeta : I \to I'$ such that the ps-characters Θ_i and $\Theta'_{\zeta(i)}$ are endo-equivalent, for all $i \in I$. We can assume by conjugation with an automorphism of V that V^i is equal to $V^{\zeta(i)}$, for all i. Taking a ζ-comparison pair, by restriction to the i-th block we have a comparison pair for which the realisations of Θ_i and $\Theta'_{\zeta(i)}$ intertwine as Θ_i and $\Theta'_{\zeta(i)}$ are endo-equivalent. We thus obtain a block diagonal intertwiner of realisations of Θ and Θ'. For the converse, take intertwining realisations of Θ and Θ'. We need to show that we can take these realizations, say $\theta\in\mathcal{C}(\Lambda,r,\phi(\beta))$ and $\theta'\in\mathcal{C}(\Lambda',r',\phi'(\beta'))$, such that Λ and Λ' have the same period and $r = r'$. Then it would follow from Theorem 9.1 the existence of ζ, such that the simple character θ_i and $\theta_{\zeta(i)}$ intertwine, have the same degree and the same group level which would finish the proof. By doubling we can modify Λ and Λ' such that both lattice sequences have the same period. So let us assume that $r < r'$. Then $\theta|_{H^{r'+1}(\phi(\beta),\Lambda)}\in\mathcal{C}(\Lambda,r',\gamma)$ for some semisimple stratum $[\Lambda,n,r',\gamma]$ in a defining sequence of $[\Lambda,n,r,\beta]$, and this character intertwines with θ'. Theorem 9.1 implies that γ and β' have the same degree and $e(\Lambda|E') = e(\Lambda'|F[\gamma])$. So γ and β have the same degree. Thus γ and β have the same number of simple blocks, say (γ_i) and (β_i), and $e(F[\gamma_i]|F) = e(F[\beta_i]|F)$ and $f(F[\gamma_i]|F) = f(F[\beta_i]|F)$ for all indexes i. Thus, $e(\Lambda|E) = e(\Lambda|F[\gamma]) = e(\Lambda'|E')$. Since θ and θ' have the same group level we obtain that $[\Lambda, n, r', \beta]$ is still semisimple so we could have chosen $\gamma = \beta$. Thus $\theta|_{H^{r'+1}(\phi(\beta),\Lambda)}$ is a realisation of Θ, i.e. we can reduce to $r = r'$. $\square$

We now prove transitivity of endo-equivalence of pss-characters, showing that endo-equivalence of pss-characters is an equivalence relation.

Proposition 10.5. Suppose we have pss-characters $\Theta \approx \Theta' \approx \Theta''$ then $\Theta \approx \Theta''$ and the matchings satisfy $\zeta_{\Theta',\Theta''} \circ \zeta_{\Theta,\Theta'} = \zeta_{\Theta,\Theta''}$. Two pss-characters are endo-equivalent if and only if there is a bijection ζ from the index set of the first to the index set of the second one such that for every ζ-comparison pair the corresponding realisations of the pss-characters intertwine in $\widetilde{G}$.

Proof. The second assertion follows Proposition 10.4 and Theorem 2.11. The first assertion follows now directly from Proposition 10.4 and the fact that endo-equivalence is an equivalence relation for ps-characters, by Theorem 2.12. $\square$

We call the equivalence classes GL-endo-classes. Since we consider the zero strata $[\Lambda, r, r, 0]$ to be simple strata, this definition includes the so-called *zero-endo-class*, for each $k \geqslant 0$: this consists of the unique ps(s)-character supported on $[k, 0]$, whose realisations are all trivial characters.

Now we can gather all the results of the previous sections to get the following:

Theorem 10.6. Let Θ_- and Θ'_- be two self dual pss-characters and Θ and Θ' their lifts. Then, the following assertions are equivalent:

(i) The self dual pss-characters Θ_- and Θ'_- are endo-equivalent;

(ii) The lifts Θ and Θ' are endo-equivalent.

(iii) There is a self dual bijection ζ between the index sets of the semisimple pairs such that for all Witt ζ-comparison pairs the corresponding realisations of Θ_- and Θ'_- intertwine over G^+.

Proof. If Θ_- and Θ'_- are endo-equivalent so are Θ and Θ' by [21, 2.5], i.e. (ii) follows from (i). Assertion (i) follows from (iii) because we can find realizations of the semisimple pairs such that for all indexes $i \in I_0$ the forms $h_{i,\phi(\beta_i)}$ and $h_{\zeta(i),\phi'(\beta'_{\zeta(i)})}$ are hyperbolic. We are left with the implication (ii)$\Rightarrow$(iii). Suppose that Θ and Θ' are endo-equivalent. Let ζ be the matching from Θ to Θ'. It is self dual because it is the matching of lifts θ and θ' of realisations of Θ_- and Θ'_- which must intertwine by Proposition 10.5. As in Lemma 8.6 we decompose Θ_- and Θ'_- into self dual ps-characters and ps-characters Θ_i and $\Theta'_{\zeta(i)}$, $i \in I_0 \cup I_+$. Take the pair of realisations of a Witt ζ-comparison pair. Then because of Proposition 10.4 and Theorem 7.2 these realisations intertwine block-wise, i.e. they intertwine. $\square$

One consequence of Theorem 10.6 and Proposition 10.5 is the transitivity of endo-equivalence, and we obtain the rather surprising result that intertwining of self dual semisimple characters with same group level is transitive, and hence equivalence relation:

Corollary 10.7. (i) For self dual pss-characters endo-equivalence is an equivalence relation, and the matchings of pairwise endo-equivalent self dual pss-characters Θ_-, Θ'_- and Θ''_- satisfy that $\zeta_{\Theta'_-,\Theta''_-} \circ \zeta_{\Theta_-,\Theta'_-}$ is equal to $\zeta_{\Theta_-,\Theta''_-}$.

(ii) Intertwining over G^+ is an equivalence relation on the class of self dual semisimple characters with the same group level and the same degree.

(iii) Intertwining over $\widetilde{G}$ is an equivalence relation on the class of semisimple characters with the same group level and the same degree.

Proof. The first and the last assertion follow from Theorem 10.6 and Proposition 10.5. The second assertion follows from Theorem 10.6 and the first assertion of this corollary. □

Definition 10.8. We call the equivalence classes of self dual pss-characters (σ, ϵ)-*endo-classes* or *classical endo-classes.*

11 Intertwining under special orthogonal groups

Here we consider endo-equivalent pss-characters Θ_- and Θ'_- for self-dual semisimple pairs $[k, \beta]$ and $[k, \beta']$ in the orthogonal setting $\sigma = 1$, $\epsilon = 1$. Let $\theta_- \in \mathcal{C}_-(\Lambda, r, \varphi(\beta))$ and $\theta'_- \in \mathcal{C}_-(\Lambda', r, \varphi'(\beta'))$ be realisations of Θ_- and Θ'_- respectively on an orthogonal space (V, h). In this section we analyse the intertwining classes of an orthogonal endo-class. Let us assume that Λ and Λ' have the same period, i.e. $e(\Lambda) = e(\Lambda')$.

Proposition 11.1. (i) Assume that all $\varphi(\beta_i)$ have the same valuation for the lattice sequence Λ, i.e. $\nu_\Lambda(\varphi(\beta_i)) = -q$ for all indexes i. The realisations θ_- and θ'_- intertwine over G if and only if the symplectic forms $h^{\varphi(\beta)}$ and $h^{\varphi'(\beta')}$ are isometric by an automorphism of V of determinant congruent 1 modulo $\mathfrak{p}_F$.

(ii) Assume that there is an index i_0 such that β_{i_0} is zero. Then, θ and θ' intertwine by an element of G.

Proof. (i) The assumption on the β_i is also true for the β'_i by the matching theorem, Theorem 9.1. As Θ_- and Θ'_- are endo-equivalent, there is an element g of G^+ which intertwines θ_- and θ'_-. Then, the fundamental strata $[\Lambda, n, n-1, \phi(\beta)]$ and $[\Lambda', n, n-1, \phi'(\beta')]$ intertwine under g and the twists of h, $h^{\varphi(\beta)}$ and $h^{\varphi'(\beta)}$, are isometric by an element of the form $u'gu$ with elements $u \in 1 + \mathfrak{a}_1(\Lambda)$ and $u' \in 1 + \mathfrak{a}_1(\Lambda')$, by [18, Proposition 3.1]. The element $u'gu$ has determinant congruent to the determinant of g modulo $\mathfrak{p}$. The fact that an isometry of a symplectic space has determinant 1 finishes the proof.

(ii) By Theorem 9.5 we can assume that both characters have the same associated splitting. The characters θ_{i_0} and θ'_{i_0} are trivial, and therefore intertwine under any element of the group $\mathrm{U}(V^{i_0}, h_{i_0})$, and in particular by an element of determinant -1. Thus, there is an element of G which intertwines the semisimple characters.

□

Theorem 11.2. (i) If there is an index i such that β_i is zero, then θ_- and θ'_- intertwine under an element of G.

(ii) If β has no zero component, then θ_- and θ'_- intertwine under an element of G if and only if $h^{\varphi(\beta)}$ and $h^{\varphi'(\beta)}$ are isometric by an automorphism of V of determinant congruent 1 modulo $\mathfrak{p}$. In this case every element of G^+ intertwining θ_- and θ'_- is in G.

Proof. (i) See Proposition 11.1.

(ii) Without loss of generality we can assume that the matching is the identity, because there is an element of G which maps the decomposition of V for $\varphi(\beta)$ to the decomposition of V for $\varphi'(\beta')$. Write the index set I in the coarsest way as the union of subsets S such that the elements $\varphi(\beta_s)$ have all the same valuation with respect to Λ, for all $s \in S$. We now apply Proposition 11.1 to all blocks S and we get the result.

□

As a corollary we have for semisimple characters an analogue of Corollary 10.7.

Corollary 11.3. Let θ_-, θ'_- and θ''_- be self dual semisimple characters with the same group level. If θ_- intertwines with θ'_- and θ'_- intertwines with θ''_- over G, then θ_- intertwines with θ''_- over G.

The next theorem is about intertwining and conjugacy of semisimple characters over special orthogonal groups. There are examples of θ and θ' which intertwine over G and are conjugate over G^+, but are not conjugate under G. This, can not happen if β has no zero-component. But if β has a zero-component $\beta_{i_0} = 0$ such that there is a lattice sequence in Λ^{i_0} which has no element of determinant -1 in its normaliser, then take an element g_{i_0} of determinant -1 in $\mathrm{U}(h_{i_0})$ and define $\theta'' := \mathrm{diag}(\mathrm{id}, g_{i_0}).\theta$, then θ and θ'' are not conjugate by an element of G, but intertwine by an element of G, by Theorem 11.2. We fix this problem in the following way:

Theorem 11.4. Suppose θ and θ' intertwine under an element g of G and there is an element g' of G^+ such that $g'\Lambda^i$ is equal to $\Lambda'^{\zeta(i)}$. We write $g = u\,\mathrm{diag}(g_i | i \in I_+ \cap I_0)v$ using Corollary 9.2. Suppose one of the following three assertions:

(i) The element β has a zero component, say β_{i_0}, Λ^{i_0} and θ_{i_0} have the same normaliser in $\mathrm{U}(h_{i_0})$, $g' \in G$ and $g_{i_0}^{-1} g'_{i_0}$ is an element of $\mathrm{SU}(h_{i_0})$.

(ii) The element β has a zero component, say β_{i_0}, and the normaliser of Λ^{i_0} contains an element of $\mathrm{U}(h_{i_0})$ with determinant -1.

(iii) The element β has no zero component.

Then θ is conjugate to θ' by an element of $G \cap P^-(\Lambda)$.

Remark 11.5. Unfortunately the theorem is very unsatisfying, because in the first part we have the condition that Λ^{i_0} and θ_{i_0} have the same normaliser in G^+. In general this is a proper condition, i.e. there are many examples where this fails. As an example consider the lattice sequence Λ on the Iwahori in $O(1,1)(F)$ but with three lattices in one period, such that for only one index j the lattice Λ_j is self dual. Then, $\mathfrak{a}_2$ is the radical of a maximal order and thus the trivial characters on $P_2(\Lambda)$ has a normaliser in $O(1,1)(F)$ bigger than the normaliser of Λ. But, in spite of the example, the normaliser condition holds if $r = 0$, we call this later *full*, see Section 13, because $\mathfrak{a}_1$ is the radical of Λ and both have the same normaliser.

Proof of Theorem 11.4. (i) We can assume g' is the identity. Therefore ζ is the identity and Λ^i and Λ'^i equal. In particular, on the block for the zero component of β the characters equal, i.e. $\theta_{i_0} = \theta'_{i_0}$. There is nothing to prove if Λ^{i_0} has an element of determinant -1 in its normaliser in $\mathrm{U}(h_{i_0})$. Thus let us assume the opposite, which by assumption implies that the $\mathrm{U}(h_{i_0})$-normaliser of θ_{i_0} is also contained in $\mathrm{SU}(h_{i_0})$. By Theorem B.1, θ and θ' are thus conjugate by an element of $P^-(\Lambda)$. Corollary 9.2 and Theorem 11.2 now imply that this conjugating element must have determinant 1.

(ii) This follows directly from Theorem B.1.

(iii) The determinant of the conjugating element and the intertwining element have to coincide by Proposition 11.1.

□

Remark 11.6. The second assumption of Theorem 11.4 is satisfied if the lattice sequence Λ^{i_0}, represents a vertex in the Bruhat Tits building of $\mathrm{U}(h_{i_0})$, i.e. (say for simplicity that $h = h_{i_0}$) there is an index l such that

$$\varpi_F \Lambda_l^{\#} \subset \Lambda_l \subseteq \Lambda_l^{\#},$$

and the image of Λ only consists of $F^\times$-multiples of Λ_l and its dual ($\Lambda_l = \Lambda_l^{\#}$ is possible).

Proposition 11.7. Let G be an $F = F_0$-form of a special orthogonal group on an F-vector space V. Suppose Λ is a self dual lattice sequence which corresponds to a vertex in the Bruhat Tits building of G^+. Then there is an element of $G^+\backslash G$ in the normaliser of Λ.

Proof. We have two cases. Let us at first assume that there is a positive anisotropic dimension. Then every lattice sequence Λ' is normalised by an orthogonal element of determinant -1, because we just have to take a Witt basis whose apartment contains the point corresponding to Λ', i.e. a Witt basis which splits the lattice sequence. Then the diagonal matrix $\mathrm{diag}(1,\ldots,1,-1)$ is an element of $P^-(\Lambda')$. Let us now assume that the anisotropic dimension is zero. We take a Witt basis which splits Λ, i.e. a basis $(v_j)_{j\in J_+\cup J_-}$ with $-J_+ = J_-$ and $h(v_j, v_{j'}) = \delta_{j,-j'}$ (Kronecker symbol) for all $j \in J_+$. We denote $M := \Lambda_l$, see Remark 11.6. Then,

$$M = \bigoplus_{j\in J_+\cup J_-} \mathfrak{p}_F^{\nu_j} v_j,$$

and we can rescale the basis the way that $\nu_j = 0$ for all $j \in J_+$. Then the condition on M rephrases as ν_j is 1 or 2, for $j \in J_-$, such that at least one exponent has to be 1, say $\nu_{-1} = 1$. We define an element of G^+ by

$$g(v_j) := v_j,\ j \neq 1,-1,\ g(v_1) := v_{-1}\varpi_F,\ g(v_{-1}) := v_1\varpi_F^{-1},$$

and g normalises Λ and has determinant -1. □

12 Intertwining implies conjugacy for cuspidal types

We recall definitions of the third author in [23] of cuspidal types in G, then prove our main theorem that two intertwining cuspidal types are conjugate in G.

A skew semisimple stratum $[\Lambda, n, 0, \beta]$ is called *cuspidal* if G_E has compact centre and $P^\circ(\Lambda_E)$ is a maximal parahoric subgroup in G_E. By [22, §3.2], associated to $[\Lambda, n, 0, \beta]$ are compact open subgroups $J(\beta,\Lambda) \supseteq J^1(\beta,\Lambda) \supseteq H^1(\beta,\Lambda)$ of G. The quotient $J(\beta,\Lambda)/J^1(\beta,\Lambda) \simeq$

$P(\Lambda_E)/P_1(\Lambda_E)$ is a finite reductive group, and we put $J^{\circ}(\beta,\Lambda)$ the preimage of its connected component in $J(\beta,\Lambda)$.

Let $[\Lambda,n,0,\beta]$ be a cuspidal skew semisimple stratum. Let $\theta\in\mathcal{C}_{-}(\Lambda,0,\beta)$ be a self dual semisimple character. By [22, Corollary 3.29], there exists a unique irreducible representation η of $J^1(\beta,\Lambda)$ containing θ. The representation η extends to $J(\beta,\Lambda)$ and we choose a particular type of extension κ as in [23, Theorem 4.1], called a *β-extension.* Let τ be an irreducible representation of $J(\beta,\Lambda)/J^1(\beta,\Lambda)$ with cuspidal restriction to $J^{\circ}(\beta,\Lambda)/J^1(\beta,\Lambda)$. Put $J=J(\beta,\Lambda)$ and $\lambda=\kappa\otimes\tau$. A pair (J,λ), constructed as above, is called a *cuspidal type* for G. The main result of [23] (noting the correction of [14, Appendix A]) can be stated as follows:

Theorem 12.1 ([23, Corollary 6.19 & Proposition 7.13])**.** Let (J,λ) be a cuspidal type for G. Then the representation $\mathrm{ind}_J^G(\lambda)$ is irreducible and cuspidal. Moreover, every irreducible cuspidal representation of G appears in this way.

Thus, it remains to determine when two cuspidal types $(J,\lambda),(J',\lambda')$ induce isomorphic cuspidal representations. Notice that, if $\mathrm{ind}_J^G(\lambda)\simeq\mathrm{ind}_{J'}^G(\lambda')$ then $(J,\lambda),(J',\lambda')$ intertwine in G.

Main Theorem (intertwining implies conjugacy). Cuspidal types intertwine in G if and only if they are conjugate in G.

Proof. As the cuspidal types intertwine so do the underlying skew semisimple characters θ,θ'. Suppose that $\theta\in\mathcal{C}_{-}(\Lambda,0,\beta)$ and $\theta'\in\mathcal{C}_{-}(\Lambda',0,\beta')$. By Theorem 9.5, there is a matching $\zeta:I\to I'$; and there exists $g\in G$ such that $gV^i=V'^{\zeta(i)}$ and the characters $\theta_{i,-}^{g_i}$ and $\theta'_{\zeta(i),-}$ intertwine in $G_{\zeta(i)}$. By conjugating by an element of G, if necessary, we can assume that $V=V'$, $\zeta=1$, and

$$I_{G_i}(\theta_{i,-},\theta'_{i,-})\neq 0.$$

As all of the arithmetical invariants are the same, we can build a lattice sequence Λ'' relative to β' such that there exists $g\in G^+$ with $g\Lambda'=\Lambda''$ respecting the block structure, as follows: For every index i there is an E_i'-lattice sequence Λ''^i and an element $g_i\in\mathrm{U}(h_i)$ such that $g_i\Lambda^i=\Lambda''^i$ by Corollary 9.4 and [17, Proposition 5.2]. And at the end we define Λ'' as the sum of the Λ''^i and $g:=\bigoplus_i g^i$. If there is some β'_{i_0} equal to zero and we are in the special orthogonal case then there is an element of determinant -1 in $P^-(\Lambda^{i_0})$ by Proposition 11.7, and Theorem 11.4 ensures the existence of an element g of G which conjugates Λ to Λ''.

We let $\theta''=\tau_{\Lambda',\Lambda'',\beta'}(\theta')$, which intertwines with θ by Corollary 10.7 and 11.3. By conjugating, see Theorem B.1 and Theorem 11.4, we can assume $\theta=\theta''$, and hence θ is a semisimple character for β'. We can now conclude by [12, Theorem 11.3]. □

Remark 12.2. In fact, due to [12], we have the analogous result in the greater generality of representations with coefficient field any algebraically closed field of characteristic prime to p.

13 Parametrisation of the intertwining classes of semisimple characters

This section is about the classification of intertwining classes of (self dual) semisimple characters. But before we start we need the following definition of endo-equivalent semisimple characters.

Definition 13.1. Two (self dual) semisimple characters are called *endo-equivalent* if their corresponding (self dual) pss-characters are endo-equivalent.

13.1 GL-endo-parameters

In this section we parametrise the intertwining classes of semisimple characters of $\widetilde{G}$. We say that a semisimple character is *full* if it lies in $\mathcal{C}(\Lambda, 0, \beta)$, for some semisimple stratum $[\Lambda, n, 0, \beta]$. Similarly, a pss-character is full if it is supported on a semisimple pair of the form $[0, \beta]$, while an endo-class of pss-characters is full if it consists of full pss-characters; note that this includes the full zero-endo-class $\mathbf{0}$.

Definition 13.2. We define $\mathcal{E}$ to be the set of all full simple endo-classes of ps-characters. A GL-*endo-parameter* for $\widetilde{G}$ is a function from the set $\mathcal{E}$ to the set of non-negative integers with finite support.

The goal of this section is to prove that a natural set of endo-parameters parametrise $\widetilde{G}$-intertwining classes of semisimple characters of $\widetilde{G}$. Let $\mathcal{E}^{fin}$ be the set of finite subsets of $\mathcal{E}$.

Proposition 13.3. There is a bijection between $\mathcal{E}^{fin}$ and the set of all full GL-endo-classes.

For the proof we need the following lemmas:

Lemma 13.4. Suppose that a semisimple stratum $[\Lambda, n, r, \beta]$ is split by $V = V^1 \oplus V^2$. Given $\tilde{\theta} \in \mathcal{C}(\Lambda, r+1, \beta)$ and an extension $\theta_1 \in \mathcal{C}(\Lambda^1, r, \beta_1)$ of $\tilde{\theta}|_{H^{r+2}(\beta_1,\Lambda^1)}$ there is a semisimple character $\theta \in \mathcal{C}(\Lambda, r, \beta)$ such that the restriction to $H^{r+1}(\beta_1, \Lambda^1)$ is θ_1 and the restriction to $H^{r+2}(\beta, \Lambda)$ is $\tilde{\theta}$.

Proof. We only need to show the case where $[\Lambda^2, n_2, r, \beta_2]$ is simple, because the general case is then obtained by induction along the simple blocks of $[\Lambda^2, n_2, r, \beta_2]$. The proof proceeds by an induction along the critical exponent $k_0 = k_0(\beta, \Lambda)$. If β is zero we take for θ the trivial character. Consider now $-k_0 \geqslant n$: At first we prove the case where r is smaller than $\lfloor \frac{-k_0}{2} \rfloor$. Then by [22, 3.15] there exists an element $\theta \in \mathcal{C}(\Lambda, r, \beta)$ such that the block restrictions are the block restrictions of θ_1 and an extension of $\tilde{\theta}|_{H^{r+2}(\beta_2,\Lambda^2)}$ to $H^{r+1}(\beta_2, \Lambda^2)$, if $[\Lambda^2, n, r, \beta_2]$ is a simple block of $[\Lambda, n, r, \beta]$. If $[\Lambda^2, n, r, \beta_2]$ is not a simple block of $[\Lambda, n, r, \beta]$ then there is a simple block $[\Lambda^{1,i}, n, r, \beta_{1,i}]$ such that β_2 and $\beta_{1,i}$ have the same minimal polynomial over F. Take for θ_2 the transfer of $\theta_{1,i}$. Then $\theta_{1,i} \otimes \theta_2$ is an element of $\mathcal{C}(\Lambda^{1,i} \oplus \Lambda^2, r, \beta_1 + \beta_2)$, and we apply [22, 3.15] to obtain the desired assertion. We consider now the case that $r \geqslant \lfloor \frac{-k_0}{2} \rfloor$. Then we take a semisimple stratum $[\Lambda, n, -k_0, \gamma]$ equivalent to $[\Lambda, n, -k_0, \beta]$ which is split under the $V^1 \oplus V^2$ and split under the associate splitting of β. By induction hypothesis we find a common extension $\theta' \in \mathcal{C}(\Lambda, r, \gamma)$ of $\theta_1 \psi_{\gamma_1 - \beta_1}$ and $\tilde{\theta} \psi_{\gamma - \beta}$ to $H^{r+1}(\beta, \Lambda)$. Put $\theta := \psi_{\beta - \gamma} \theta'$. □

Lemma 13.5. Let $\theta_1 \in \mathcal{C}(\Lambda^1, r, \beta_1)$ and $\theta_2 \in \mathcal{C}(\Lambda^2, r, \beta_2)$ be two semisimple characters, then there exists a semisimple stratum $[\Lambda := \Lambda^1 \oplus \Lambda^2, n = \max(n_1, n_2), r, \beta_1 \oplus \beta_2']$ such that the groups $H^{r+1}(\beta_2, \Lambda^2)$ and $H^{r+1}(\beta_2', \Lambda^2)$ coincide, and an element of $\mathcal{C}(\Lambda, r, \beta_1 \oplus \beta_2')$ with restrictions θ_1 and θ_2 on $H^{r+1}(\beta_1, \Lambda^1)$ and $H^{r+1}(\beta_2, \Lambda^2)$, respectively.

Proof. Because of Lemma A.8 we can accomplish the proof successively in adding simple characters to θ_1: Say θ_2 is the product of simple characters $\theta_{2,i}$ using its associated splitting, and suppose that we have found a character $\theta \in \mathcal{C}(\Lambda, r, \beta_1 + \sum_i \beta_{2,i}')$ with restrictions θ_1 and $\theta_{2,i}$ for all indexes i. Then θ and θ_2 have the same block restrictions on the $(2,i)$-blocks, and thus by Corollary A.9 $\theta|_{H^{r+1}(\Lambda^2, \sum_i \beta_{2,i}')}$ and θ_2 coincide. So thus let us assume without loss of generality that θ_2 is simple. We prove the statement by induction on r. If r is n then we take for θ the trivial characters. If $r < n$ then by induction hypothesis there is a character $\tilde{\theta} \in \mathcal{C}(\Lambda, r+1, \gamma_1 \oplus \gamma_2')$ with restrictions $\theta_i|_{H^{r+2}(\beta_i, \Lambda^i)}$, $i = 1, 2$. By the translation principle [18, 9.16] there is a simple stratum $[\Lambda^2, n, r, \beta_2'']$ with γ_2' in the first member of a defining sequence such that $\mathcal{C}(\Lambda^2, r, \beta_2)$ is equal to $\mathcal{C}(\Lambda^2, r, \beta_2'')$. Suppose that we could have chosen β_2'' such that $[\Lambda, n, r, \beta_1 \oplus \beta_2'']$ is semisimple. Now we can apply Lemma 13.4 which provides a semisimple character $\theta' \in \mathcal{C}(\Lambda, r, \beta_1 + \beta_2'')$ with restrictions θ_1 and $\tilde{\theta}$. Thus there is an element $a \in \mathfrak{a}^2_{-1-r}$ such that θ_2 is equal to $\theta'\psi_a$ on the domain of θ_2. We define $\theta := \psi_a\theta'$ and we choose $\beta_2' \in A^2$ such that $[\Lambda, n, r, \beta_1 \oplus \beta_2']$ is a semisimple stratum equivalent to $[\Lambda, n, r, \beta_1 \oplus (\beta_2'' + a)]$, which would finish the proof by Lemma A.8.

Now we have to show that β_2'' can be chosen such that $[\Lambda, n, r, \beta_1 \oplus \beta_2'']$ is semisimple. If it is not semisimple, then there is an index i for the associated splitting of β_1 such that $[\Lambda^{1,i} \oplus \Lambda^2, n_2, r, \beta_{1,i} \oplus \beta_2]$ is equivalent to a simple stratum. Then we can find a stratum $[\Lambda^2, n_2, r, \tilde{\beta}_2]$ with $\tilde{\beta}_2$ having the same minimal polynomial as $\beta_{1,i}$ such that $F[\tilde{\beta}_2]^\times$ normalizes Λ^2. The stratum is simple by the formula for the critical exponent which only depends on the period of Λ^2, which is the same as the period of Λ^1, and the minimal polynomial of $\tilde{\beta}_2$. The strata $[\Lambda^{1,i} \oplus \Lambda^2, n_2, r, \beta_{1,i} \oplus \tilde{\beta}_2]$ and $[\Lambda^{1,i} \oplus \Lambda^2, n_2, r, \beta_{1,i} \oplus \beta_2]$ are equivalent to simple strata. Thus the sum $[\Lambda^2 \oplus \Lambda^2, n_2, r, \beta_2 \oplus \beta_2]$ is equivalent to a simple stratum by a small combinatorial argument, i.e. the corresponding strata intertwine [18, 6.16]. And thus they are conjugate by intertwining implies conjugacy for simple strata, see [18, 8.1]. So we can replace β_2'' by a conjugate of $\tilde{\beta}_2$. This finishes the proof. □

Proof of Proposition 13.3. By Proposition 10.4, given a semisimple endo-class, the restriction to the blocks gives a tuple of simple endo-classes. Conversely, given a tuple of simple endo-classes, take from every class a value of a ps-character, so that one gets a tuple of simple characters. By Lemma 13.5 there is a semisimple character which has the given simple characters as restrictions. The corresponding pss-character defines a semisimple endo-class. □

Recall that the degree of a simple character is independent of intertwining and transfer. Thus we define the degree of a simple endo-class $c \in \mathcal{E}$ to be the degree of the values of the ps-characters in c, and we denote it by $\deg(c)$.

Theorem 13.6. The set of intertwining classes of full semisimple characters for $\tilde{G} = \mathrm{GL}_F(V)$ is in bijection to the set of those endo-parameters f which satisfy

$$\sum_{c \in \mathcal{E}} \deg(c) f(c) = \dim_F V. \tag{13.7}$$

Proof. For a full semisimple character $\theta \in \mathcal{C}(\Lambda, r, \beta)$ for $\tilde{G}$, denote by c_i the endo-classes of its simple block restrictions θ_i, and put $f_i = \dim_F V^i / \deg(c_i)$ (which are integers), for $i \in I$. The endo-classes c_i and the integers f_i are invariant under intertwining, by Theorem 9.1. Thus we get an endo-parameter f given by

$$f(c) = \begin{cases} f_i & \text{if } c = c_i, \text{ for some } i \in I, \\ 0 & \text{otherwise,} \end{cases}$$

which depends only on the intertwining class of θ and satisfies (13.7).

Conversely, given an endo-parameter f, write $\mathrm{supp}(f) = \{c_i \mid i \in I\}$. Then Proposition 13.3 gives us a full semisimple endo-class corresponding to $\mathrm{supp}(f)$. We pick any pss-character Θ in this endo-class, with corresponding semisimple pair $[0, \beta]$ and $\beta = \sum_{i \in I} \beta_i$ numbered such that the simple block restriction Θ_i has endo-class c_i, for each $i \in I$. Since $\deg(c_i) = [F[\beta_i] : F]$, we can choose a realisation of each β_i on an F-vector space V^i of dimension $f(c_i)\deg(c_i)$. The sum of these gives us a realisation of the semisimple pair $[0, \beta]$ on (a space isomorphic to) V, and the realisation of Θ on this space gives us a full semisimple character.

It is straightforward to check that the two maps described are the inverses of each other. □

13.2 Classical endo-parameters

Here we want to describe the intertwining classes of G^+ and of G. We start at first with G^+. Let us fix (σ, ϵ). We call a self-dual semisimple character *elementary* if it is simple (hence skew simple) or non-skew and the index set contains two elements. If a self dual pss-character has an elementary value, then all its values are elementary and if there is a non-skew value then all of its values are non-skew. Thus we call a self-dual pss-character *elementary* if one (equivalently all) of its values is elementary. Moreover, if a self dual pss-character is endo-equivalent to an elementary one then it too is elementary and either both are skew or both are non-skew. Thus, the notions of *skew elementary* and *non-skew elementary* depend only on the endo-class, and we apply them to endo-classes as well. Let $\mathcal{E}_- = \mathcal{E}_{\sigma,\epsilon}$ be the set of full elementary (σ, ϵ)-endo-classes (including the (σ, ϵ)-zero-endo-class).

One invariant of an intertwining class of skew semisimple characters comes from the theory of matching Witt towers, i.e. if two skew semisimple characters θ and θ' intertwine by an element of G^+ then block-wise the Witt towers match. We now encode this into the invariants in the following way.

We consider the class of pairs (β, t) where β is zero or β generates a self dual field extension $F[\beta]$ and t is an element of $W_{\sigma,\epsilon}(F[\beta])$. Two such pairs (β_1, t_1) and (β_2, t_2) are called equivalent if:

(i) t_1 and t_2 are hyperbolic;

(ii) t_1 and t_2 are not hyperbolic, the β_i are non-zero, $\lambda_{\beta_1}(t_1)$ is equal to $\lambda_{\beta_2}(t_2)$, and the Witt towers match;

(iii) t_1 and t_2 are not hyperbolic, $\beta_1 = \beta_2 = 0$ and $t_1 = t_2$.

The equivalence classes are called *Witt types*. The set of Witt types is denoted by $\mathcal{W}_{\sigma,\epsilon}$ and we denote the class of hyperbolic Witt towers by 0.

Definition 13.8. A (σ, ϵ)-*endo-parameter* is a map $f = (f_1, f_2) : \mathcal{E}_{\sigma,\epsilon} \to (\mathbb{N} \times \mathcal{W}_{\sigma,\epsilon})$ of finite support, such that for all non-skew elements $c \in \mathcal{E}_{\sigma,\epsilon}$ the Witt type $f_2(c)$ is zero.

The aim now is to prove that a natural set of (σ, ϵ)-endo-parameters parametrises the G^+-intertwining classes of self-dual semisimple characters of G^+ (and similarly for G). As in the case of $\tilde{G}$, we write $\mathcal{E}_-^{fin}$ for the set of finite subsets of $\mathcal{E}_-$, and we have:

Proposition 13.9. There is a bijection between $\mathcal{E}_-^{fin}$ and the set of all full (σ, ϵ)-endo-classes.

Proof. The proof mimics that of Proposition 13.3. Using a completely analogous proof, or by the Glauberman correspondence, we get the following two G^+-version of Lemma 13.5.

Lemma 13.10. Let $\theta_1 \in \mathcal{C}_-(\Lambda^1, r, \beta_1)$ and $\theta_2 \in \mathcal{C}_-(\Lambda^2, r, \beta_2)$ be two self dual semisimple characters, then there exists a self dual semisimple stratum $[\Lambda := \Lambda^1 \oplus \Lambda^2, n = \max(n_1, n_2), r, \beta_1 \oplus \beta_2']$ and an element of $\mathcal{C}_-(\Lambda, r, \beta_1 \oplus \beta_2')$ with restrictions θ_1 and θ_2 on $H_-^{r+1}(\beta_1, \Lambda^1)$ and $H_-^{r+1}(\beta_2, \Lambda^2)$, respectively.

Proof. The proof is completely analogous. The only difference is that one has to add elementary self dual characters and the element a is skew-symmetric. We use the translation principle Theorem A.1 for elementary self dual characters. □

This also completes the proof of Proposition 13.3, which follows directly from Lemma 13.10. □

To state the main theorem we need two more notions: For an elementary endo-class c we define $\deg(c)$ to be the degree of the restriction to the first block. Secondly, we attach to a Witt type $T := [(\beta, t)]$ the Witt tower $WT_F(T) := \lambda_\beta(t)$ $(\lambda_0 := \mathrm{id}_F)$ of $W_{\sigma,\epsilon}(F)$.

Theorem 13.11. The set of intertwining classes of full self-dual semisimple characters for G^+ are in bijection with the set of (σ, ϵ)-endo-parameters $f = (f_1, f_2)$ which satisfy

$$\sum_{c \in \mathcal{E}_-} \deg(c)(2f_1(c) + \mathrm{diman}(f_2(c))) = \dim_F V$$

and

$$\sum_{c \in \mathcal{E}_-} WT_F(f_2(c)) = h_{\equiv}.$$

Proof. The proof is completely analogous to the proof of Theorem 13.6; we just need to keep track of Witt towers, using Theorem 9.5, while the multiplicities for the classical simple endo-classes have the meaning of the $F[\beta]$-Witt index instead of the $F[\beta]$-dimension of the block. □

We conclude the classification of the intertwining classes of semisimple characters for a special orthogonal groups G. For that let us fix a symplectic form h_{sym} on V if the F-dimension is even. We take a system of representatives $\mathcal{R}$ of $F^\times/(\pm 1 + \mathfrak{p}_F)$.

Corollary 13.12. Suppose $\sigma = \mathrm{id}$ and $\epsilon = 1$, then the set of G-intertwining classes of maximal self dual semisimple characters is in bijection with the union of the two following sets:

(i) The set of orthogonal endo-parameters whose support contains the orthogonal zero-endo-class, and

(ii) The set of pairs (f, y), where the first coordinate is an orthogonal endo-parameter whose support does not contain the orthogonal zero-endo-class and where the second coordinate is either 1 or -1.

The bijection is given in the following way: Let Φ be the bijection from Theorem 13.11 and let $\theta_- \in \mathcal{C}_-(\Lambda, 0, \beta)$ be a self dual semisimple character in V. Let $[\theta_-]^+$ and $[\theta_-]$ be the G^+- and G-intertwining class of θ_-, respectively. Then,

(i) if β has a zero-component then $[\theta_-]^+ = [\theta_-]$ and is mapped to $\Phi([\theta_-]^+)$, otherwise

(ii) β has no zero-component and $[\theta_-]$ is mapped to the pair $(\Phi([\theta_-]^+), y)$ where y equals 1 if only if there is an isometry from h_{sym} to h^β of determinant in $(1 + \mathfrak{p}_F)\mathcal{R}$.

Proof. Theorem 13.11 and Theorem 11.2. □

Remark 13.13. If V is odd dimensional then G^+-intertwining classes are the same as G-intertwining classes, in the orthogonal case, because there is an element of determinant -1 in the centre. Another way of seeing this is that the support of the corresponding orthogonal endo-parameter always contains the zero-endo-class.

A The translation principle for self dual semisimple characters

In this section we generalize the translation principle from semisimple and skew-semisimple characters, see [18, 9.28], to self dual semisimple characters.

For a stratum $\Delta = [\Lambda, n, r, \beta]$ we write $\mathfrak{m}(\Delta)$ for the set $\mathfrak{n}_{-r}(\beta, \Lambda) \cap \mathfrak{a}_{-(r+k_0(\beta,\Lambda))}$, where the set $\mathfrak{n}_{-r}$ is defined as in section 2.4 with arbitrary β. The pro-p-subgroup $(1+\mathfrak{m}(\Delta))$ of $P_{-k_0-r}(\Lambda)$ normalises the stratum as already $1+\mathfrak{n}_{-r}$ does and therefore in the case of a semisimple stratum it also normalises every character in $\mathcal{C}(\Lambda, m, \beta)$.

Theorem A.1. Let $\Delta = [\Lambda, n, r+1, \gamma]$ and $\Delta' = [\Lambda, n, r+1, \gamma']$ be self dual semisimple strata with the same associated splitting, say $(V_j)_{j\in J}$, such that

$$\mathcal{C}(\Lambda, r+1, \gamma) = \mathcal{C}(\Lambda, r+1, \gamma').$$

Let $[\Lambda, n, r, \beta]$ be a self dual semisimple stratum, with associated splitting $V = \bigoplus_{i\in I} V^i$, such that $[\Lambda, n, r+1, \beta]$ is equivalent to $[\Lambda, n, r+1, \gamma]$ and γ is an element of $\prod_{i\in I} A^{i,i}$. Then, there exists a self dual-semisimple stratum $[\Lambda, n, r, \beta']$, with splitting $V = \bigoplus_{i'\in I'} V'^{i'}$ and an element u of $(1+\mathfrak{m}(\Delta)) \cap \prod_j A^{j,j} \cap G^+$, such that $[\Lambda, n, r+1, \beta']$ is equivalent to $[\Lambda, n, r+1, \gamma']$, with $u\gamma' u^{-1} \in \prod_{i'\in I'} A^{i',i'}$ and

$$\mathcal{C}(\Lambda, r, \beta) = \mathcal{C}(\Lambda, r, \beta').$$

A.1 Idempotents and self dual minimal strata

Lemma A.2 ([18] 7.13)**.** Let $(\mathfrak{k}_r)_{r\geqslant 0}$ be a decreasing sequence of σ-invariant o_F-lattices in A such that $\mathfrak{k}_r\mathfrak{k}_s \subseteq \mathfrak{k}_{r+s}$, for all $r, s \geqslant 0$, and $\bigcap_{r\geqslant 1} \mathfrak{k}_r = \{0\}$. Suppose there is an element α of $\mathfrak{k}_0$ which satisfies $\alpha^2 - \alpha \in \mathfrak{k}_1$. Then there is an idempotent $\tilde{\alpha} \in \mathfrak{k}_0$ such that $\tilde{\alpha} - \alpha \in \mathfrak{k}_1$. Moreover, if $\sigma_h(\alpha) = \alpha$ then we can choose $\tilde{\alpha}$ such that $\sigma_h(\tilde{\alpha}) = \tilde{\alpha}$.

For the self dual setting we also have to consider idempotents e which satisfy $\sigma_h(e)e = 0$.

Lemma A.3. Let $(\mathfrak{k}_r)_{r\geqslant 0}$ be as in Lemma A.2 and suppose that α is an element of $\mathfrak{k}_0$ such that $\alpha^2 \equiv \alpha$ and $\sigma_h(\alpha)\alpha \equiv \alpha\sigma_h(\alpha) \equiv 0$ modulo $\mathfrak{k}_1$. Then there is a idempotent $\tilde{\alpha}$ such that $\sigma_h(\tilde{\alpha})\tilde{\alpha}$ and $\tilde{\alpha}\sigma_h(\tilde{\alpha})$ are zero.

Proof. Lemma A.2 provides a symmetric idempotent e congruent to $\alpha + \sigma_h(\alpha)$ modulo $\mathfrak{k}_1$. Then the element $\alpha' = e\frac{1+\alpha-\sigma_h(\alpha)}{2}e$ satisfies $\alpha' + \sigma_h(\alpha') = e$. We follow now the idea of the proof of Lemma [18, Lemma 7.13]. It is easy to check that $\alpha'' := 3\alpha'^2 - 3\alpha'^3$ also satisfies $\alpha'' + \sigma(\alpha'') = 1$ and the result follows in the same way as in *loc. cit.*. □

Corollary A.4. Let $(\mathfrak{k}_r)_{r\geqslant 0}$ be as in Lemma A.3. Suppose that $\alpha_1, \ldots, \alpha_l$ are elements of $\mathfrak{k}_0$ such that $\alpha_i^2 \equiv \alpha_i$ and $\alpha_i\alpha_j \equiv 0$ modulo $\mathfrak{k}_1$ for all i and all j different from i. Suppose further that $\sum_i \alpha_i \equiv 1$ modulo $\mathfrak{k}_1$ and that there is an action of σ on $I = \{1, \ldots, l\}$ such that $\sigma_h(\alpha_i)$ is congruent to $\alpha_{\sigma(i)}$ for all indexes $i \in I$. Then there are idempotents $\tilde{\alpha}_i$ congruent to α_i modulo $\mathfrak{k}_1$ which are pairwise orthogonal and such that $\sum_i \tilde{\alpha}_i$ is 1 and such that $\sigma_h(\tilde{\alpha}_i) = \tilde{\alpha}_{\sigma(i)}$ for all indexes i.

Proof. The proof is the same as in [18, Corollary 7.14]. Using the Lemmas A.2 and A.3. □

Corollary A.5. Suppose $\Delta = [\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$ are equivalent skew-simple strata and suppose that Δ is split by $V = \oplus_i V^i$, and suppose that the set of idempotents of this splitting is invariant under σ_h. Then there is an element u of $(1 + \mathfrak{m}(\Delta)) \cap G$ such that $u\beta' u^{-1}$ is an element of $\prod_i A^{i,i}$.

Proof. The action of σ_h on the idempotents gives a sum $1 = \sum_{i \in I_0} 1^i + \sum_{i \in I_+}(1^i + \sigma(1^i))$ and by [18, 9.25] there is an element g of $(1+\mathfrak{m}(\Delta)) \cap G$ such that $g\beta' g^{-1}$ is split by $\oplus_{\mathfrak{o} \in I/\sigma_h} V^{\mathfrak{o}}$. Thus we only need to consider the case where I is one orbit with two elements. In this case $I = \{+, -\}$ take σ_h-permuted idempotents $1'^+$ and $1'^-$ commuting with β', which exist by Corolary A.4. Then the case follows now from [18, 9.15] which provides an element (g_+, g_-) and we take the element $u := (g_+, \sigma_h(g_+)^{-1})$. ☐

Proposition A.6. Suppose $[\Lambda, n, n-1, \alpha]$ is a skew-stratum which is equivalent to a semisimple stratum and let ϕ be the characteristic polynomial of the stratum with primary factors ϕ_i. Then it is equivalent to a self dual semisimple stratum $[\Lambda, n, n-1, \beta']$, and the action of σ on the index set I' (for the associated splitting) satisfies

$$\sigma_h(\phi_i)(-X) = (-1)^{\deg(\phi_i)} \phi_{\sigma(i)}(X), \tag{A.7}$$

for all indexes $i \in i$.

Proof. The stratum is equivalent to a semisimple stratum $[\Lambda, n, n-1, \beta]$, say with associated idempotents $(1^i)_{i \in I}$. The skew-symmetry of α implies that the strata $[\Lambda, n, n-1, \beta]$ and $[\Lambda, n, n-1, -\sigma_(\beta)]$ are equivalent and thus by [18, 7.17] the idempotents are permuted by σ_h modulo $\mathfrak{a}_1$, which defines an action of σ on I such that (A.7) is satisfied. Corollary A.4 provides pairwise orthogonal idempotents e_i congruent to 1^i which sum up to 1 and satisfy $\sigma_h(e_i) = e_{\sigma(i)}$. The map $g := \sum_i e_i 1^i \in P_1(\Lambda)$ conjugates $[\Lambda, n, n-1, \beta]$ to a semisimple stratum which is split by $(e_i)_i$. For the indexes i fixed by σ the stratum $[\Lambda, n, n-1, \beta' = g\beta_i g^{-1}]$ is equivalent to a skew and a simple stratum, i.e. to a skew-simple stratum [21, 1.10], and for the remaining indexes take a section I_+ through the non-singleton orbits and replace $\beta'_{\sigma(i)}$ by $-\sigma_h(\beta_i)$ for all $i \in I_+$. ☐

A.2 Equal sets of semisimple characters

Lemma A.8 (see [18, 9.11])**.** Suppose that $V = \bigoplus_k V^k$ is a splitting which refines the associated splitting of a semisimple strata $[\Lambda, n, r, \beta]$. Suppose further that θ is an element of $\mathcal{C}(\Lambda, r, \beta)$ and a is an element of $\mathfrak{a}_{-r-1} \cap \prod_k A^{k,k}$ such that $\theta_k \psi_{a_k}$ is a semisimple character in $\mathcal{C}(\Lambda^k, r, \beta_k)$. Then $[\Lambda, n, r, \beta + a]$ is equivalent to a semisimple stratum $[\Lambda, n, r, \beta']$ which is split by $V = \bigoplus_k V^k$, and the sets $\mathcal{C}(\Lambda, r, \beta')$ and $\psi_a \mathcal{C}(\Lambda, r, \beta)$ coincide. If there is a semisimple character $\theta' \in \mathcal{C}(\Lambda, r, \beta'')$ such that $V = \bigoplus_k V^k$ refines the associated splitting of $[\Lambda, n, r, \beta'']$ such that $\theta'_k = \theta_k \psi_a$ for all k and $H^{r+2}(\beta, \Lambda)$ is equal to $H^{r+2}(\beta'', \Lambda)$. Then $\theta\psi_a$ and θ' coincide and $\mathcal{C}(\Lambda, r, \beta'')$ is equal to $\mathcal{C}(\Lambda, r, \beta')$.

Proof. The proof is the same as in [18, 9.11] (The statement is slightly different, but the proof is the same.) One just shows that $[\Lambda^k, n, r, \beta_k + a_k]$ is equivalent to a simple stratum for every k, using derived strata. ☐

Corollary A.9 ([18, 9.12])**.** Suppose that $V = \bigoplus_k V^k$ is a splitting which refines the associated splittings of two semisimple strata $[\Lambda, n, r, \beta]$ and $[\Lambda, n, r, \beta']$ and suppose that there are characters $\theta \in \mathcal{C}(\Lambda, r, \beta)$ and $\theta' \in \mathcal{C}(\Lambda, r, \beta')$ such that θ_k and θ'_k coincide, for all k. Then, the set of semisimple characters coincide and θ is equal to θ'.

Proposition A.10. Suppose $\mathcal{C}_-(\Lambda, r, \beta)$ coincides with $\mathcal{C}_-(\Lambda, r, \beta')$, say with associate splittings $(1^i)_{i\in I}$ and $(1^{i'})_{i'\in I'}$. Then there is an element of $g \in P_1^-(\Lambda)$ which normalizes every character of $\mathcal{C}_-(\Lambda, r, \beta)$ such that $g1^i g^{-1} = 1'^{\tau(i)}$ where τ is the bijection from I to I' such that $1^i \equiv 1'^{\tau(i)}$ modulo $\mathfrak{a}_1$.

We denote the normalizer of a character $\theta \in \mathcal{C}(\Lambda, r, \beta)$ by $\mathfrak{n}(\theta)$. Note that all elements of $\mathcal{C}(\Lambda, r, \beta)$ have the same normalizer.

Proof. The bijection τ exists by [18, 9.9]. Take a decomposition $I = I_0 \cup I_+ \cup I_-$ which gives $1 = 1^0+1^++1^- = 1^0+1^++1^-$ and the same for $I' = I'_0 \cup \tau(I_+) \cup \tau(I_-)$. Then $1^0 \equiv 1'^0$, $1^+ \equiv 1'^+$ and $1^- \equiv 1'^-$ modulo $S_r(\beta, \Lambda)$. By [18, 9.23(iii)] there is an element g of $P_1^-(\Lambda) \cap \mathfrak{n}(\theta)$ which sends V^0 to V'^0. Thus by Corollary A.9 we only have to prove the proposition for the cases where I_0 or I_+ is empty. The case where I_+ is empty is [18, 9.23(iii)]. So let us assume that I_0 is empty. By [18, 9.23(iii)] there is an element $g = (g_+, g_-) \in S_r(\beta, \Lambda)$ which sends the splittings to each other with respect to τ. Then $g' = (g_+, \sigma_h(g_+)^{-1}) \in P_1^-(\Lambda)$ maps V^i to $V'^{\tau(i)}$. Take a character $\theta \in \mathcal{C}(\Lambda, r, \beta)$, then $\theta_{i'}^{g'^{-1}} = \theta_{i'}$ for all $i' \in I'_+$ and thus by the isometry property of g' this equality extends to the whole of I'. By Corollary A.9 the sets $\mathcal{C}(\Lambda, r, g'\beta g'^{-1})$ and $\mathcal{C}(\Lambda, r, \beta')$ are the same. □

A.3 Proof of the translation principle

Here we prove Theorem A.1 granting that we have already the Theorem for the skew-case ([18, 9.28]) and the $\tilde{G}$-case ([18, 9.16]).

Let $J = J_0 \cup J_+ \cup J_-$ be a partition with respect to σ and write J_{+-} for $J_+ \cup J_-$.

(i) At first we assume that J_0 is empty. By [18, 9.16] (the $\tilde{G}$-case) there is a semisimple stratum $[\Lambda^{J_+}, n, r, \beta'_{J_+}]$ such that $\mathcal{C}(\Lambda^{J_+}, r, \beta_{J_+})$ is equal to $\mathcal{C}(\Lambda^{J_+}, r, \beta'_{J_+})$ and such that γ'_{J_+} satisfies the desired conjugation property. Thus $[\Lambda, n, r, \beta' = \beta'_{J_+} + \sigma_h(\beta'_{J_-})]$ is a self dual stratum and $\gamma'_{J_+\cup J_-}$ satisfies the desired conjugation property such that its set of semisimple characters coincides on the blocks of V^{J_+} and V^{J_-} with the corresponding restrictions of $\mathcal{C}(\Lambda, r, \beta)$. Take $\theta \in \mathcal{C}(\Lambda, r, \beta)^\sigma$ and an extension $\theta' \in \mathcal{C}(\Lambda, r, \beta')^\sigma$ of $\theta|_{H^{r+2}(\beta,\Lambda)}$. By Proposition A.10 we can assume that β and β' have the same associated splitting. Take a self dual $a \in \mathfrak{a}_{-r-1} \cap \prod_i A^{i,i}$ such that $\theta = \theta'\psi_a$. Then by Lemma A.8 the stratum $[\Lambda, n, r, \beta' + a]$ is equivalent to a self dual semisimple stratum, say with entry β'', with the same associated splitting as β' and such that $\mathcal{C}(\Lambda, r, \beta'')$ and $\mathcal{C}(\Lambda, r, \beta)$ coincide.

(ii) Here we prove how we can reduce to the case where J only consists of one element. Suppose we have proven the Theorem for simgleton case. By (i) and the singleton case we obtain $\mathcal{C}(\Lambda, r, \beta')$ which coincides with $\mathcal{C}(\Lambda, r, \beta)$ on every simple block for $j \in J_0$ and on the block corresponding to J_{+-}. Using Proposition A.10 we can assume that β and β' have the same associated splitting and we finish the proof by Lemma A.8 in the same manner as at the end of (i).

(iii) Here we prove the case where J is a singleton. We follow the part (iv) of the proof of [18, 9.16]. Note, that by Corollary A.5 we can replace Δ' by any equivalent stratum. Thus by Proposition [18, 9.24] we can assume that $\mathcal{C}(\Lambda, r, \gamma)$ and $\mathcal{C}(\Lambda, r, \gamma')$ coincide. Take σ_h-equivariant tame corestrictions s and s' for γ and γ', respectively, which satisfy the assertions of [10, 5.2], in particular $s(x)$ is congruent to $s'(x)$ modulo $\mathfrak{a}_l$ for all $x \in \mathfrak{a}_{l-1}$ and all integers l. The stratum $[\Lambda, r, r+1, s(\beta-\gamma)]$ is equivalent to a semisimple stratum, by [18, 6.15] , and as in [18, 9.16 proof (iv)] it follows that $[\Lambda, r, r+1, s'(\beta-\gamma)]$ is equivalent to a semisimple stratum . Further $s'(\beta-\gamma)$ is skew-symmetric, and by Proposition A.6 the stratum is equivalent to self dual semisimple stratum, say with associated splitting $(V^{i''})$. Thus $[\Lambda, q, r, \beta'' := \gamma' + \sum_{i''} 1^{i''}(\beta-\gamma)1^{i''}]$ is equivalent to a self dual semisimple stratum with associated splitting $(V^{i''})$ by [18, 6.15] and [21, 1.10], and by [18, 9.6] there is an element $u \in (1+\mathfrak{m}(\Delta')) \cap G$ such that $\beta' := u\beta''u^{-1}$ is congruent to $\gamma'+\beta-\gamma$ modulo $\mathfrak{a}_{-r}$. This finishes the proof.

B Intertwining and Conjugacy for self dual semisimple characters

Here we generalize "Intertwining implies conjugacy", to self dual non-skew characters. It is known for skew-semisimple characters and for semisimple characters, see [18, 10.2,10.3].

Theorem B.1. Let $\theta \in \mathcal{C}_-(\Lambda, r, \beta)$ and $\theta' \in \mathcal{C}_-(\Lambda, r, \beta')$ be two semisimple self dual characters which intertwine over G^+ such that the matching $\zeta: I \to I'$ satisfies

$$\Lambda^i_j/\Lambda^i_{j+1} \cong \Lambda'^{\zeta(i)}_j/\Lambda'^{\zeta(i)}_{j+1},$$

for all indexes $i \in I$ and integers j. Then, there is an element of $P^-(\Lambda) \cap \prod_i A^{i,\zeta(i)}$ which conjugates θ to θ'.

Proof. The involution σ acts on the index sets I and I' and this action commutes with the map ζ by the matching theorem [18, 10.1]. We therefore obtain for the decompositions $I = I_0 \cup I_{+-}$ and $I' = I'_0 \cup I'_{+-}$ that ζ sends I_0 to I'_0 and I_{+-} to I'_{+-}. Then consider the splittings $V^0 \oplus V^{+-}$ and $V'^0 \oplus V'^{+-}$. The hyperbolic spaces V^{+-} and V'^{+-} are isometric, i.e. V^0 and V'^0 are isometric. Take an isometry g of (V, h) which sends V^0 to V'^0 and V^{+-} to V'^{+-}. By [17, 5.2] we can modify g such that g is an element of $P^-(\Lambda)$. So we assume without loss of generality that V^0 and V'^0 coincide. We show next that there is an element of $G^+ \cap \mathrm{Aut}_F(V^0) \times \mathrm{Aut}_F(V^{+-})$ which intertwines θ with θ'. By Theorem [18, 10.2] there is an element $\tilde{g} \in P(\Lambda) \cap \prod_i A^{i,\zeta(i)}$ which conjugates θ to θ'. Taking the intertwining formula [22, 3.22] and conjugating back with $\tilde{g}$ we obtain that $I(\theta, \theta')$ coincides with $S_r(\beta', \Lambda)\tilde{g}^{-1}B^\times_{\beta'}S(\beta, \Lambda)$ which is a subset of

$$S_r(\beta', \Lambda)(\mathrm{Aut}_F(V^0) \times \mathrm{Aut}_F(V^{+-}))S(\beta, \Lambda).$$

As in [21, 4.14] we prove that the intersection of $I(\theta, \theta')$ with G^+ is contained in

$$(S_r(\beta', \Lambda) \cap G^+)((\mathrm{Aut}_F(V^0) \times \mathrm{Aut}_F(V^{+-})) \cap G^+)(S_r(\beta, \Lambda) \cap G^+)$$

and thus we obtain that the restriction of θ and θ' on V^0 and on V^{+-} intertwine by an element of $\mathrm{U}(V^0)$ and $\mathrm{U}(V^{+-})$, respectively. By Corollary A.9 we can restrict to the cases where $I = I_0$ or $I = I_{+-}$. The first case is precisely [18, 10.3] and the second case is an easy exercise using Theorem [18, 10.2] (on V^+ and V'^+) and Corollary A.9. □

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