Übung 1 28.4.2017.
p-adische Darstellungstheorie

Satz 12: Every l.c.t.d. space has a baso of clopen (closed - open) subsets.

Proof: X is regular $\Rightarrow$ We can assume without loss of generality that X is compact, because we only need to find for all $x \in X$ a neighbourhood base ~~of~~ consisting of clopen sets. ~~$\mathcal{B}$~~

Take $x_0 \in X$. $\mathcal{B}_{x_0} := \{ U \subseteq X \,/\, U \text{ clopen and } x_0 \in U \}$

$B := \bigcap \mathcal{B}_{x_0}$

We show: 1) $\forall F \subseteq X$ closed with $F \cap B = \emptyset$: $\exists U \in \mathcal{B}_{x_0}$: $F \cap U = \emptyset$, i.e. $\mathcal{B}_{x_0}$ is a ~~neil~~ neighbourhood base for B.

2) $B = \{x_0\}$.

Proof of 1): Take F closed with $F \cap B = \emptyset$.

$$\Rightarrow \bigcup_{U \in \mathcal{B}_{x_0}} (X \setminus U) \text{ covers } F.$$

X compact, F closed $\Rightarrow F$ compact

$$\Rightarrow \exists U_1, \ldots, U_m \in \mathcal{B}_{x_0} : \bigcup_{i=1}^{m} (X \setminus U_i) \supseteq F.$$

$$\bigcup_{i=1}^{m} (X \setminus U_i) = X \setminus \left(\bigcap_{i=1}^{m} U_i\right)$$

Thus $U := \bigcap_{i=1}^{m} U_i \in \mathcal{B}_{x_0}$ satisfies $U \cap F = \emptyset$.

Proof of 2) It is enough to prove that B is connected, because X is a t.d. space.

Take W, V open disjoint and $x_0 \in W$ s.t. $B \subseteq V \dot\cup W$.

$F := X \setminus (V \cup W)$. ~~is close~~ is disjoint to B and by 1) $\exists U \in \mathcal{B}_{x_0}$: $F \cap U = \emptyset$.

$$\Rightarrow U \subseteq V \dot\cup W \Rightarrow U = (U \cap W) \dot\cup (U \cap V)$$

$\Rightarrow U \cap W$ is clopen in $U \underset{U \text{ clopen}}{\Rightarrow} U \cap W$ clopen in X.

$$\Rightarrow B \subseteq U \cap W \subseteq W \underset{V, W \text{ arbitrary}}{\Rightarrow} B \text{ is connected}$$

□

Exercise session II

Representations of $GL_2(\mathbb{F})$, $\mathbb{F}$ finite field

<u>Motivation</u>: "Level zero representations" of $GL_2(F)$:

$$\tau \in \mathcal{R}(GL_2(\mathbb{F}))$$

Inflate τ to $\tilde{\tau} \in \mathcal{R}(GL_2(\mathcal{O}_F))$ $\quad \kappa_F =: \mathbb{F}$, $\#\mathbb{F} =: q$

and induce: $\pi := c\text{-}\mathrm{ind}_{GL_2(\mathcal{O}_F)}^{GL_2(F)} \tilde{\tau}$.

~~and~~ and consider the subquotients.

<u>To $GL_2(\mathbb{F})$</u>:

$$T = \begin{pmatrix} * & \\ & * \end{pmatrix} \quad \text{torus}$$

$$B := \begin{pmatrix} * & * \\ & * \end{pmatrix} \quad \text{Borel}$$

$$= T \ltimes N$$

$$N := \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \qquad G := GL_2(\mathbb{F}), \quad Z = \left\{ \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \right\}_{\lambda \in \mathbb{F}^\times}$$

$$w := \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

1. Principal series representation of $GL_2(\mathbb{F})$.

These are the subquotients of $\mathrm{Ind}_B^G \chi$

$\chi : T \to \mathbb{C}^\times$ character inflated to B.

<u>Prop:</u> $\pi \in \mathcal{R}(G)$ is contained in $\mathrm{Ind}_B^G \chi$ for some $\chi: T \to \mathbb{C}^\times$ char. iff

$\mathbb{1}_N \subseteq \pi$.

<u>Proof:</u> Frob. ~~Reciprocity~~ reciprocity:

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \chi) \simeq \mathrm{Hom}_B(\mathrm{Res}_B^G \pi, \chi)$$

G, B, are finite, so all representations are semisimple by Maschke's theorem. (See also Ex sheet 3. Problem 1)

We have

$$\exists \chi \in \mathcal{R}_{irr}(T) : \pi \subseteq \mathrm{Ind}_B^G \chi$$

$$\Leftrightarrow \exists_\chi : \mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \chi) \neq \{0\}$$

$$\Leftrightarrow \exists_\chi : \mathrm{Hom}_B(\mathrm{Res}_B^G \pi, \chi) \neq \{0\}$$

↑ Frob. rec.

$$\Leftrightarrow \exists_\chi : \chi \mid \mathrm{Res}_B^G \pi$$

↑ semisimplicity

$$\Leftrightarrow \mathbb{1}_N \subseteq \mathrm{Res}_N^G \pi$$

↑ χ is inflated to B

"⇐" If $1\!\!1_N \subseteq \pi$. Then

$$1\!\!1_N \mid \mathrm{Res}^G_B \pi$$ by semisimplicity

Let W be the $1\!\!1_N$-isotypic component of $\pi|_N$

Because of $N \trianglelefteq B$, we have that

~~W is a B repr~~ $\pi(B)W = W$.

$\Rightarrow$ On W $\mathrm{Res}^G_B \pi$ factors through $B/N = T$

Semisimplicity $\Rightarrow$ W is a direct sum of characters

of T. $\Rightarrow \exists \chi : \chi \mid W \mid \mathrm{Res}^G_B \pi$. $\square$

Put ~~Prob.:~~ $I(\chi) := \mathrm{Ind}^G_B \chi$

<u>Classification of principal series repr.:</u> $\chi, \xi \in \mathcal{R}_{in}(T)$

1) $\mathrm{Hom}(I(\chi), I(\xi)) \neq \{0\} \Leftrightarrow \chi = \xi \vee \chi^\omega = \xi$

2) ~~If χ~~

$$\dim_{\mathbb{C}} \mathrm{Hom}(I(\chi), I(\chi)) = \begin{cases} 1, & \text{if } \chi \neq \chi^\omega \\ 2, & \text{if } \chi = \chi^\omega \end{cases}$$

$$\| $$

$$\dim_{\mathbb{C}} \mathrm{Hom}(I(\chi), I(\chi^\omega)).$$

3) $I(\chi)$ irred $\Leftrightarrow$ $\chi \neq \chi^w$

$\dim_{\mathbb{C}} I(\chi) = q$ in this case.

4) $I(\chi)$ has length 2 $\Leftrightarrow \chi = \chi^w$.

One factor is 1-dim and one is q-dim.

<u>Proof</u>: $\cdot$ $\mathrm{Hom}_G(I(\chi), I(\xi)) \simeq \mathrm{Hom}_B(\mathrm{Res}^G_B(I(\chi)), \xi)$

$\simeq \mathrm{Hom}_B\left(\bigoplus_{B \backslash G/B} \mathrm{Ind}^B_{B \cap {}^gB} \mathrm{Res}^{{}^gB}_{B \cap {}^gB} {}^g\chi, \xi\right)$
↑
Mackey

$\simeq \bigoplus_{B \backslash G/B} \mathrm{Hom}_B\left(\mathrm{Ind}^B_{B \cap {}^gB} \mathrm{Res}^{{}^gB}_{B \cap {}^gB} \chi, \xi\right)$

$\simeq \bigoplus_{B \backslash G/B} \mathrm{Hom}_{B \cap {}^gB}\left(\mathrm{Res}^{{}^gB}_{B \cap {}^gB} {}^g\chi, \xi\right)$
↑
Frob. rec.

$B \backslash G/B = \{B, BwB\}$ by the Bruhat decomposition.

$\simeq \mathrm{Hom}_B(\chi, \xi) \oplus \mathrm{Hom}_T({}^w\chi, \xi)$

This gives 1) and 2).

3). $I(\chi)$ is irred $\Leftrightarrow \dim_{\mathbb{C}} \mathrm{Hom}_G(I(\chi), I(\chi)) = 1$

↑ semisimplicity

$\Leftrightarrow \omega\chi \neq \chi.$

4) $I(\chi)$ has length 2 $\Leftrightarrow$

— " — and $I(\chi)$ is not irred.

$\Leftrightarrow$ (3) — " — $\omega\chi = \chi$

$\Leftrightarrow \omega\chi = \chi.$

↑

If $\chi = {}^{\omega}\chi$, then length$(I(\chi)) = 2$

We need the dimensions

$\chi : T \longrightarrow \mathbb{C}^{\times}$ $\chi\begin{pmatrix} a & \\ & b \end{pmatrix} = \chi_1(a)\,\chi_2(b)$

$\chi^{\omega} = \chi \Rightarrow \chi_1 = \chi_2$, Thus $\chi\begin{pmatrix} a & \\ & b \end{pmatrix} = \chi_1(\det\begin{pmatrix} a & \\ & b \end{pmatrix}))$

$\Rightarrow \mathrm{Hom}_B(\chi, \underbrace{\mathrm{Res}^G_B \chi_1 \circ \det}_{=\chi}) \neq \{0\}$

Frob. recipr.
$\Rightarrow \mathrm{Hom}_G(I(\chi), \chi_1 \circ \det) \neq \{0\}$

$\Rightarrow$ One factor is 1-dimensional. $\square$

Example: Steinberg repr. $\mathbb{1}_G \oplus St_G = I(\mathbb{1}_T)$.

Cuspidal representations:

An irred repr. π of G, s.t. $\mathbb{1}_N \notin \pi$, is called <u>cuspidal</u>.

Take a non-trivial character of ψ of N.

and character $\theta_0 : \mathbb{F}^\times \longrightarrow \mathbb{C}^\times$.

Take a quadratic field extension $\mathbb{L}$ of $\mathbb{F}$ in $M_2(\mathbb{F})$ and extend θ_0 to $\theta : \mathbb{L}^\times \to \mathbb{C}^\times$ s.t. $\theta^q \neq \theta$. $\quad q := \#\mathbb{F}$.

Define $\theta_\psi : ZN \longrightarrow \mathbb{C}^\times$ via $\theta_\psi(zn) = \theta_0(z)\psi(u)$

Thm: $\pi_\theta := \dfrac{\operatorname{Ind}_{ZN}^G \theta_\psi}{\operatorname{Ind}_{\mathbb{L}^\times}^G \theta}$ is cuspidal

and of dimension $q-1$.

And all cuspidal representations of G are of this form.

- $\pi_\theta \simeq \pi_{\theta'} \iff \theta^q = \theta' \vee \theta = \theta'$

Proof: [Theorem 6.4, Bushnell, Henniart "The local Langlands conjecture for GL(2)"]

A counting argument with character formula.

$\square$

Vorlesung: p-adische Darstellungstheorie

0. Motivation zur Darstellungstheorie

Gegeben ist eine diophantische Gleichung $P(X) = 0$
$P \in \mathbb{Q}[X]$ und es sind die Lösungen $\in \mathbb{R}$ gesucht.

Idee: Für $P \in \mathbb{Z}[X]$ betrachte $P(X)$ mod p, d.h.
$\overline{P} \in \mathbb{Z}/p\mathbb{Z}[X]$ (p Primzahl)
d.h. wir suchen global eine Lösung und schauen lokal nach.

Für quadratische Gleichung ist das sehr erfolgreich.

P sei ein quadratisches Polynom
$p(x) = X^2 + XY + Y + 1$
⇝ homogenisiere $P_h(X) = X^2 + XY + YZ + Z^2$

~~⇝ Wende folgenden Satz an~~

~~Satz (Hasse - Minkowski):~~
~~Es sei ein homogenes quadr. Polynom $P \in \mathbb{Q}[X]$ gegeben. Dann sind äquivalent~~
~~1° P hat eine nichttriviale~~

Statt mod p verwenden wir die analytische Variante, d.h. wir betrachten einen vollständigen Körper $\mathbb{Q}_p \supseteq \mathbb{Q}$ mit Betrag $|\cdot|_p$, der misst wie stark ein Element (z.B. aus $\mathbb{Q}$) durch p teilbar ist.

~~$|15|_5$~~

$$\left(z \in \mathbb{Z}: \quad |z|_p := \left(\frac{1}{p}\right)^{\nu_p(z)}, \text{ wobei } z = \prod_{q \text{ Primzahl}} q^{\nu_q(z)} \varepsilon \quad \varepsilon \in \{\pm 1\} \right)$$

$$|15|_5 = \frac{1}{5}, \quad |24|_2 = \frac{1}{8}.$$

Es ist viel einfacher in $\mathbb{Q}_p$ Lösungen zu suchen als in $\mathbb{Q}$.

Zurück zu unserem P_h.

<u>Satz</u> (Hasse-Minkowski): Es sei P_h ein homogenes quadratisches Polynom in $\mathbb{Q}[X]$. Dann sind äquivalent

1) P_h hat eine nichttriviale Lsg in $\mathbb{Q}$

2) P_h hat — " — in $\mathbb{R}$ und
$\forall p$: P_h — " — in $\mathbb{Q}_p$.

bei Grad ≥ 3 ist die Aussage ~~t~~ im Allgemeinen leider falsch.

Statt nur $\mathbb{Q}_p$ zu betrachten ist es ~~tud~~ sinnvoll meromorphe Abb. zu betrachten

E sei eine elliptische Kurve.

$E: Y^2 = X^3 + X + 1.$

E kann eine L-Reihe zugeordnet werden,

~~wobei~~ $$L(E,s) = \prod_{p \text{ Primzahl}} (1 - a_p p^{-s} + \varepsilon(p) p^{1-2s})^{-1}$$

wobei a_p nur von $E(\mathbb{F}_{p^n})$, $n \in \mathbb{N}$ abhängt. $\mathbb{F}_{p^n}$ endliche Körper mit p^n Elementen.

Interessant ist die Frage ob $L(E,s)$ holomorph ist. In den 90 zigern wurde bewiesen, dass das der Fall ist

Ein anderer Weg L-~~funk~~ Reihen zu konstruieren wurde durch Artin gezeigt.

$\rightsquigarrow$ über Darstellungen von $Gal(\overline{\mathbb{Q}}_p | \mathbb{Q}_p)$

<u>Def</u>: 1) Es sei G eine Gruppe. Eine Darstellung von G über $\mathbb{C}$ ist ein Gruppenhomomorphismus

$$G \to \text{~~GL~~}\ \mathrm{Aut}_{\mathbb{C}}(V),$$ wobei V ein $\mathbb{C}$-Vektorraum ist.

<u>Bem</u>: ~~#~~ Wir interessieren uns für Darstellungen von $\mathrm{Gal}(\overline{\mathbb{Q}}_p / \mathbb{Q}_p) =: G_{\mathbb{Q}_p}$

Die Körpererweiterungen von $\mathbb{Q}_p$ definieren eine Topologie auf $G_{\mathbb{Q}_p}$, die Krulltopologie.

⇝ Wir fordern ~~Stet~~ Stetigkeit von Darstellungen.

<u>Def</u>: ~~Eine Darstellung heißt glatt~~

~~Eine D~~

Es sei G eine topologische Gruppe.

Eine Darstellung (G, V) heißt <u>glatt</u>, falls

$$\forall v \in V: \quad \mathrm{stab}_G(v) = \{g \in G \mid gv = v\}$$ ist offen.

Die Darstellungstheorie von $\mathcal{G}_{\mathbb{Q}_p}$ steht mit der Darstellungstheorie von $(GL_n(\mathbb{Q}_p))_{n\in\mathbb{N}}$ in Beziehung.

Dazu betrachtet man eine lokal kompakte dichte UG $W_{\mathbb{Q}_p}$ von $\mathcal{G}_{\mathbb{Q}_p}$ ($W_{\mathbb{Q}_p}$ hat mehr Darstellungen)

<u>Def:</u> Eine Darstellung V von G heißt irreduzibel, falls sie keine Unterdarstellung $\neq 0, V$ besitzt.

<u>Langlands-Korrespondenz:</u> (Harris-Taylor-Henniart)

$$\left\{\begin{matrix}\text{irreduzible glatte Dg} \\ \text{von } W_{\mathbb{Q}_p} \text{ mit } \dim V = n\end{matrix}\right\}_{/\cong} \longleftrightarrow \left\{\begin{matrix}\text{irreduzible gl.} \\ \text{Dgn von } GL_n(\mathbb{Q}_p)\end{matrix}\right\}$$

Das Bild kann man genau beschreiben.

Wir schauen uns das für $n = 2$ an.

⇝ ~~Haben die Verbindung zur Darstellungstheorie von $GL_n(F)$.~~

Deshalb diese Vorlesung:

<u>Themen der VL:</u>

a) p-adische Bew.

b) lokal kompakte Gruppen

c) Grundlagen zur Darstellungstheorie lokal kompakter Gruppen

d) Darstellungstheorie von $GL_2(F)$.

e) 〃 von klassischen Gruppen als Beispiele

f) Etwas zur lokalen Langlands Korrespondenz von $GL_2(F)$.

I Nichtarchimedische lokale Körper

Es sei F ein Körper beliebiger Charakteristik.

<u>Def 1:</u> Eine nicht-archimedische multiplikative Bewertung auf F (bzw. Absolutbetrag $|\cdot|$ auf F) ist eine Abbildung $|\cdot| : F \longrightarrow \mathbb{R}^{\geq 0}$ ~~mit~~, die die folgenden Bedingungen erfüllt:

B1) $|x| = 0 \Leftrightarrow x = 0$ — Definitheit

B2) $|xy| = |x||y|$ — Multiplikativität

B3) $|x+y| \leq \max\{|x|, |y|\}$ — Ultrametrische Ungleichung.

Als Beispiel sei die triviale Bewertung genannt.

$|x| := 1 \quad \forall x \in F^\times$.

<u>Satz 2:</u> $(F, |\cdot|)$ sei ein nicht-Archimedisch bewerteter Körper. Dann gelten

a) $|1| = 1$

b) $|-x| = |x|$

c) $|x| \neq |y| \Rightarrow |x+y| = \max\{|x|, |y|\}$

d) $\mathcal{O}_F := \{x \in F \mid |x| \leq 1\}$ ist ein Ring.

e) $\mathfrak{p}_F := \{x \in F \mid |x| < 1\}$ ist ein Maximalideal in $\mathcal{O}_F$.

Bew: a) $|1| = |1 \cdot 1| = |1||1| \wedge |1| \in \mathbb{R}^{>0}$ $\Rightarrow |1| = 1$

b) $|-1|^2 = |-1||-1| = |(-1)^2| = |1| = 1 \quad |-1| \in \mathbb{R}^{>0}$ $\Rightarrow |-1| = 1$

c) Ann $|x| < |y|$

$\Rightarrow |y| = |-x + (x+y)| \underset{B3}{\leq} \max\{|x|, |x+y|\}$

$\underset{|x|<|y|}{\leq} |x+y| \underset{B3}{\leq} \max\{|x|, |y|\}$

$\underset{|x|<|y|}{=} |y|$

d) · $B1 \Rightarrow \mathcal{O}_F \ni 0$.

· b) ~~und B2~~ und B3 folgt nach Untergruppenkriterium, dass $(\mathcal{O}_F, +)$ eine UG von F ist.

· Aus B2 folgt, dass $(\mathcal{O}_F, +, \cdot)$ ein Unterring ist.

e) · $B_1 \Rightarrow \mathfrak{p}_F \ni 0$

· b), B2, B3 ~~und~~ $\Rightarrow \mathfrak{p}_F$ ist ein Ideal in $\mathcal{O}_F$

· Maximalität: $\mathfrak{p}_F \subseteq J \subsetneq \mathcal{O}_F$

$x \in J$ Falls $x \notin \mathfrak{p}_F \Rightarrow |x| = 1 \underset{B2, a)}{\Rightarrow} |x^{-1}| = 1$

$\underset{J \text{ Ideal}}{\Rightarrow} 1 = x^{-1} \cdot x \in J \underset{J \text{ Ideal}}{\Rightarrow} J$ ~~Ideal~~ $= \mathcal{O}_F$ ↯ $\square$

Def 3: Eine nicht-archim. Bewertung $|\cdot|$ heißt diskret, falls $\Gamma_F := |F^\times|$ eine diskrete UG in $\mathbb{R}^{>0}$ ist.
($\Leftrightarrow \exists \varepsilon > 0 : \Gamma_F \cap]1-\varepsilon, 1+\varepsilon[= \{1\}$)

Satz 4: $(F, |\cdot|)$ n-a. ist diskret $\Leftrightarrow$ $\mathfrak{P}_F$ ist ein Hauptideal, d.h. $\exists\, \varpi_F \in \mathfrak{P}_F$: $(\varpi_F) = \mathfrak{P}_F$.

Bew: ÜA

Zu einer multiplikativen Bewertung kann man eine additive Bewertung zuordnen.

$0 < \lambda < 1$ $\qquad \nu(x) := \begin{cases} \log_\lambda |x|, & x \in F^\times \\ \infty, & x = 0 \end{cases}$

Eigenschaften: $\nu(x) = \infty \Leftrightarrow x = 0$ $\quad (B'1)$

$\nu(xy) = \nu(x) + \nu(y)$ $\quad (B'2)$

$\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$ $\quad (B'3)$

Def 5: Eine Abb $\nu : F \to \mathbb{R} \cup \{\infty\}$ mit $(B'1)(B'2)(B'3)$ heißt nicht-archimedische additive Bewertung (oder auch Exponent)

Satz 6: $(F, |\cdot|)$ diskret $\Rightarrow$ $\mathcal{O}_F$ ist ein Hauptidealring.

Bew: $\Gamma_F = \{|\varpi_F|^z \mid z \in \mathbb{Z}\}$, da Γ_F diskret ist.
(Die Existenz von $|\varpi_F|^z < |x| < |\varpi_F|^{z-1}$ führt zum Widerspruch)

$\Rightarrow$ Für $x \in F^\times$ $\exists\, z \in \mathbb{Z}$: $\exists\, y \in \mathcal{O}_F^\times$: $x = \varpi_F^z y$. Def. $\nu(x) := z$

Es sei J ein Ideal in $\mathcal{O}_F$. $z_0 := \inf \nu(J) \geq 0$

$\nu(J) \subseteq \mathbb{N}_0$ $\underset{\uparrow}{\Rightarrow}$ $z_0 \in \nu(J)$ $\underset{(\text{ÜA})}{\Rightarrow}$ $J = (\varpi_F^{z_0})$ $\square$

Minimalitätssatz

Satz von Ostrowski: Alle nicht-archimedischen Bewertungen auf $\mathbb{Q}$ sind diskret und für ~~jedes solche ν gibt es~~ Exponenten ν gibt es ein $\lambda \in \mathbb{R}^{>0}$, sodass

$\lambda\nu$ entweder die triviale Bewertung oder gleich einem ν_p mit einer Primzahl p ist, wobei ν_p durch

$$n = \prod_q q^{\nu_q(n)}$$ definiert ist.

Hat man eine n-a. Bew. $|\cdot|$ auf F so kann man F vervollst. $\bar{F} := \{(x_n)_{\mathbb{N}} \mid (x_n) \text{ ist eine Cauchyfolge}\} / \sim$

$$(x_n)_{\mathbb{N}} \sim (y_n)_{\mathbb{N}} \Leftrightarrow \sout{\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n}$$

$$\lim_{n\to\infty} |x_n - y_n| = 0.$$

$\|(x_n)_\sim\| := \lim_{n\to\infty} |x_n|$ ist wohldefiniert, da $(|x_n|)_{n\in\mathbb{N}}$ eine ~~beschr~~ Cauchyfolge in $\mathbb{R}^{\geq 0}$ ist und für $(y_n) \sim (x_n)$: $\big||x_n| - |y_n|\big| \leq |x_n - y_n| \to 0$ gilt.

Vervollständige $(\mathbb{Q}, \nu_p) \rightsquigarrow (\mathbb{Q}_p, \nu_p)$

<u>Weiteres Beispiel:</u> K sei ein endlicher Körper

$K((T))$ der Laurentreihenkörper in T

$$K((T)) = \left\{ \sum_{i \geq z} a_i T^i \;\middle|\; z \in \mathbb{Z},\ a_i \in K \right\}$$

$$\nu\left(\sum_{i \geq z} a_i T^i\right) := z_0 \quad \text{definiert einen Exponenten.}$$

$\mathcal{O}_{K((T))} = K[[T]]$ Ring der Taylorreihen in T.

<u>Def 8:</u> ~~F~~ $(F, |\cdot|)$ sei ein nicht-arch. bew. Körper.

$\kappa_F := \mathcal{O}_F / \mathfrak{p}_F$ heißt Restkörper von F.

$(F, |\cdot|)$ ~~hat~~ heißt nicht-archimedischer lokaler Körper, falls es die folgenden Bedingungen erfüllt:

- Vollständigkeit
- κ_F ist endlich
- $|\cdot|$ ist diskret.

<u>Satz 9:</u> Die nicht-archimedischen lokalen Körper sind genau die endlichen Körpererweiterungen von $\mathbb{Q}_p$ und

und die Laurentreihenkörper $K((T))$ mit K endlich.

Satz 10: (Henselsches Lemma)

Es sei $(F, |\cdot|)$ ein nichtarchimed. lokaler Körper, dann gilt für jedes $f \in \mathcal{O}_F[X] \setminus \mathfrak{p}_F[X]$, dass sich in $\kappa_F[X]$ in teilerfremde Faktoren $a(X), b(X)$ faktorisiert, $\bar{f}(X) = a(X)\, b(X)$, dass $h(X), g(X) \in \mathcal{O}_F[X]$ existieren mit

- $f(X) = h(X)\, g(X)$,
- $\bar{h}(X) = a(X)$
- $\bar{g}(X) = b(X)$, und
- $\deg(g) = \deg(b)$.

Bew: Schritt 1: Lifte a und b zu $h^{(1)}, g^{(1)}$

$$f - h^{(1)} g^{(1)} \in \mathfrak{p}_F[X]$$

Schritt $i+1$: $f \equiv h^{(i)} g^{(i)} \mod \mathfrak{p}_F^i[X]$

Ansatz $h^{(i+1)} := h^{(i)} + \bar{\omega}_F^i \cdot \tilde{h}$ $\qquad \deg \tilde{h} \leq \deg f - \deg g$

$g^{(i+1)} := g^{(i)} + \bar{\omega}_F^i \cdot \tilde{g}$ $\qquad \deg \tilde{g} \leq \deg g$

$$f \equiv h^{(i+1)} g^{(i+1)} \mod \mathfrak{p}_F^{i+1}[X]$$

$\Leftrightarrow \tilde{f} := \frac{1}{\omega_F^i}(f - g^{(i)}h^{(i)}) \equiv g^{(i)}\tilde{h} + h^{(i)}\tilde{g} \mod \mathfrak{p}_F[X]$

Lemma von Bezout in $K_F[X]$:

$\Rightarrow \exists\, e, d \in K_F[X]: \quad \overline{\tilde{f}} = e a + b d$

Division mit Rest $\quad \overline{\tilde{f}} = b d + a(b \cdot q + r)$

$= b(d + q a) + a r$

$\deg(r) < \deg(b)$

$\Rightarrow \deg(b) + \deg(d + q a) \leq \max\{\deg \overline{\tilde{f}}, \deg(a r)\}$

$\leq \deg f.$

Lifte $\quad d + q a \rightsquigarrow \tilde{h}$

$r \rightsquigarrow \tilde{g}.$

$g := \lim_{i \to \infty} g^{(i)} \qquad h = \lim_{i \to \infty} h^{(i)} \qquad \square$

II Locally compact totally disconnected spaces and groups.

If we say top. space we always implicitely mean Hausdorffsch.

<u>Def 11</u>: A topological space X is called 1) totally disconnected if all connected components are singletons.

2) locally compact if $\forall x \in X \ \exists K \subseteq X$ compact and $U \subseteq X$ open, such that $x \in U \subseteq K$. (We call such a K compact neighbourhood of x)

<u>Prop 12</u>: (Exercise session):

A loc. comp. tot. disc. ~~spact~~ space has
(lctd)
a base of compact open subsets.

<u>Def 13</u>: A topological group is called <u>locally profinite</u> if the group is t.d.l.c. space under its topology. We also say t.d.l.c.-group.

Prop 14: A topological group G is t.d.l.c. iff $e \in G$ has a neighbourhood base consisting of compact open subgroups.

Proof: Exercise sheet 2. $\square$

Prop 15: Let G be a ~~topological~~ locally compact group and $H \leq G$ be a closed subgroup. Then

1) G/H is locally compact
2) If G is t.d. then G/H is t.d.

Remark: G top. group, $H \leq G$ closed $\Rightarrow$ G/H is Hausdorff.

Proof of Prop 15: 1) $\pi : G \to G/H$ is open and surjective $\Rightarrow$ π maps open (compact) sets to open (compact) sets.

Thus G/H is locally compact.

b) Take $zH \in G/H$. Let $V \subseteq G/H$ be open s.t. $zH \in V$.

G t.d.l.c. $\underset{12}{\Longrightarrow}$ $\exists\, U \subseteq G$ compact open: $z \in U \subseteq \pi^{-1}(V)$

$\Longrightarrow$ $zH \in \pi(U) \subseteq \pi(\pi^{-1}(V)) = V$

($\uparrow$ π surj + open)

and $\pi(U)$ is compact and open. □

Def 16: A compact t.d.l.c. group is called profinite

Prop 17: 1) Let $(G_i)_{i\in I, \leq}$ be a projective system of finite subgroups. (G_i with discrete topology)

Then $\varprojlim_{I} G_i = \{ (g_i)_{i\in I} \mid g_i \in G$ and $f_{ij}(g_j) = g_i \ \forall_{i \leq j} \}$

is a profinite group.

2) Let G be a profinite group then

$G \simeq \varprojlim_{N\in \mathcal{C}} G/N$ as topological groups, where $\mathcal{C}$ is a nbhd base of e consisting of open normal subgroups.

Proof: 1) You take on $\varprojlim_{i \in I} G_i$ ~~de~~ the topology induced from the product topology of $\prod_{i \in I} G_i$.

Consider $\prod_{i \in I} G_i \xrightarrow{\pi_i} G_i$, $\prod_{i \in I} G_i \xrightarrow{\pi_j} G_j$, $G_j \xrightarrow{f_{ij}} G_i$, $i \leq j$

$$\varprojlim_{i \in I} G_i = \bigcap_{i \leq j} \{ f_{ij} \circ \pi_j = \pi_i \}$$

is closed and thus compact because $\prod_{i \in I} G_i$ is compact.

$\{ \pi_i^{-1}(1) \mid i \in I \}$ is a nbhd base of $(e_i)_{i \in I}$ consisting of open ~~and~~ normal subgroups.

2) We have $G \xrightarrow{\phi} \varprojlim_{N \in \mathcal{C}} G/N$ continuous.

Surjectivity: Take $(\bar{g}_N)_{N \in \mathcal{C}} \in \varprojlim_{N \in \mathcal{C}} G/N$.

The net $(g_N)_{N \in \mathcal{C}}$ has a culmination point g_0 because G is compact.

Let N be an element of $\mathcal{C}$

Take $N' \in \mathcal{C}$, $N' \subseteq N$, s.t. $g_{N'} \in N g_0$

$\Rightarrow N g_N = N g_{N'} = N g_0$

$\Rightarrow (\bar{g}_N)_N$ ~~is ... value of~~

$\quad \shortparallel$

$\varphi(g)$. $\square$

<u>Examples 18:</u>

- $GL_n(F)$ F n.a.l.f. and
- closed subgroups of $GL_n(F)$

are locally profinite w.r.t. the $|\cdot|$ topology on $GL_n(F)$.

- $Gal(\bar{F}^{sep}/F)$ is profinite w.r.t. the Krull topology.

Functions on locally profinite groups and Hecke algebras

Def 18: Let X be a t.d. l.c. space.

$$\mathcal{C}^\infty(X) := \{ f : X \longrightarrow \mathbb{C} \mid f \text{ is locally constant} \}$$

⌜Def: $f : X \to \mathbb{C}$ is called locally constant if $\forall_{x \in X}\ \exists\ U$ nbhd of x: $f|_U = \text{constant}$.⌟

$Y \subseteq X$: $\mathbb{1}_Y : X \longrightarrow \mathbb{C}$. $\mathbb{1}_Y(x) := \begin{cases} 1, & x \in Y \\ 0, & \text{else} \end{cases}$

$$\mathcal{C}_c^\infty(X) := \{ f \in \mathcal{C}^\infty(X) \mid \text{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}} \text{ is compact} \}$$

An $f \in \mathcal{C}_c^\infty(X)$ has the form $f = \sum_{i=1}^{m} a_i \mathbb{1}_{U_i}$ where $a_i \in \mathbb{C}$ and U_i compact open and pairwise disjoint.

Now let G be a locally profinite group.

$f \in \mathcal{C}_c^\infty(G)$ has the form $\sum_{i=1}^{m} a_i \mathbb{1}_{Kg_i} = \sum_{j=1}^{n} e_j \mathbb{1}_{K'g_j'}$

with small enough comp. open subgroups K, K' of G.

So for $f \in \mathcal{C}_c^\infty(G)$ exists a $K \leq G$ copen, s.t. $\forall g \in G \; \forall_{k \in K}$: $f(gk) = f(g) = f(kg)$

i.d. $$\mathcal{C}_c^\infty(G) = \bigcup_{\substack{K \leq G \\ \text{copen}}} \mathcal{C}^\infty({}_K\backslash G/_K).$$

We want to have a product on $\mathcal{C}_c^\infty(G)$. $\rightsquigarrow$ Need Haar measure

<u>Haar measure:</u> A measureon G is a ~~linar~~ linear form on $\mathcal{C}_c^\infty(G)$.

We write $\mu(f) =: \int_G f \, dg =: \int_G f \, d\mu$.

G has two left actions on $\mathcal{C}^\infty(G)$:

$$(\ell(g).f)(g') := f(g^{-1}g')$$

$$(r(g).f)(g') := f(g'g)$$

μ is called left invariant if it invariant under the action of ℓ.

A left invariant non-zero measure on G is called Haar measure on G.

Remark 20: $\mu(f)$ is just a finite sum.

$$f = \sum_{i=1}^{m} a_i \mathbb{1}_{g_i K} \Rightarrow \mu(f) = \sum a_i \mu(\mathbb{1}_{g_i K})$$

$$\underset{\substack{\uparrow \\ \text{left invariant}}}{=} \sum_{i=1}^{m} a_i \mu(\mathbb{1}_K)$$

Rk 21: $\mu(Y) := \mu(\mathbb{1}_Y)$ for $Y \subseteq G$ c-open. volume of Y.

Prop 22: 1) μ Haar measure $\Rightarrow \forall_{K \leq G}$ c-open: $\mu(K) \neq 0$

2) $K \leq G$ c-open given

$\Rightarrow \exists!$ Haar measure on G: $\mu(K) = 1$.

<u>Proof:</u> Define for $K, K' \leq G$ copen:

$$(K:K') := \frac{(K : K \cap K')}{(K' : K \cap K')} = \frac{|K / K \cap K'|}{|K' / K \cap K'|}$$

index of K' in K.

2) Take $f \in \mathcal{C}_c^\infty(G)$ $f = \sum a_i \mathbb{1}_{g_i K'}$

$$\mu(f) := \sum_{i=0}^{m} a_i (K':K).$$

<u>Exercise:</u> This definition is independent of the choices made.

$$\mu(K) = \mu(\mathbb{1}_K) = (K:K) = 1.$$

<u>Uniqueness:</u> Say $\tilde{\mu}$ is a Haar measure with $\tilde{\mu}(K) = 1$.

Take K' copen $\Rightarrow \tilde{\mu}(K') = \tilde{\mu}(K)(K':K^*)$

$\qquad \parallel \leftarrow \tilde{\mu}(K) = 1$

$\mu(K') = (K':K)$ ← Def of μ.

1) μ non-zero $\Rightarrow \exists K^* \leq G$ copen: $\mu(K) \neq 0$

Take $K' \leq G$ copen $\Rightarrow \mu(K') = \mu(K)(K':K) \neq 0$ □

Def 23: (Modulus character)
~~We me~~ The ~~character~~ map $\delta_G : G \to \mathbb{C}$ which satisfies $\int_G f(gh)\,dg = \delta_G(h) \int_G f(g)\,dg$

is called the modulus character of G.

G is called unimodular, if δ_G is trivial.

~~Cat~~
Remark 24: Take $K \leq G$ c open and $h \in G$.

$$\int_G \mathbb{1}_K(gh)\,dg = \int_G \mathbb{1}_{Kh^{-1}}(g)\,dg$$

$$= \mu(Kh^{-1}) = \mu(hKh^{-1})$$

$$\int_G \mathbb{1}_K(g)\,dg = \mu(K)$$

$$\Rightarrow \quad \delta_G(h) = \mu(hKh^{-1})\,\mu(K)^{-1}.$$

δ_G is a character, because

$$\delta_G(h_1 h_2) = \mu(h_1 h_2 K h_2^{-1} h_1^{-1})\,\mu(K)^{-1}$$

$$= \mu(h_1 h_2 K h_2^{-1} h_1^{-1})\,\mu(h_2 K h_2^{-1})^{-1}$$

$$\mu(h_2 K h_2^{-1})\,\mu(K)^{-1}$$

$$= \delta_G(h_1)\,\delta_G(h_2).$$

Example 24: $G = \begin{pmatrix} F^\times & F \\ & F^\times \end{pmatrix} = B \subseteq GL_2(F)$

Borel subgroup.

What is δ_G? $\quad G = F^\times \begin{pmatrix} \varpi_F^{\mathbb{Z}} & \\ & 1 \end{pmatrix} \begin{pmatrix} \mathcal{O}_F^\times & F \\ & \mathcal{O}_F^\times \end{pmatrix}$

$F^\times \subseteq Z(G) \Rightarrow \delta_G(F^\times) = \{1\}$

$\begin{pmatrix} \mathcal{O}_F^\times & F \\ & \mathcal{O}_F^\times \end{pmatrix}$ is a union of ~~compact subgroups.~~ compact open subgroups of G.

δ_G is trivial on compact open subgroups.

$\Rightarrow \delta_G \begin{pmatrix} \mathcal{O}_F^\times & F \\ & \mathcal{O}_F^\times \end{pmatrix} = \{1\}$.

So we are left with $\begin{pmatrix} \varpi_F^{\mathbb{Z}} & \\ & 1 \end{pmatrix}$.

$K := \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ & \mathcal{O}_F^\times \end{pmatrix}$, Take μ with $\mu(K) = 1$.

$$\mu\left(\begin{pmatrix} \varpi_F & \\ & 1 \end{pmatrix} K \begin{pmatrix} \varpi_F^{-1} & \\ & 1 \end{pmatrix}\right) = \mu\left(\begin{pmatrix} \mathcal{O}_F^\times & \mathfrak{p}_F \\ & \mathcal{O}_F^\times \end{pmatrix}\right)$$

$$= \frac{1}{q} = \|\varpi_F\| \qquad q := \# k_F.$$

$\|\cdot\|$ is the normalized multiplicative valuation on F.

$$\Rightarrow \quad \delta_G\left(\begin{pmatrix} a & c \\ & b \end{pmatrix}\right) = \frac{\|a\|}{\|b\|}.$$

Hecke algebra:

We have a product on $\mathcal{C}_c^\infty(G)$ "convolution":

$$(f_1 * f_2)(g) := \int_G f_1(h) f_2(h^{-1}g)\, dh.$$

$(\mathcal{C}_c^\infty(G), *)$ is an algebra over $\mathbb{C}$.

This definition depends on μ.
We want to have it μ ~~indepandant~~ independent.

Def 25: A linear form $T : \mathcal{C}_c^\infty(G) \to \mathbb{C}$ is called a distribution (or measure)

$$\mathrm{supp}(T) = \overline{\{ g \in G \mid \exists \text{ nbhd base } \mathcal{B}_g \text{ of } g : \forall U \in \mathcal{B}_g : T(\mathbb{1}_U) \neq 0 \}}$$

~~T is called~~

~~$\mathcal{H}(G) = \{ T \mid T \text{ distribution } \mathrm{supp}(T) \text{ compact} \}$~~

We write $T(f) =: \int_G f(g)\, dT(g)$

T is called $K \leq_{\text{copen}} G$ invariant if $\forall f \in \mathcal{C}_c^\infty(G)\ \forall_{k \in K}$:

$$T(r(k).f) = T(\ell(k).f) = T(f)$$

$$\mathcal{H}(G/\!/K) := \{T \mid T \text{ distribution}, ~~Flat~~ \operatorname{supp}(T) \text{ compact}, T\ K\text{-invariant}\}$$

$$\mathcal{H}(G) = \bigcup_{K \leq_{\text{copen}} G} \mathcal{H}(G/\!/K)$$ „Hecke algebra" of G.

Remark 26: We have an isomorphism

$$\mathcal{C}_c^\infty(G) \xrightarrow{\sim} \mathcal{H}(G)$$
$$f \longmapsto \underbrace{f\,dg}_{=T_f}$$

$$T_f(f') := \int_G f f'\,dg.$$

where the Product on $\mathcal{H}(G)$ is defined by

$$(T * S)(f) := \int_G \int_G f(g_1 g_2)\, dT(g_1)\, dS(g_2)$$

Additional remark to the last time

G locally profinite $\quad \mathcal{C}_c^\infty(G) \overset{\Phi_\mu}{\cong} \mathcal{H}(G)$

$$f \longmapsto f\,d\mu$$

μ Haarmeasure.

We define for $A \subseteq G$, A open compact subset of G,

$I_A \in \mathcal{H}(G)$ as $\quad I_A := \Phi_\mu\left(\frac{1}{\mu(A)} 1\!\!1_A\right)$

If A is a copen sub-group we write e_A.

The are nice for calculations with $*$.

Remark A1:

0) If $A, B \subseteq G$ copen disjoint, then $I_{A \dot\cup B} = \frac{\mu(A)}{\mu(A \cup B)} I_A + \frac{\mu(B)}{\mu(A\cup B)} I_B$

1) If $K', K \leq G$ copen and $K' \leq K$ then $e_{K'} * e_K = e_K * e_{K'} = e_K$

2) ~~δ_g~~ let $g \in G$. $\delta_g : \mathcal{C}_c^\infty(G) \to \mathbb{C}$ Dirac distribution. Take $A \underset{\text{copen}}{\subseteq} G$.

Then $\quad \delta_g * I_A = I_{gA}$

and $\quad I_A * \delta_g = I_{Ag}$.

3) $A \subseteq_{\text{copen}} G$ and $K, K' \leq_{\text{copen}} G$ s.t. $AK = A$ and $K'A = A$, then

$$I_A * e_K = I_A \text{ and } e_{K'} * I_A = I_A.$$

4) Take $K, K' \leq G$ open. Then

$$e_K * e_{K'} = I_{KK'}, \text{ and for } g \in G$$

$$e_K * \delta_g * e_{K'} = I_{KgK'}$$

$$\delta_g * e_K * e_{K'} = I_{gK} * e_{K'} = I_{gKK'}$$

Proof: 1) last time

2) analogue to last time

3) $I_{A \dot\cup B} = \Phi\left(\frac{1}{\mu(A \dot\cup B)} 1\!\!1_{A \dot\cup B}\right)$

$$= \Phi\left(\frac{\mu(A)}{\mu(A \dot\cup B)} \cdot \frac{1}{\mu(A)} 1\!\!1_A + \frac{\mu(B)}{\mu(A \dot\cup B)} \frac{1}{\mu(B)} 1\!\!1_B\right)$$

$$= \frac{\mu(A)}{\mu(A \dot\cup B)} I_A + \frac{\mu(B)}{\mu(A \dot\cup B)} I_B.$$

3) $A = \dot\bigcup_{i=1}^{\ell} g_i K.$

$$\Rightarrow I_A = \sum \frac{\mu(K)}{\mu(A)} I_{g_i K}$$

$$= \sum \frac{\mu(K)}{\mu(A)} (\delta_{g_i} * e_K)$$

$$= \left(\sum \frac{\mu(K)}{\mu(A)} (\delta_{g_i} * e_K)\right) * e_K$$

$$= I_A * e_K$$

4) $K = \dot{\bigcup}_{i=1}^{\ell} g_i(K \cap K')$ $\quad g_i \in K$

$$\Rightarrow KK' = \dot{\bigcup}_{i=1}^{\ell} g_i K' \qquad (*)$$

$$e_K * e_{K'} \underset{0)}{=} \left(\sum_{i=1}^{\ell} \delta_{g_i} * e_{K \cap K'} \right) \frac{1}{\ell} * e_{K'}$$

$$\underset{1)}{=} \left(\sum_{i=1}^{\ell} \delta_{g_i} * e_{K'} \right) \frac{1}{\ell}$$

$$\underset{(*)+2)}{=} I_{KK'}$$

$$e_K * \delta_g * e_{K'} = e_K * \delta_g * e_{K' g^{-1} g}$$

$$\underset{1)}{=} e_K * e_{gK'g^{-1}} * \delta_g \underset{\uparrow \text{ see above}}{=} I_{KgK'g^{-1}} * \delta_g$$

$$\underset{2)}{=} I_{KgK'} .$$

$\square$

<u>Corollary A2:</u> If $A, B \subseteq G$ are cosets of open subgroups of G, then

$$I_A * I_B = I_{AB}$$

<u>CAUTION</u>!: I think this is not true for A, B not cosets in general!

You can think of it as $(\mathcal{C}^\infty_c(G), *)$

$\mathcal{H}(G)$ has lots of idempotents ~~in general~~ if G is not compact.

$e_K := \frac{1}{\mu(K)} \mathbb{1}_K$. Take $K' \subseteq K$ open.

$$(e_K * e_{K'})(g) = \frac{1}{\mu(K)\mu(K')} \int_G \mathbb{1}_K(h) \mathbb{1}_{K'}(h^{-1}g)\, d\mu(h)$$

$$= \frac{1}{\mu(K)\mu(K')} \int_G \mathbb{1}_K(h) \mathbb{1}_{gK'}(h)\, dh$$

$$= \frac{1}{\mu(K)\mu(K')} \mu(K \cap gK') = \frac{1}{\mu(K')} \mathbb{1}_{K'}(g).$$

$$\mu(K \cap gK') = \begin{cases} 0, & \text{if } g \notin K', \text{ because then } G \setminus gK' \supseteq K' \supseteq K \\ \mu(K), & \text{if } g \in K', \text{ because then } gK' = K' \supseteq K. \end{cases}$$

$\Rightarrow e_K * e_{K'} = e_{K'}$

<u>Prop 27:</u> $K \leq G$ open, then

$$e_K * \mathcal{H}(G) * e_K = \mathcal{H}(G /\!/ K).$$

<u>Proof:</u> We have

$$e_K * T * e_K(f) = \frac{1}{\mu(K)^2} \iiint \mathbb{1}_K(h_1)\, \mathbb{1}_K(h_3)\, f(h_1 h_2 h_3)\, dh_3\, dh_1\, dT(h_2).$$

1) If T is K-invariant, we have

$$e_K * T * e_K(f) = \frac{1}{\mu(K)^2} \int_K dh_1 \int_K dh_3 \int_G f(h_2)\, dT(h_2)$$

$$= T(f)$$

2) If T is arbitrary then $e_K * T * e_K$ is K-invariant. $k \in K$

$$(e_K * T * e_K)(r(k).f) = \int_G \int_K \int_K f(h_1 h_2 h_3 k)\, dh_3\, dh_1\, dT(h_2)$$

$$= \underbrace{\delta_G(k)}_{=1,\text{ because } \delta_G(K) = \{1\}.} (e_K * T * e_K)(f)$$

□

Remark 28: If T is a distribution of G and $S \in \mathcal{H}(G)$, then $T * S$ is still defined because take $f \in \mathcal{C}_c^\infty(G)$, then

$$g \longmapsto \int_G f(\overset{\frown}{gh})\, dT(h)$$

is an element of $\mathcal{C}^\infty(G)$ and S ~~is defin~~ can be extendended to $\mathcal{C}^\infty(G)$ using the restriction

$$\mathcal{C}^\infty(G) \xrightarrow[\text{res}]{} \mathcal{C}^\infty(\operatorname{supp}(S))$$

$$\begin{array}{c}\Psi \\ \tilde{f}\end{array}$$

$$S(\tilde{f}) := S(\tilde{f}\,|\,\operatorname{supp}(S)).$$

So $$T * S(f) := \int_G \int_G f(h_1 h_2)\, dT(h_1)\, dS(h_2)$$

makes sence.

III Smooth representations of locally profinite groups

We only consider complex representations.

Let G be a group

<u>Def 29</u>: 1) A representation (π, V) of G is a group homomorphism

$$G \longrightarrow \mathrm{Aut}_{\mathbb{C}} V$$

where V is a $\mathbb{C}$-vector space.

2) A morphism $\varphi : V \longrightarrow V'$, between repr. of G is an $\mathbb{C}G$-linear map.

$$\varphi(\pi(g)v) = \pi(g)\varphi(v).$$

3) representations of $G \overset{1-1}{\longleftrightarrow}$ $\mathbb{C}G$-modules

⇝ notion of quotient, subrepresentation, finite length, finite type (which means finitely generated), indecomposable ($\nexists$ decomposition $V = V_1 \oplus V_2$ as $\mathbb{C}G$ modules)

irreducible (= simple $\mathbb{C}G$-module, i.e. $\nexists$ submodul $W_{\neq 0} : 0 \neq W \subsetneqq V$)

Prop. 32: Let (π, V) be a repr. of G. Then are equivalent

1°) (π, V) is a direct sum of irred sub-repr.

2°) (π, V) is a sum of irred sub-repr.

3°) Any subrepr. W of V has a direct summand $W' \leq_G V$ $W \oplus W' = V$.

(↑ $\leq_G$ means sub-representation)

Proof: $1° \Rightarrow 2°$ ✓

$2° \Rightarrow 1°$ Let $\{U_i \mid i \in I\}$ - be the set of all irred subrepr. of V.

$M = \{ J \mid J \subseteq I$ and $\sum_{j \in J} U_j$ is direct. $\}$

Zorn's Lemma $\Rightarrow \exists$ maximal element in M, say J

Suppose $\bigoplus_{j\in\hat{J}} U_j \neq V$

Then, since V is a sum of irred sub-modules, we have $\exists j_0 \in I : U_{j_0} \nsubseteq \bigoplus_{j\in\hat{J}} U_j$

$\Rightarrow$ $U_{j_0} + \bigoplus_{j\in\hat{J}} U_j$ is direct, because U_{j_0} is irreducible.

- $\Rightarrow$ $\{j_0\} \cup \hat{J} \in \mathcal{M}$ and bigger than $\hat{J}$ ↯.

$2°) \Rightarrow 3°)$ Take $W \leq_G V$.

~~Consider $\mathcal{M} = \{(J_1, J_2) \mid J_i \subseteq I$~~

- Consider $\mathcal{M} = \{J \subseteq I \mid \forall_{j\in J} : W \cap U_j = \{0\}\}$ ~~$\exists_1$~~

Zorns Lemma $\Rightarrow$ $\exists \hat{J}$ maximal

$\Rightarrow$ $W \oplus \left(\sum_{j\in\hat{J}} U_j\right) = V$.

$3°) \Rightarrow 2°)$ Part 1: Every non-zero sub module W of V contains an irreducible subrepresentation. Take $w \in W \setminus \{0\}$

~~Take~~ $U := \mathbb{C}Gw$.

Let $\tilde{U}$ be a maximal subrep. of U s.t. $w \notin \tilde{U}$ and let T be a complement of U, i.e. $U \oplus T = V$.

Take a complement S of $\tilde{U} \oplus T$ in V.

$$\Rightarrow S \simeq V / \tilde{U} \oplus T \simeq U / \tilde{U} \text{ is irreducible.}$$

Consider $S \hookrightarrow V \twoheadrightarrow U$

This is non-zero because $S \not\subseteq T$.

Thus U has an irred subrepres..

From Part 1 follows directly 2°) using 3°):

Def. $W := \sum_{j \in I} U_j$. Let W' be a direct complement of W. Suppose $W' \neq \{0\}$.

Part 1 $\Rightarrow$ W' has an irred subrepresentation $\Rightarrow W \cap W' \neq \{0\}$ ↯ □

From now on let G be a locally profinite group.

Def 33: A representation (π, V) of G is called **smooth**, if $\forall\, v \in V$:

$$\mathrm{Stab}_G(v) := \{ g \in G \mid \pi(g)\, v = v \}$$

is open.

$\mathcal{R}(G) :=$ "category of smooth representations of G"

Examples 34:

1) trivial representation of G.

~~$(\mathbb{1}_r, \mathbb{C})$ $\mathbb{1}_r(g).z := z$.~~

$\parallel$

$(\mathbb{1}, \mathbb{C})$ $\mathbb{1}(g)\, z := z$.

2) A character $\chi : G \to \mathbb{C}^\times$ is smooth iff its kernel is open.

~~**Def 34:** All notions of Def 29 are copied to ~~smooth~~ the category of smooth representations, i.e. ~~ind.~~ means no~~

- $V^K := \{ v \mid kv = v \ \forall k \in K \}$

$K \leq G$.

Relation to Hecke algebras

We have an action of $\mathcal{H}(G)$ on $(\pi, V) \in \mathcal{R}(G)$.

$f \in C_c^\infty(G) \cong \mathcal{H}(G)$, $v \in V^K \subseteq V$.

$f = \sum_{i=1}^{\ell} \mathbb{1}_{g_i K}$, f - K invariant.

$$\pi(f)v := \int_G f(g)\pi(g)v\,dg$$

$$:= \mu(K) \sum_{i=1}^{\ell} f(g_i)\pi(g_i)v.$$

Prop 35: For $f, f' \in C_c^\infty(G)$ we have

$$\pi(f * f') = \pi(f) \circ \pi(f').$$

Proof: Exercise. $\square$

Def 36: An $\mathcal{H}(G)$ module V is called **unital** if for all $v \in V$ $\exists$ $e \in \mathcal{H}(G)$ idempotent s.t. $ev = v$.

Remark: This is equivalent in saying that

$$\forall v \in V \ \exists \underset{\text{c open}}{K \leq G} : \ e_K v = v,$$

- because every element of $\mathcal{H}(G)$, in particular idempotents are invariant under a c open subgroup.

$\ulcorner e \in \mathcal{H}(G/K) \Rightarrow e_K * e = e \lrcorner$

- $\Rightarrow e_K v = e_K (e v) = (e_K e) v = e v = v \lrcorner$

Theorem 37: The functor

$$\mathcal{R}(G) \xrightarrow{\mathcal{F}} \mathcal{H}(G)\text{ mod} := \text{„Category of unital } \mathcal{H}(G)\text{-modules"}$$

$$(\pi, V) \longmapsto (\pi, V)_{\mathcal{H}(G)}$$

is an equivalence of categories

Proof: We give $\mathcal{H}(G)\text{-mod} \xrightarrow{\mathcal{G}} \mathcal{R}(G)$

Let V be a unital $\mathcal{H}(G)$ module

Define: $\pi(g)v := I_{gK} \circ v$ for $v \in V$ with $e_K v = v$, $K \leq_{\text{c open}} G$.

- Independent of the choice of K:

$K' \leq K$ c open. $I_{gK'} \circ v = I_{gK'} \circ (e_K \circ v)$

$= (I_{gK'} * e_K) \circ v = I_{gK'K} \circ v$

$= I_{gK} \circ v$

- (π, V) is a representation $g, h \in G$.

~~$\pi(g)(\pi(h) v) = \ldots I_{ghKh^{-1}}$~~

$\pi(h) v = I_{hK} \circ v =: w$ satisfies

$e_{hKh^{-1}} w = w$, because

$e_{hKh^{-1}} * I_{hK} = I_{hK}$.

$\Rightarrow \pi(g)(\pi(h) \cdot v) = I_{ghKh^{-1}} \circ (I_{hK} \circ v)$

$$= (I_{ghKh^{-1}} * I_{hK}) \circ v$$

$$= (\delta_g * I_{hKh^{-1}} * \delta_h * I_K) \circ v$$

$$= I_{ghK} \circ v$$

(π, V) is smooth: Take $v \in V$ and $K, K' \leq G$ compact open st. $e_{K'} v = v$ and $K' \leq K$.

- Then $v \in V^K \iff \forall_{k \in K}: \pi(k) v = v$.

$$\iff \forall_{k \in K}: I_{kK'} \circ v = v$$

$$\Rightarrow I_K \circ v = \frac{\mu(K')}{\mu(K)} \sum_{i=1}^{\ell} I_{k_i K'} \circ v$$

$$= \frac{1}{\ell} \sum_{i=1}^{\ell} v = v$$

- And if $e_K \cdot v = v$, then

$$\forall_{k \in K}: \pi(k) v = I_{kK} \circ v = e_K \circ v = v.$$

<u>We have:</u>

$$\mathcal{R}(G) \xrightarrow{F} \mathcal{H}(G)\text{-mod}$$
$$\circlearrowleft$$
$$\mathcal{G}$$

To show: they are inverse to each other.

<u>$\mathcal{F} \circ \mathcal{G}$:</u> $(\pi, V) := \mathcal{G}(\circ, V)$

to show $\pi(f) v = f \circ v$.. Take $K \leq_{\text{copen}} G$, s.t. $v \in V^K$

$$\pi(\mathbb{1}_{gK}) v = \int_G \mathbb{1}_{gK}(h) \pi(h) v \, dh$$

$$= \int_G \mathbb{1}_{gK}(h) \pi(g) v \, dh$$

$$= \int_G \mathbb{1}_{gK}(h) \, dh \; \pi(g) v$$

$$= \mu(K) \cdot \pi(g) v = \mu(K) (\mathbb{1}_{gK} \circ v)$$

↑ $e_K \circ v = v$ just shown.

$$= (\mathbb{1}_{gK} \circ v).$$

<u>$\mathcal{G} \circ \mathcal{F}$:</u> $(\pi', V') \in \mathcal{R}(G)$

$(\pi'', V'') := \mathcal{G} \circ \mathcal{F}(\pi, V)$.

<u>Part 1:</u> $K \leq G$ copen; $v \in V$.

$v \in V'^{K \text{ w.r.t. } \pi'} \iff \pi'(e_K) v = v$

"⇐" If $\pi'(e_K)v = v$, then

$$\pi'(k)v = \pi'(k)\,\pi'(e_K)v = \pi'(k)\int_G \frac{1}{\mu(K)} \mathbb{1}_K(h)\,\pi'(h)v\,dh$$

$$= \int_G \frac{1}{\mu(K)} \mathbb{1}_K(h)\,\pi'(kh)\,v\,dh$$

$$= \int_G \frac{1}{\mu(K)} \mathbb{1}_K(k^{-1}h)\,\pi'(h)v\,dh$$

$$\underset{\substack{\uparrow \\ k\in K}}{=} \int_G \frac{1}{\mu(K)} \mathbb{1}_K(h)\,\pi'(h)v\,dh = \pi(e_K)v = v$$

"⇒" Exercise. □

<u>Part 2:</u> ~~$\pi''(g)$~~ Take $v \in V$, s.t. $v \in V^{K}$ w.r.t π'

$$\pi''(g)v := \pi'(I_{gK})v$$

$$= \int_G \frac{1}{\mu(K)} \mathbb{1}_{gK}(h)\,\pi'(g)v\,dh$$

$$= \pi'(g)v.$$

<u>Exercise:</u>

- $\mathrm{Hom}_G(V_1, V_2) \xrightarrow{\mathcal{F}} \mathrm{Hom}_{\mathcal{H}(G)}(\mathcal{F}(V_1), \mathcal{F}(V_2))$

$$\mathcal{F}(\varphi) := \varphi.$$

$\mathcal{G}(\varphi) := \varphi.$ • $\mathrm{Hom}_{\mathcal{H}(G)}(W_1, W_2) \xrightarrow[\sim]{\mathcal{G}} \mathrm{Hom}_G(\mathcal{G}(W_1), \mathcal{G}(W_2))$ □

Prop 38: $(\pi, V) \in \mathcal{R}(G)$, $K \leq_{\text{open}} G$

Then $\pi(e_K) : V \twoheadrightarrow V^K$ has kernel

$V(K) = \langle \pi(k)v - v \mid k \in K, v \in V \rangle$

in particular $V^K \oplus V(K) = V$.

Proof: We have $\pi(e_K) \circ \pi(k) = \pi(e_K * \delta_k)$

$= \pi(\mathbb{1}_{Kk}) = \pi(e_K)$.

(Bitte wenden)

Thus $\pi(e_K)V(K) = \{0\}$.

Take $v \in \ker(\pi(e_K))$ and $K' \leq_{\text{open}} K$ such that $v \in V^{K'}$. $K = \overset{\ell}{\underset{i=1}{\dot\bigcup}} g_i K'$

Then $0 = \pi(e_K) v = \frac{1}{\mu(K)} \sum_{i=1}^{\ell} \int_{g_i K'} \pi(h) v \, dh$

$$= \sum_{i=1}^{\ell} \pi(g_i) v \cdot \frac{\mu(K')}{\mu(K)}$$

$$\Rightarrow \quad v = \sum_{i=1}^{\ell} \Big(-\pi(g_i) v \frac{\mu(K')}{\mu(K)} - \underbrace{\Big(-\frac{\mu(K')}{\mu(K)}\Big) v}_{v_i} \Big)$$

$$= \sum_{i=1}^{\ell} \big(\pi(g_i) v_i - v_i \big) \in V(K). \quad \square$$

The exact functor $V \mapsto V^K$.

Recap: A functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$

is called

left-exact, if $\forall_{0 \to A \to B \to C \to 0 \text{ exact}}$:

$0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C)$ exact
(covariant case)

$0 \to \mathcal{F}(C) \to \mathcal{F}(B) \to \mathcal{F}(A)$ exact
(contravariant case)

Analogously right-exact.

exact $:=$ left- and right-exact.

Prop 39: Let $H \leq G$ be closed. Then

(a) $\mathcal{R}(G) \longrightarrow \mathcal{R}(1) =$ "$\mathbb{C}$-vector spaces"

$$V \longmapsto V^H$$

is left exact

and if H is compact it is exact.

(b) Conversely: If $V_1 \xrightarrow{\varphi} V_2 \xrightarrow{\psi} V_3$ is ~~exact~~ a sequence s.t. $\forall K \leq G$ compact:

$$V_1^K \to V_2^K \to V_3^K \text{ is exact.}$$

Then $V_1 \to V_2 \to V_3$ is exact.

Proof: (a) Suppose $0 \to V_1 \xrightarrow{\varphi} V_2 \xrightarrow{\psi} V_3 \to 0$ is exact. $\Rightarrow$ $0 \to V_1^H \hookrightarrow V_2^H \longrightarrow V_3^H$

Take $v_2 \in V_2^H$ s.t. $\psi(v_2) = 0$

$\Rightarrow \exists\, v_1 \in V_1 : \varphi(v_1) = v_2$

$$\Rightarrow \varphi(\pi_1(h) v_1) = \pi_2(h)\varphi(v_1) = \pi_2(h) v_2 = v_2$$
$$= \varphi(v_1) \quad \forall h \in H$$

$\Rightarrow$ $\pi_1(h) v_1 = v_1 \; \forall h \in H \Rightarrow v_1 \in V_1^H$

↑ injectivity

(b) If H is compact, then

$$V_3^H = \psi(V_2)^H = \pi_3(e_H)\,\psi(V_2)$$
$$= \psi(\pi_2(e_H) V_2) = \psi(V_2^H)$$

(iii) Let W ~~a~~ $\in \mathcal{H}(G/K)$-mod be irreducible. Then $\exists\ (\pi, V) \in \mathcal{R}(G)$: $V^K \simeq W$.

<u>Proof:</u> (i). „$\Rightarrow$" Suppose $V^K \neq 0$. Take ~~$W \leq_K V^K$~~

$0 \neq W \leq_{\mathcal{H}(G/K)} V^K$

~~$\Rightarrow \mathcal{H}(G/K) W =$~~

$\Rightarrow \mathcal{H}(G) W = V$ by irreducibility of V.

and $V^K = \pi(e_K) V = e_K * \mathcal{H}(G) W$

$\overset{W \subseteq V^K}{=} e_K * \mathcal{H}(G) * e_K W$

$= \mathcal{H}(G/K) W = W.$

„$\Leftarrow$" Take $0 \neq V' \leq_G V$. ~~Take $K \leq_{\text{open}} G$~~

~~o.E.~~ Let $v \in V^K$. Take $K' \leq_{\text{open}} K$

s.t. $V'^{K'} \neq 0$.

Then $V'^{K'} = V^{K'}$, because $V^{K'}$ is irred.

$\Rightarrow \quad v \in V^{K'} \subseteq V' \Rightarrow$ (v arbitrary) $V' = V$.

(ii) „$\Rightarrow$" ✓

„$\Leftarrow$" Take an isomorphism $f: V_1^K \to V_2^K$.

$0 \neq W = \{(v_1, f(v_1)) \; v_1 \in V_1^K\}$

$\subseteq V_1^K \oplus V_2^K \subseteq V_1 \oplus V_2$

$V' := \mathcal{H}(G) W$

$V' \xrightarrow{p_i} V_i \qquad i = 1, 2.$

If we know that p_i is an isomorphism then $V_1 \simeq V_2$. To show $\ker(p_i) = 0$.

~~$\ker(p_1) = \{(0, f(0))\} = \{(0,0)\}$~~

~~$\ker(p_2)$~~

If $\ker(p_1) \neq \{0\}$, then $0 \oplus V_2 \subseteq \ker(p_1)$, because V_2 is irreducible.

$\Rightarrow \quad 0 \oplus V_2^K \subseteq \ker(p_1)^K \subseteq (\mathcal{H}(G) W)^K$

$$(\mathcal{H}(G) W)^K = W$$

But $(0 \oplus V_2) \cap W = \{(0, f(0))\} = \{(0,0)\}$

Thus $V_2^K = 0$ ↯

Analogously $\ker(P_2) \neq 0$. ∎

(iii) W irred. $\mathcal{H}(G/K)$ module.

⟹ $W \cong \mathcal{H}(G)/I$ I left ideal.

$V_1 := \mathcal{H}(G) * \mathcal{H}(G/K)$

$V_2 := \mathcal{H}(G) * I$.

$\Rightarrow V_1^K = \mathcal{H}(G/K)$, $V_2^K = I$.

$\Rightarrow$ (↑ exactness) $W \simeq V_1^K / V_2^K \simeq (V_1/V_2)^K$.

~~In particular $\mathcal{H}(G)(V_1/V_2)$ is a finitely generated $\mathcal{H}(G)$-module.~~

$V_3 := V_2/V_1$ Take $v \in V_3^K \setminus \{0\}$.

$\Rightarrow \mathcal{H}(G) v = \mathcal{H}(G)\mathcal{H}(G/K) v$

$= \mathcal{H}(G) V_1^K / V_2^K = V_1/V_2 = V_3$

because $\mathcal{H}(G)\, V_1^K = V_1$.

Take a maximal sub-rep. V_4 of V_3 which does not contain v.

Then V_3 / V_4 is irreducible.

and $0 \neq \left(V_3 / V_4 \right)^K = V_3^K / V_4^K$

V_3^K is irred. and $V_3^K / V_4^K \neq 0 \Rightarrow V_4^K = \{0\}$.

$\Rightarrow \left(V_3 / V_4 \right)^K \simeq V_3^K \simeq W$ □

Coinvariants

$(\pi, V) \in \mathcal{R}(G)$. $\quad V(H) := \langle \pi(h)v - v \mid h \in H, v \in V \rangle$

$H \leq G$ closed

$$V_H := V / V(H) \quad \text{space of } H\text{-coinvariants.}$$

The functor $\mathcal{R}(G) \longrightarrow \mathcal{R}(1)$ $V \longrightarrow V_H$

is ~~rei~~ right exact.

(Exercise)

<u>Proposition 40</u>: Suppose H is ~~an increasing~~ a union of compact open subgroups, ~~i.e.~~ s.t.

$\forall_{h_1, \ldots, h_\ell} h_\ell \in H \;\; \exists K \underset{\text{c.open (in } H)}{\leq} H : h_1, \ldots, h_\ell \in K$.

Then

(i) $\forall_{(\pi, V) \in \mathcal{R}(G)} : V(H) = \bigcup_{\substack{K \leq H \\ \text{c.open}}} V(K)$

(ii) $V \longrightarrow V_H$ is exact.

Important example:

$$GL_n(F) \geq H = \begin{pmatrix} 1 & F & \cdots & F \\ & \ddots & & \vdots \\ & & & F \\ & & & 1 \end{pmatrix}$$

$$= \bigcup_i \begin{pmatrix} 1 & \varphi_F^i & \cdots & \varphi_p^i \\ & \ddots & \ddots & \\ & & & \varphi_p^i \\ & & & 1 \end{pmatrix}$$

Proof: (i) ✓

(ii) Let $V_1 \overset{\varphi}{\hookrightarrow} V_2$ be given.

Take $\bar{v}_1 \in V_{1\,H} : \bar{\varphi}(\bar{v}_1) = \bar{0} \in V_{2\,H}$

$\Rightarrow \varphi(v_1) \in V_2(H)$.

$\overset{(i)}{\Rightarrow} \exists\, K \leq_{\text{c.open}} H : \varphi(v_1) \in V_2(K)$.

$\Rightarrow \varphi(\pi_1(e_K) v_1) = \pi_1(e_K)\varphi(v_1) = 0$.

$\Rightarrow \pi_1(e_K) v_1 = 0 \Rightarrow v_1 \in V_1(K)$.

↑ φ injective.

$\Rightarrow \bar{v}_1 = \bar{0} \in V_1 / V_1(H)$ □

Remark: (i) The co invariant - functor $()_H$ is the left adjoint to the inflation functor.

Contragredient representation:

$(\pi, V) \in {}^{\infty}\mathcal{R}(G)$

$V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is not smooth with

$$(\pi^*(g)\, v^*)(v) := v^*(g^{-1} v).$$

So take the smooth part

$$(V^*)^{\infty} = \{ v^* \in V^* \mid \exists\, K \leq G \text{ (compact)} : v^* \in V^{*K} \}$$

$\| \colon$

$\tilde{V}$ <u>Contragredient</u> ~~representation~~ of (π, V).

<u>Properties</u>: From $V(K) \oplus V^K = V$ follows.

$\tilde{V}^K \simeq (V^K)^*$; because just take a linear form of V^K and extend it to V by 0 on $V(K)$.

<u>Prop 41:</u> $V \longmapsto \tilde{V}$ $\mathcal{R}(G) \longrightarrow \mathcal{R}(G)$

is contravariant and exact.

<u>Proof</u>: Take $K \leq_{\text{open}} G$.

$0 \to V_1 \to V_2 \to V_3 \to 0$ exact

$\Rightarrow 0 \to V_1^K \to V_2^K \to V_3^K \to 0$ exact

$\Rightarrow 0 \to V_3^{K*} \to V_2^{K*} \to V_1^{K*} \to 0$ —"—

$\Rightarrow 0 \to \tilde{V}_3^K \to \tilde{V}_2^K \to \tilde{V}_1^K \to 0$ —"—

K arbitrary $0 \to \tilde{V}_3 \to \tilde{V}_2 \to \tilde{V}_1 \to 0$ exact $\square$

Admissibility

$(\pi, V) \in \mathcal{R}(G)$ is called admissible if
for all $K \leq_{\text{open}} G$: V^K is finitedimensional.

$\mathcal{R}_{adm}(G) := \{ (\pi, V) \in \mathcal{R}(G) \mid (\pi, V) \text{ admissible} \}$

Properties 41: (π, V) admissible $\Leftrightarrow$ $(\tilde{\pi}, \tilde{V})$ is admissible.

$\Leftrightarrow$ $\pi \simeq \tilde{\tilde{\pi}}$, via $v \mapsto (v^* \mapsto v^*(v))$

Proof: • $\dim_{\mathbb{C}} V^K < \infty \Leftrightarrow \dim_{\mathbb{C}} (V^{K*}) < \infty$.

• ~~V^K~~ ~~$\Rightarrow$~~ If (π, V) is adm.

then $\dim_{\mathbb{C}} V^K < \infty \Rightarrow (V^K)^{**} \simeq V^K$

$$\begin{matrix} \cup \\ \tilde{\tilde{V}}^K \end{matrix}$$

~~Take K s.t. $V^K \neq 0$~~ Take K arbitrary

~~$\Leftarrow$)~~

$\Rightarrow$ $V \longrightarrow \tilde{\tilde{V}}$ is surjective.

If $V \to \tilde{V}$ is surjective

$\Rightarrow \forall K : V^K \to \tilde{V}^K$ is bijective.
$\tilde{V}^K = (V^K)^{*r}$

$\Rightarrow \forall K : \dim_{\mathbb{C}} V^K < \infty$. □

Exercise: ~~V~~ (π, V) adm. Then

V irred $\Leftrightarrow$ $\tilde{V}$ irreducible.

Shur's lemma 42: Suppose that $(\pi, V) \in \mathcal{R}(G)$ is irreducible s.t. $\dim_{\mathbb{C}} \mathrm{End}_G V < |\mathbb{C}|$

then $\mathrm{End}_G V = \mathbb{C}$.

Proof: • $\mathrm{End}_G V$ is a skew field, because V is irreducible:

$\varphi \in \mathrm{End}_G V \Rightarrow$ (φ is not bijective
$\Leftrightarrow \ker(\varphi) = V$ or $\mathrm{im}(\varphi) = 0$
$\Leftrightarrow \varphi = 0$) [V irred]

• Suppose $(\mathrm{End}_G V) \setminus \mathbb{C} \neq \emptyset$. Take $\varphi \in \mathrm{End}_G V \setminus \mathbb{C}$

Then

Then $(\varphi - \lambda\, id_V)^{-1}$, $\lambda \in \mathbb{C}$ are linearly independent:

Say $\sum_{i=1}^{\ell} \alpha_i (\varphi - \lambda_i\, id_V) = 0$

$\Rightarrow$ $P(\varphi) = 0$ for some $P \in \mathbb{C}[X]$

multiply with $\prod_{i=1}^{\ell} (\varphi - \lambda_i\, id_V)$

[P is non-zero, because λ_1 is not a zero]

$\Rightarrow$ φ is algebraic over $\mathbb{C}$

$\Rightarrow$ $\varphi \in \mathbb{C}$. ↯ $\square$

↑

$\mathbb{C}$ algebraically closed

Consequence 4.3: If (π, V) is irred and $\dim_{\mathbb{C}} \mathrm{End}_G V < |\mathbb{C}|$

Then $Z(G) :=$ center of $G = \{g \in G \mid \forall g' \in G : gg' = g'g\}$

acts as a scalar on V via π.

$\omega_\pi : Z(G) \longrightarrow \mathbb{C}$ is defined by

$\pi(z)\, v = \omega_\pi(z)\, v \quad \forall v \in V$.

"central character of π".

Def 44: Tensor products of repr.

G_1, G_2 loc. profinite grps. $G := G_1 \times G_2$

Then we have a functor

$$\mathcal{R}(G_1) \times \mathcal{R}(G_2) \xrightarrow{\otimes} \mathcal{R}(G)$$

$$((\pi_1, V_1), (\pi_2^*, V_2)) \longmapsto (\pi_1 \otimes \pi_2, V_1 \otimes_{\mathbb{C}} V_2)$$

$$(\pi_1 \otimes \pi_2)(g).(v_1 \otimes v_2) = \pi_1(g) v_1 \otimes_{\mathbb{C}} \pi_2(g) v_2.$$

~~Two cases are very interesting:~~

~~1) $\dim_{\mathbb{C}} V_2 = 1$. $\Rightarrow \pi_2$ is a character χ_2~~

Later we will be interested in the case

$G_1 = GL_{n_1}(F)$ and $G_2 = GL_{n_2}(F)$.

Prop 45 (without proof) Let $(\pi_i, V_i) \in \mathcal{R}(G_i)$ be admissible representations. Then

1° $\pi_1 \otimes \pi_2$ is admissible

2° $\tilde{\pi}_1 \otimes \tilde{\pi}_2 \cong \widetilde{\pi_1 \otimes \pi_2}$

3° If π_1 and π_2 are irred. then $\pi_1 \otimes \pi_2$ is too.

4° If $(\pi, V) \in \mathcal{R}_{adm}(G)$ is irr.

Then $\exists\, \pi_i \in \mathcal{R}_{adm}(G)$ unique up to isom. s.t. $\pi \cong \pi_1 \otimes \pi_2$.

Induction

Given G loc. profinite ~~disc~~ and $H \leq G$ closed we want to construct a functor.

$$\mathcal{R}(H) \longrightarrow \mathcal{R}(G).$$

Def 4.6: ~~Ind_H^G~~ For $(\rho, W) \in \mathcal{R}(H)$ define

$$\mathrm{Ind}_H^G \rho := \{ f : G \to W \mid \exists_{K \leq G \text{ open}}\ \forall_{h \in H} \forall_{g \in G} \forall_{k \in K} : f(hgk) = \rho(h) f(g). \}$$

"induced representation of ρ from H to G"

$$c\text{-}\mathrm{Ind}_H^G \rho := \{ f \in \mathrm{Ind}_H^G \rho \mid \exists_{E \text{ compact}}\ \mathrm{supp}(f) \subseteq HE. \}$$

The action is defined via $(g \cdot f)(\tilde{g}) := f(\tilde{g} g)$

The condition with K means smoothness.

1st important statement:

Prop 47: (Frobenius reciprocity)

1) The functor Ind_H^G is right adjoint to $\mathrm{Res}_H^G : \mathcal{R}(G) \longrightarrow \mathcal{R}(H)$

2) If H is open, then $c\text{-}\mathrm{Ind}_H^G$ is left adjoint to Res_H^G.

Proof: Take $(\pi, V) \in \mathcal{R}(G)$ and $(\sigma, W) \in \mathcal{R}(H)$

1) To show:

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\sigma)) \simeq \mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \sigma)$$

$$\varphi \longmapsto \psi_\varphi \qquad \psi_\varphi(v) := \varphi(v)(1)$$

$$\varphi_\psi \longleftarrow\!\!\shortmid \psi$$

$$(\varphi_\psi(v))(g) := \psi(\pi(g) v).$$

Have to show: a) $\psi_\varphi \in \mathrm{Hom}_H(\ldots)$

$$\Psi_\varphi(\pi(h)v) = \varphi(\pi(h)v)(1) \underset{\uparrow \atop \varphi \in \mathrm{Hom}_G(-,-)}{=} \varphi(v)(h)$$

$$= \rho(h)(\varphi(v)(1)) = \rho(h)\Psi_\varphi(v).$$

e) $\varphi_\Psi(v) \in \mathrm{Ind}_H^G(\sigma)$. Take $K \leq_{\text{open}} G$ s.t. $v \in V^K$.

Then for all $h \in H, g \in G, k \in K$:

$$\varphi_\Psi(v)(hgk) = \Psi(\pi(hgk)v) \overset{\Psi \in \mathrm{Hom}_H(-,-)}{\underset{v \in V^K}{=}} \rho(h)\Psi(\pi(g)v)$$

$$= \rho(h)(\varphi_\Psi(v)(g)).$$

c) $\varphi_\Psi \in \mathrm{Hom}_G(-,-)$ Exercise

d) $\Psi_{\varphi_\Psi} = \Psi$ and $\varphi_{\Psi_\varphi} = \varphi$.

We only show $\Psi_{\varphi_\Psi} = \Psi$.

$$\Psi_{(\varphi_\Psi)}(v) = \varphi_\Psi(v)(1) = \Psi(\pi(1)v) = \Psi(v).$$

2) To show

$$\mathrm{Hom}_G(c\text{-}\mathrm{Ind}_H^G \sigma, \pi) \simeq \mathrm{Hom}_H(\sigma, \mathrm{Res}_H^G(\pi))$$

$$\varphi \longmapsto \Psi_\varphi \qquad \Psi_\varphi(w) := \varphi(f_w)$$

where $f_w \in c\text{-}\mathrm{Ind}_H^G \sigma$ is defined via

$$f_w(g) := \begin{cases} 0, & \text{if } g \notin H \\ \sigma(g)w, & \text{if } g \in H. \end{cases}$$

$$\varphi_\psi \longleftarrow\!\!\mapsto \psi$$

$$\varphi_\psi(f) := \sum_{H\backslash G \ni Hg} \pi(g^{-1})\psi(f(g)) \quad , \text{ well defined}$$

because supp(f) is compact mod H and H is open, and because $\psi \in \mathrm{Hom}_H$.

<u>To show:</u>

e) $\psi_\varphi \in \mathrm{Hom}_H(-,-)$:

$$\psi_\varphi(\sigma(h)\omega) = \varphi(f_{\sigma(h)\omega}) = \varphi(\sigma(h) f_\omega) \overset{\varphi \in \mathrm{Hom}_G(-,-)}{=} \pi(h)\varphi(f_\omega) = \pi(h)\psi_\varphi(\omega).$$

$$\left\lceil f_{\sigma(h)\omega}(g) = \begin{cases} 0, & g \notin H \\ \underbrace{\sigma(g)\sigma(h)}_{\sigma(gh)}\omega, & g \in H \end{cases}\right.$$

$$= f_\omega(gh) = (\sigma(h) f_\omega)(g) \rfloor$$

f) $\varphi_\psi \in \mathrm{Hom}_G(-,-)$

$$\varphi_\psi(g \bullet f) = \sum_{H\backslash G\, H\tilde{g}} \pi(\tilde{g}^{-1})\psi(f(\tilde{g}g))$$

$$= \pi(g)\sum_{H\backslash G \ni H\tilde{g}} \pi((\tilde{g}g)^{-1})\psi(f(\tilde{g}g))$$

$$= \pi(g)\varphi_\psi(f).$$

g) $\Psi_{\Phi_\Psi} = \Psi$ and $\Phi_{\Psi_\varphi} = \varphi$.

We only show $\varphi_{\Psi_a} = \varphi$.

$$\varphi_{(\Psi_\varphi)}(\tilde{f}) = \sum_{H\backslash G \ni Hg} \pi(g^{-1})\, \Psi_\varphi(\tilde{f}(g))$$

$$= \sum_{H\backslash G\, Hg} \pi(g^{-1})\, \varphi(f_{\tilde{f}(g)})$$

$$\underset{\varphi \in \mathrm{Hom}_G(-,-)}{=} \varphi\Big(\sum_{H\backslash G \ni Hg} g^{-1} \cdot f_{\tilde{f}(g)}\Big) = \varphi(\tilde{f})$$

$$\Big(\sum_{H\backslash G} g^{-1} \cdot f_{\tilde{f}(g)}\Big)(g')$$

$$= \sum_{H\backslash G} f_{\tilde{f}(g)}(g' g^{-1})$$

$$= f_{\tilde{f}(g')}(1) = \sigma(1)\tilde{f}(g') = \tilde{f}(g')$$

$\big(f_{\tilde{f}(g)}(g'g^{-1}) \neq 0 \Leftrightarrow g'g^{-1} \in H\big)$

q.e.d.

Mackey decomposition

<u>Prop 48:</u> For $K \leq G$ open, $H \leq G$ closed and $(\sigma, W) \in \mathcal{R}(H)$ we have

$$\operatorname{Res}^G_K \operatorname{Ind}^G_H \sigma \xrightarrow{\cong} \prod_{K\backslash G/H} \operatorname{Ind}^K_{K\cap {}^gH} \operatorname{Res}^{{}^gH}_{K\cap {}^gH} {}^g\sigma .$$

- and

$$\operatorname{Res}^G_K \operatorname{c-Ind}^G_H \sigma \simeq \prod_{K\backslash G/H} \operatorname{c-Ind}^K_{K\cap {}^gH} \operatorname{Res}^{{}^gH}_{K\cap {}^gH} .$$

<u>Proof:</u> (Sketch) $\operatorname{Ind}_{HgK} \sigma := \{ f \in \operatorname{Ind}^G_H \sigma \mid \operatorname{supp}(f) \subseteq HgK \}$

- We have

$$\operatorname{Ind}_{HgK} \sigma \xrightarrow{\sim} \operatorname{Ind}^K_{K\cap g^{-1}H} \operatorname{Res}^{g^{-1}H}_{K\cap g^{-1}H} {}^{g^{-1}}\sigma .$$

$$f \longmapsto \alpha_f$$

$$\alpha_f(k) := f(gk)$$

$$f_\alpha(hgk) := \sigma(h)\alpha(k) \longleftarrow\!\!| \ \alpha$$

We have $\alpha_{f_\alpha} = \alpha$ and $f_{(\alpha_f)} = f$.

e.g. $f_{(\alpha_f)}(hgk) = \sigma(h)\,\alpha_f(k) = \sigma(h)\, f(gk)$
$= f(hgk)$.

To show: • α_f is smooth: This is, because f is smooth.

• $\alpha_f(kk') = f(gkk')$ $k \in K \cap {}^{g^{-1}}H$, $k' \in K$

$= \sigma(gkg^{-1})\, f(gk')$

$= \sigma(gkg^{-1})\, \alpha_f(k')$

• f_α is smooth because α is smooth.

$f_\alpha(hgk) = \sigma(h)\,\alpha(k) = \sigma(h)\, f(gk)$.

We have ~~Thus~~

$$f \longmapsto \sum_{H\backslash G/K} f|_{HgK}$$

$$\mathrm{Res}^G_K \mathrm{Ind}^G_H \sigma \xrightarrow{\sim} \prod_{HgK \in H\backslash G/K} \mathrm{Ind}_{HgK}\, \sigma$$

$$\cong \prod_{K\backslash G/H} \mathrm{Ind}^K_{K\cap {}^gH}\, \mathrm{Res}^{{}^gH}_{K\cap {}^gH}\, {}^g\sigma$$

↗ Substitute g by g^{-1}

Exercise: This restricts to $\mathrm{Res}^G_K\, c\text{-}\mathrm{Ind}^G_H \sigma \cong \bigoplus_{K\backslash G/H} c\text{-}\mathrm{Ind}^K_{K\cap {}^gH}\, \mathrm{Res}^{{}^gH}_{K\cap {}^gH}\, {}^g\sigma$ □

The image of

(*) $$\mathrm{Res}^G_K \, \mathrm{Ind}^G_H \sigma \hookrightarrow \prod_{H\backslash G/K} \mathrm{Ind}^\sigma_{H^gK}$$

is equal to ~~consists of of~~ $$\bigcup_{K' \subseteq_{\text{open}} G} \Big(\prod_{H\backslash G/K} \mathrm{Ind}^\sigma_{H^gK} \Big)^{K'}$$

$$\parallel$$

$$\bigcup_{K' \subseteq_{\text{open}} G} \prod_{H\backslash G/K} \big(\mathrm{Ind}^\sigma_{H^gK}\big)^{K'}.$$

~~*~~ <u>Corollary ~~#~~ 9:</u> $\left(\mathrm{Ind}^G_H \sigma\right)^K \simeq \prod_{H\backslash G/K} W^{H \cap {}^gK}$,

and $\left(c\text{-}\mathrm{Ind}^G_H \sigma\right)^K \simeq \bigoplus_{H\backslash G/K} W^{H\cap {}^gK}$

<u>Proof</u>: We only prove it for $\mathrm{Ind}^G_H \sigma$.

(*) gives $\left(\mathrm{Ind}^G_H \sigma\right)^K \simeq \prod_{H\backslash G/K} \left(\mathrm{Ind}^\sigma_{H^gK}\right)^K$

and we have further

$$\left(\mathrm{Ind}^\sigma_{H^gK}\right)^K \underset{\mathbb{C}\text{-v.s.}}{\simeq} W^{H\cap {}^gK}$$ via

The following maps

$$f \longmapsto f(g)$$

$$f_\omega \longleftarrow\!\!\!| \ \omega$$

$f_\omega(hgk) \overset{\text{''}}{=} \sigma(h)\,\omega$

We have to show:

- $f(g) \in W^{H \cap {}^gK}$:

$\sigma(h) f(g) = f(hg) = f(g g^{-1} h g)$

thus if $g^{-1} h g \in K$, then

$\sigma(h) f(g) = f(g)$, because

$f \in (\mathrm{Ind}_{H^gK}^{\sigma})^K$.

- They are inverse to each other:

- $f_{f(g)}(hgk) \underset{\uparrow}{=} \sigma(h) f(g) \underset{\uparrow}{=} f(hg) \underset{\uparrow}{=} f(hgk)$.

(Def of f_ω; Property of f; f K-fixed)

- $f_\omega(g) = \omega$.

- We have forgotten to check that f_ω is well-defined:

$hgk = h'gk' \Rightarrow h'^{-1}h \in H \cap gKg^{-1}$

$\underset{\uparrow}{\Rightarrow} \sigma(h'^{-1}h)\,\omega = \omega$

($\omega \in W^{H \cap gKg^{-1}}$)

$\Rightarrow \sigma(h)\omega = \sigma(h')\omega$.

□

Corollary 50: Let H be a closed subgroup of G

(i) If $(\sigma, W) \in \mathcal{R}(H)$ is admissible then

$\mathrm{Ind}_H^G \sigma$ is adm. $\Leftrightarrow$ $c\text{-}\mathrm{Ind}_H^G \sigma$ is adm.

and in then $\mathrm{Ind}_H^G \sigma \simeq \text{c-}\mathrm{Ind}_H^G \sigma$.

(ii) If $H \backslash G$ is compact then $\mathrm{Ind}_H^G = \text{c-}\mathrm{Ind}_H^G$ and Ind_H^G respects admissibility.

Proof: (i) This follows directly from Corollary 49. and by $(\text{c-}\mathrm{Ind}_H^G \sigma)^K \xrightarrow{\simeq} (\mathrm{Ind}_H^G \sigma)^K$ forall $K \leq G$ copen we ~~get~~ get that $\text{c-}\mathrm{Ind}_H^G \sigma \hookrightarrow \mathrm{Ind}_H^G \sigma$ is surjective.

(ii) $H \backslash G / K$ is finite □

Remark 51: $(\text{c-}\mathrm{Ind}_H^G \sigma)^\sim \simeq \mathrm{Ind}_H^G (\delta_G^{-1} \cdot \delta_H \tilde{\sigma})$.
(without proof)

Ende VL 5

Z - compact representations

G: locally profinite group.

Assumption: G is countable at ∞, i.e. G is a countable union of compact open subsets.

Def 52: $(\pi, V) \in \mathcal{R}(G)$. Take $v \in V$ and $\tilde{v} \in \tilde{V}$.

The map $\varphi_{v\tilde{v}}: g \mapsto \tilde{v}(\pi(g)v) =: \langle \tilde{v}, \pi(g)v \rangle \in \mathbb{C}$

is called a matrix coefficient of G.

(π, V) is called $Z = Z(G)$ - compact if all matrix coefficients are compactly supported mod Z, i.e. $\forall_{v,\tilde{v}} \exists_{E_{v,\tilde{v}}}$ compact:

$\mathrm{supp}(\varphi_{v,\tilde{v}}) \subseteq Z E_{v,\tilde{v}}$.

Prop 53: Let π be irreducible. (π, V) is Z - compact $\Leftrightarrow$ $\forall_{K \leq G \text{ copen}}$

$\forall_{v \in V}: f_{K,v}: G \to V$, $g \mapsto \underbrace{\pi(e_K)\pi(g)v}_{= \pi(I_{Kg})v}$

has compact support mod Z.

Proof "$\Leftarrow$" If every $f_{K,v}$ is compactly supported mod Z, then take $v \in V$ and $\tilde{v} \in \tilde{V}$ and $K \leq G$ copen, s.t.

$\tilde{v} \in \tilde{V}^K$. Then $\operatorname{supp} \varphi_{v,\tilde{v}} \subseteq \operatorname{supp} f_{K,v}$.

"$\Rightarrow$": <u>Part 1:</u> We show at first that (π, V) is admissible.
Take $K \leq G$ c.open and $v \in V \setminus \{0\}$.

$$_{ZK}\backslash^{G} = \{ZK g_\alpha \mid \alpha \in A\}$$

Then $V^K \overset{\uparrow}{=} \pi(e_K)\langle\pi(G)v\rangle$ ~~$= \langle \pi(e_K) g_\alpha v$~~

V is irreducible

$$= \langle \pi(e_K)\pi(g_\alpha)v \mid \alpha \in A\rangle$$

$$= \langle \pi(e_K)\pi(g_\beta)v \mid \beta \in B\rangle$$

where $\{\pi(e_K)\pi(g_\beta)v \mid \beta \in B\} =: \mathcal{B}$
is a $\mathbb{C}$-basis of V^K. $v_\beta := \pi(e_K)\pi(g_\beta)v$.

Define $\tilde{v}_{\mathcal{B}} \in \tilde{V}^K$ via $\tilde{v}_{\mathcal{B}}(v_\beta) = 1 \ \forall_{\beta \in B}$

and $\tilde{v}_{\mathcal{B}}\big|_{V(K)} = 0$.

Then $ZKg_\beta \subseteq \operatorname{supp}(\varphi_{v,\tilde{v}_{\mathcal{B}}})$ because

~~$\varphi_{v,\tilde{v}_{\mathcal{B}}}(zkg_\beta) = \omega_\pi(z)\,\tilde{v}_{\mathcal{B}}(\pi(k)\pi(g_\beta)v)$~~

$$\forall_{z \in Z}\forall_{k \in K}: \quad \varphi_{v,\tilde{v}_{\mathcal{B}}}(zkg_\beta) = \omega_\pi(z)\,\tilde{v}_{\mathcal{B}}(\pi(k)\pi(g_\beta)v)$$

$$= \omega_\pi(z) \cdot 1 \neq 0.$$

Thus $\dot{\bigcup}_{\beta \in B} ZKg_\beta \subseteq \operatorname{supp} \varphi_{v,\tilde{v}_{\mathcal{B}}}$.

$1^\circ \Rightarrow$ B is finite.

Part 2: Take $K \leq G$ c.open and $v \in V$.

$\ell := \dim_{\mathbb{C}} V^K < \infty$ by Part 1. $\Rightarrow \exists g_{i_1}, \dots, g_{i_\ell} \in G$:

$\langle \underbrace{\pi(g_i)v}_{v_i} \mid i = 1, \dots, \ell \rangle = V^K$.

Define $\tilde{v}_i \in \tilde{V}^K$ via $\tilde{v}_i|_{V(K)} = 0$ and $\tilde{v}_i(v_j) = \delta_{ij}$.

Then $\operatorname{supp} f_{K,v} \subseteq \bigcup \operatorname{supp}(\varphi_{v_i \tilde{v}_i})$

because if $g \in \operatorname{supp}(f_{K,v})$, then

$\pi(K)\pi(g)v \neq 0$, thus $\exists i \in \{1, \dots, \ell\}$: $\tilde{v}_i(\pi(K)\pi(g)v) \neq 0$.

$$\varphi_{v_i \tilde{v}_i}(g) = \tilde{v}_i(\pi(g)v)$$

$\Rightarrow \exists_i$: $g \in \operatorname{supp}(\varphi_{v_i \tilde{v}_i})$

From 1° follows now 2° $\square$

Cor. 54: A Z-compact irred repr. is admissible.

Some algebraic structure theory for $GL_n(F)$.

$G = GL_n(F)$.

Torus: A subgroup conjugate to $T := \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \,\middle|\, \lambda_i \in F^\times \right\}$

Parabolic subgroup: " $\quad n_1\{, n_2\{, \ldots, n_e\{ \begin{pmatrix} * & & * \\ & * & \\ & & \ddots \\ & & & * \end{pmatrix}^{n_1,\ldots,n_e} =: P_{n_1,\ldots,n_e}$

Borel subgroup: " $\quad P_{1,\ldots,1}$.

Levi subgroup of $P_{n_1,\ldots,n_e}$:

A subgroup of $P_{n_1,\ldots,n_e}$ conjugate under an element of $P_{n_1,\ldots,n_e}$ to $\begin{pmatrix} * & & \\ & * & \\ & & \ddots & \\ & & & * \end{pmatrix} = L_{n_1,\ldots,n_e}$.

$N_{n_1,\ldots,n_e} := R_U(P_{n_1,\ldots,n_e}) := \begin{pmatrix} I_{n_1} & & * \\ & \ddots & \\ & & I_{n_e} \end{pmatrix}$ unipotent radical of $P_{n_1,\ldots,n_e}$.

We call $P_{n_1,\ldots,n_e}$ the standard parabolic subgroups.
$L_{n_1,\ldots,n_e}$ the standard Levi subgroup of $P_{n_1,\ldots,n_e}$.

$$\overline{P}_{n_1,\dots,n_\ell} := {}^t P_{n_1,\dots,n_\ell}$$

Conjugation defines $\overline{P}$ for P.

<u>Prop. 55:</u> Every maximal compact subgroup of $GL_n(F)$ is conjugate to $GL_2(\mathfrak{o}_F)$. We call $GL_2(\mathfrak{o}_F)$ the standard maximal compact.

<u>Proof:</u> Take the standard basis ~~of~~ $e_1,\dots,e_n$ of F^n.

Let K be a compact open subgroup of $GL_n(F)$.

~~$\Gamma := FKe_1 + FKe_2$~~

$\Gamma := \mathfrak{o}_F K e_1 + \mathfrak{o}_F K e_2 + \dots + \mathfrak{o}_F K e_\ell$ is as a sum of compact subsets compact. Thus $\exists\, \nu_1,\dots,\nu_\ell \in \mathbb{N}_0$:

$$\Gamma \subseteq \mathfrak{p}_F^{-\nu_1} e_1 \oplus \dots \oplus \mathfrak{p}_F^{-\nu_\ell} e_\ell$$

On the other side $\Gamma \supseteq \mathfrak{o}_F^{\,n}$.

$\Rightarrow$ Γ is a full $\mathfrak{o}_F$-lattice, say $\Gamma = \mathfrak{o}_F e_1 \oplus \dots \oplus \mathfrak{o}_F e_\ell$

Thus $K \leq \{ g \in GL_n(F) \mid g\Gamma \subseteq \Gamma \}$, where ~~and~~ the latter is a conjugate of $GL_n(\mathfrak{o}_F) =: K_0$ $\square$

Prop 5.6: Iwahori decomposition.

Let $P = LN$ be a standard parabolic with standard Levi and $K_i := 1 + M_n(\varpi_F^i)$, $i \geq 1$.

Then
$$K_i = \underbrace{(\bar{N} \cap K_i)}_{=: K_{i-}} \underbrace{(L \cap K_i)}_{=: K_{iL}} \underbrace{(N \cap K_i)}_{=: K_{i+}}.$$

Proof: ~~Gaußian~~ Gaussian Algorithm. □

Exercise sheet 5: $e_{K_i} = e_{K_{i-}} * e_{K_{iL}} * e_{K_{i+}}$.

Parabolic induction and the Jacquet functor

$P = LN$ parabolic subgroup of G.

$i_L^G : \mathcal{R}(L) \xrightarrow{\text{inflation}} \mathcal{R}(P) \xrightarrow[\text{ind}_P^G]{\text{induction}} \mathcal{R}(G)$ Parabolic induction

$r_L^G : \mathcal{R}(G) \xrightarrow{\text{restriction}} \mathcal{R}(P) \xrightarrow{N\text{-coinvariants}} \mathcal{R}(L)$ parabolic restriction

r_L^G is called the Jacquet functor and V_N the Jacquet module.

Normalized functors: $i^u{}_L^G \sigma := i_L^G(\delta_P^{\frac{1}{2}} \sigma)$ norm. parabolic induction

$r^u{}_L^G \pi := \delta_P^{-\frac{1}{2}} \cdot r_L^G \pi$ normalized Jacquet functor.

<u>Prop 57</u>. 1) i_L^G, $i^u{}_L^G$, r_L^G, $r^u{}_L^G$ are exact

2) r_L^G and $r^u{}_L^G$ preserve finite type

3) i_L^G, $i^u{}_L^G$ preserve admissibility

4) i_L^G, $i^u{}_L^G$, r_L^G and $r^u{}_L^G$ are transitive.

5) $\forall \sigma \in \mathcal{R}(L)$ $i^u{}_L^G \tilde{\sigma} = \widetilde{(i^u{}_L^G \sigma)}$

Remark: In fact i_L^G, $i^{u}{}_L^G$, r_L^G, $r^{u}{}_L^G$ preserve finite type, admissibility, finite length, and are exact, but for some of these properties one needs more theory to prove them.

Proof (Prop 57): 1) They are made by using exact functors, reminding that N is a countable increasing union of open compact subgroups.

2) Suppose (π, V) has finite type, say $B = \{v_1, \dots, v_\ell\}$ generates V. Take $K \leq G$ open s.t. $B \subseteq V^K$. ${}_P\backslash^G/_K$ is finite, because ${}_P\backslash^G$ is compact. $\{P g_i K \mid i \in I\}$

Then $\{g_i v_j \mid i \in I, j \in \mathbb{N}^{\leq \ell}\}$ generates $\mathrm{Res}_P^G \pi$. Thus $r_L^G \pi$ is finitely generated.

3) ${}_P\backslash^G$ is compact by the Iwasawa decomposition $(K_0 \underset{\sim}{B} = K_0 \underset{\sim}{P} = G)$

Thus i_L^G and i^{uG}_L preserve admissibility by Corollary 50 (ii).

5) Remark 51 using $\mathrm{Ind}_P^G = c\text{-}\mathrm{Ind}_P^G$ (${}_P\backslash^G$ is compact)

4) · The transitivity for ~~in~~ induction is an exercise.

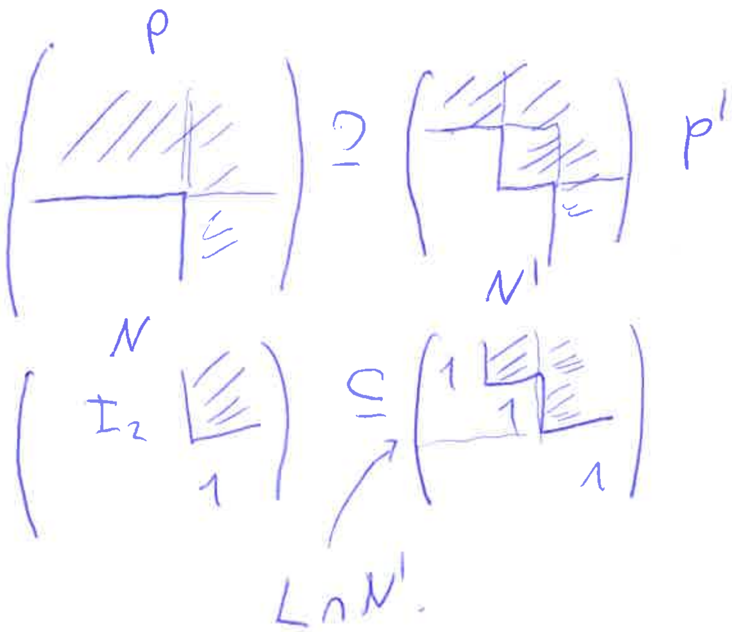
· For r_2^G: Consider $G > P > P' > B$ with $P = LN$, $P' = L'N'$

with $L > L' > T$ and $N' > N$

$\Rightarrow$ $V(N') \supseteq V(N)$ and we have

$$V_N \xrightarrow{\Phi} V_{N'}$$

To show that $V_N(N' \cap L)$ is ~~in~~ the kernel of this map.



To show $V_N(N' \cap L) = \ker \Phi$

"$\subseteq$": ✓, because $N' \cap L \subseteq N' = N(N' \cap L) = (N' \cap L)N$

"$\supseteq$" We have $V(N') = V(N' \cap L) + V(N)$

because for $\ell \in N' \cap L$ and $n \in N$ we have

$$\ell n v - v = \ell n v - n v + n v - v.$$

thus $\ker \Phi = V(N')/V(N) = \dfrac{V(N'\cap L)+V(N)}{V(N)}$

$= V_N(N'\cap L)$

Exercise $\delta_{P'}(\ell') = \delta_P(\ell')\,\delta_{L\cap P'}(\ell')$ □

<u>Def 58:</u> $(\pi,V) \in \mathcal{R}(G)$ is called cuspidal if for all proper parabolic subgroups $P < G$ and all decompositions $P = LN$ we have $r^G_{P,L}\pi = 0$.

<u>Remark:</u> By conjugacy it is enough to show $r^G_{P,L}\pi = 0$ for standard Levis.

<u>Proof:</u> Def 58 just says $V(N) = V$ for all proper parabolic subgroups. We have for $g \in G$ and all P:

$V(gNg^{-1}) = \pi(g)V(N).$ □

<u>Thm 59:</u> Suppose $(\pi,V) \in \mathcal{R}(G)$ is irreducible. Then

1) $\exists$ parabolic subgroup $P = LN$:

- $r^G_{P,L}\pi \neq 0$
- $\forall P' < P$ $\forall L'$ with $P' = L'N'$: $r^G_{L'}\pi = 0$.

i.e. $r^G_{P,L}\pi$ is cuspidal.

2) $\exists$ $P = LN$ and $\sigma \in {}^{\varphi}\mathcal{R}(L)$ irred. cuspidal s.t. $\pi \subseteq i_L^G \sigma$.

3) Fact 1) and 2) are also true ~~b~~ if one replaces $r_{P,L}^G$ and $i_{P,L}^G$ by $r_{P,L}^{uG}$ and $i_{P,L}^{uG}$.

But then we have that if $\sigma_1 \in {}^{\varphi}\mathcal{R}(L_1)$ and and $\sigma_2 \in {}^{\varphi}\mathcal{R}(L_2)$ are irred. and cuspidal s.t. $\pi \subseteq i_{P_1,L_1}^G \sigma_1$ and $\pi \subseteq i_{P_2,L_2}^G \sigma_2$

Then $\exists$ $g \in G$: $gL_1g^{-1} = L_2$ and ${}^g\sigma_1 = \sigma_2$. and one calls the

G-conjugacy class of (σ_1, L_1) the cuspidal support of π.

<u>Proof</u>: We are not going to prove 3). ~~beca~~

1) This follows by the transitivity of parabolic restriction: By the remark we only need to consider standard parabolics with standard levis. In the standard ~~wor~~ world we have

$$L \overset{1-1}{\longleftrightarrow} P \overset{1-1}{\longleftrightarrow} N.$$

There are only finitely many parabolics P s.t. $B \subseteq P$, because they are in

1-1 correspondence to the subsets of the base Φ_B of the root system of G.

Take a standard parabolic $P = LN$ with std. s.t. $r^G_{P,L}\pi \neq 0$ and $r^G_{P',L'}\pi = 0$ forall $P' < P$ std.

A parabolic subgroup $\tilde{P} < P$ contains a Borel $\tilde{B}$ which is conjugate to B by an element of P. Thus $\tilde{P}$ is conjugate to a standard parabolic. Thus $r^G_{\tilde{P},\tilde{L}}\pi = 0$

~~Tran~~ To show $r^G_{P,L}\pi$ is cuspidal. ~~Take a standard parabolic~~ We have

$\{P' \cap L \mid P' < P \text{ standard}\}$

$= \{\text{standard parabolics of } L\} \overset{1:1}{\longleftrightarrow} \{L' < L \mid \text{standard levis}\}$

$$r^L_{L\cap P', L'}(r^G_{P,L}\pi) \underset{\substack{\uparrow \\ \text{transitivity}}}{=} r^G_{P',L'}\pi = 0.$$

Thus $r^G_{P,L}\pi$ is cuspidal.

2) Consider P, L from 1). (π, V) is irred, thus of finite type. $r^G_{P,L}$ respects finite type.

$\Rightarrow$ $r^G_{P,L}\pi$ is of finite type and has thus ~~an~~ an irred. quotient σ.

Frobenius reciprocity $\Rightarrow$ $\pi \subseteq i^G_{P,L}\sigma$.

σ is still cuspidal because parabolic restriction is exact. $\square$

<u>Theorem 60:</u> An irred representation $(\pi, V) \in \mathcal{R}(G)$ is cuspidal iff it is Z-compact.

<u>Corollary 61:</u> Every irreducible smooth repres. (π, V) of G is ~~cuspidal~~ admissible.

<u>Proof:</u> Thm 59 $\Rightarrow$ $\exists\ P, L,\ \sigma \in \mathcal{R}(L)$ cuspidal: $\pi \subseteq i^G_{P,L}\sigma$.

Thm 60 $\Rightarrow$ σ is $Z(L)$-compact and thus by Corollary 54 admissible.

$i^G_{P,L}$ respects admissibility (because $P \backslash G$ is compact)

Thus $i^G_{P,L}\sigma$ is admissible $\Rightarrow$ ~~$\pi \subseteq$ something adm~~ $\Rightarrow$ π is admissible. $\square$

Proof of Thm 60:

"⇒" Let (π, V) be cuspidal. Take $K \subseteq G$ open and $v \in V$.
We need to show that $\mathrm{supp}(f_{K,v})$ is compact modulo center.

Cartan decomposition

$$G = \coprod_{\delta \in \Delta^{+}} K_0 \, \delta \, K_0$$

with

$$\Delta^{+} = \left\{ \begin{pmatrix} \varpi_F^{a_1} & & \\ & \ddots & \\ & & \varpi_F^{a_n} \end{pmatrix} \;\middle|\; \begin{array}{l} a_1 \geq a_2 \geq \dots \geq a_n \\ \text{integers} \end{array} \right\}$$

Let $\tilde{\Delta}$ be a set of representatives of $\Delta^{+} / Z(G)$

It is enough to show: $\mathrm{supp}(f_{K,v} | \tilde{\Delta}) =: \tilde{\Delta}_{K,v}$

is finite.

The representatives are given by the steps

$$\begin{pmatrix} \varpi^4 & & & & & & \\ & \ddots & & & & & \\ & & \varpi^4 & & & & \\ & & & \varpi^2 & & & \\ & & & & \varpi^2 & & \\ & & & & & \varpi^{-3} & \\ & & & & & & \varpi^{-3} \\ & & & & & & & \varpi^{-3} \end{pmatrix} \begin{matrix} \} \; 4-2=2 \\ \} \; 2-(-3)=5 \end{matrix}$$

Cone picture for $GL_3(F)$:

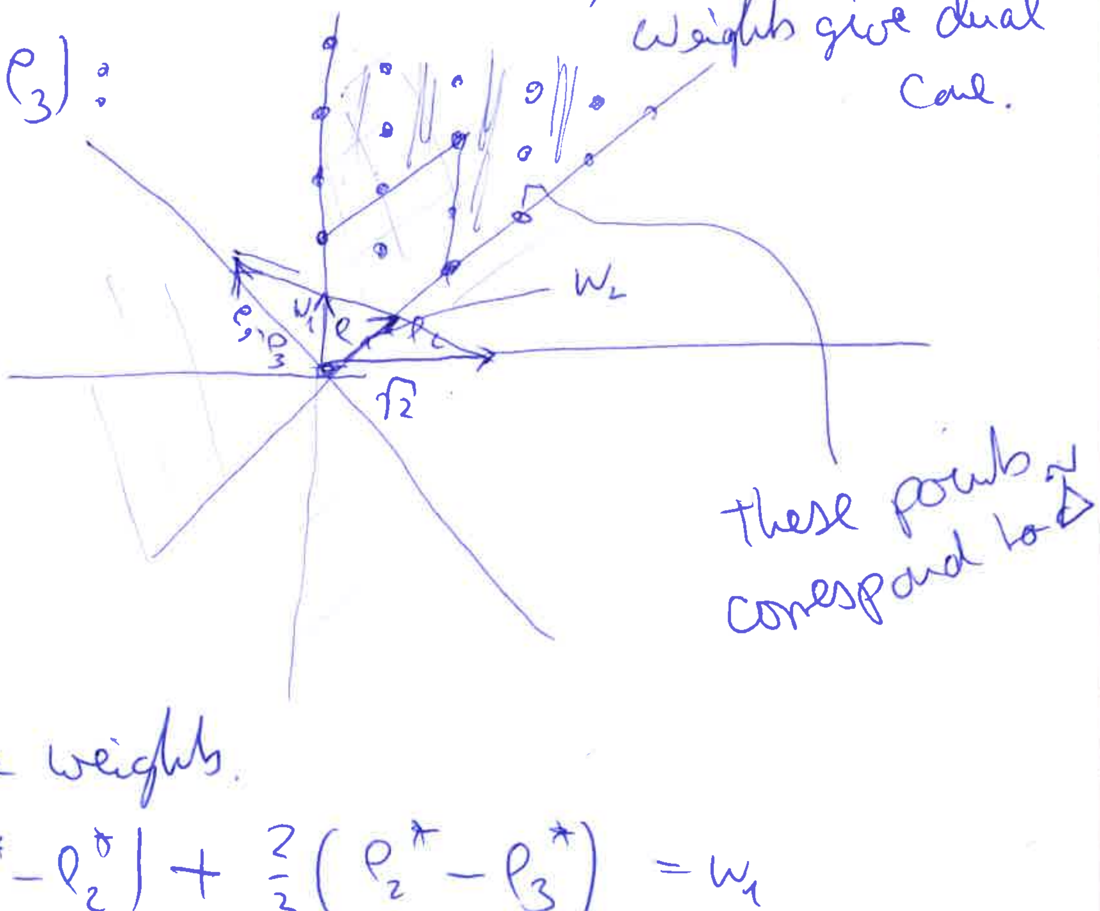
Roots: $e_1 - e_2$, $e_2 - e_3$

weights: w_1, w_2 dual basis

angle between the roots $\frac{\langle e_1 - e_2, e_2 - e_3 \rangle}{\sqrt{2} \cdot \sqrt{2}} = \frac{-1}{2}$

$= \cos(120°)$

$\text{span}_{\mathbb{R}} \{e_1 - e_2, e_2 - e_3\}$:



$\| t_p \geq 2$

$\forall p$ ~~odd~~ std.

Calculate the weights.

$$\frac{1}{3}(e_1^* - e_2^*) + \frac{2}{3}(e_2^* - e_3^*) = w_1$$

$$\frac{2}{3} \qquad\qquad \frac{1}{3} \qquad = w_2$$

Why is it enough to show that

$\mathrm{supp}(f_{K,v}|_{\tilde{\Delta}}) =: \Delta_{K,v}$ is finite

for all K, v.

It is enough to consider $K \in \{ 1 + M_n(p_F^i) \mid i \geq 1 \}$

$K \trianglelefteq K_0$. $K_0/K = \{ [k_\alpha] \mid \alpha \in A \}$ finite w.l.o.g. $v \in V^K$.

<u>Claim</u>: $\mathrm{supp}(f_{K,v}) \subseteq \bigcup_{\alpha \in A} Z K_0 \tilde{\Delta}_{K,k_\alpha v} K_0$

From the claim follows that $f_{K,v}$ is compactly supported modulo centre.

<u>Proof of the Claim</u>: $g \in \mathrm{supp}(f_{K,v})$

$g \in \bigcup_{\delta \in \tilde{\Delta}} Z K_0 \delta K_0$.

$\Rightarrow \exists$ ~~k_α, k_β~~ $\alpha, \beta \in A$, $\exists z$: $g \in z K k_\alpha \delta k_\beta K = z k_\alpha K \delta k_\beta K$ ($K_0 \trianglerighteq K$)

$\Rightarrow f_{K,v}(g) = \omega_\pi(z) \pi(k_\alpha) \pi(e_K) \pi(\delta) \pi(k_\beta) v$

$= \underbrace{\omega_\pi(z)}_{\neq 0} \underbrace{\pi(k_\alpha)}_{\text{invertible}} f_{K,k_\beta v}(\delta)$.

$\Rightarrow g \in Z K_0 \tilde{\Delta}_{K,k_\alpha v} K_0$. □

We define $t_p(\delta) := \min_{\substack{i,j \text{ in different} \\ p\text{-blocks } i<j}} |a_i - a_j|$, $\delta \in \tilde{\Delta}$

~~We have~~

We have $S \subseteq \tilde{\Delta}$ is finite $\Leftrightarrow$ ($\Leftarrow$ steps are seen by max parabolics)

$\exists t \geq 0 \ \forall_{\substack{P<G \\ \text{maximal std.}}} \ \forall \delta \in S: \ t_p(\delta) \leq t \Leftrightarrow$ ($\Leftarrow$ ~~steps a~~)

$\exists t \geq 0 \ \forall_{\substack{p<G \\ \text{std}}} \ \forall_{\delta \in S}: t_p(\delta) \leq t \Leftrightarrow$ ($\Leftarrow$ only finitely many std parabolics)

$\forall_{\substack{p<G \\ \text{std}}} \exists_{t \geq 0} \forall \delta \in S: t_p(\delta) \leq t \Leftrightarrow$

$\forall_{\substack{p<G \\ \text{std}}} \exists_{t \geq 0} \forall_{\delta \in S}: (t_p(\delta) > t \Rightarrow \delta \notin S)$ (*)

We show (*) for $\tilde{\Delta}_{K,\sigma} = \text{supp}(f_{K,\sigma})$

We only need to consider K in a neighbourhood base for identity, so let K be a K_i. ~~$K = K_t$~~

Take $P < G$ std. $V(N) = V$ because $r^G_{P,L}\pi = 0$.

$\Rightarrow v \in V(N) \Rightarrow K_N \subseteq N$ c-open : $v \in V(K_N)$

$K = K_- K_L K_+$. $K_+ \subseteq N$

Look at

$$\begin{pmatrix} I_{n_1} & & * \\ & \ddots & \\ & & I_{n_r} \end{pmatrix} \doteq N.$$

Conjugation with $\delta \in \tilde{\Delta}$ ~~...~~ with $t_p(\delta) > 0$

increase the valuation of ~~some entry~~. all entrys

$$\begin{pmatrix} \varpi^{a_1} & \\ & \varpi^{a_2} \end{pmatrix} \begin{pmatrix} 1 & \varphi_p^n \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi^{-a_1} & \\ & \varpi_p^{-a_2} \end{pmatrix} = \begin{pmatrix} 1 & \varphi_p^{n+a_1-a_2} \\ & 1 \end{pmatrix}$$

Thus $\exists_{t \geq 0} \forall_{\delta \in \tilde{\Delta}}$ with $t_p(\delta) > t$: $\delta K_N \delta^{-1} \subseteq K_+$

For these δ we have $\pi(e_K)\pi(\delta)v$

$$= \pi(e_K * e_{\delta K_N \delta^{-1}})\pi(\delta)v = \pi(e_K)\pi(\delta)\pi(K_N)v$$

$\uparrow$ $\delta K_N \delta^{-1} \subseteq K$

$= 0.$ $\uparrow$ $v \in V(K_N)$

$\Rightarrow \delta \notin \hat{\Delta}_{K,v}$.

„$\Leftarrow$" Take $P < G$ old. and $v \in V$. We have to show $v \in V(N)$. Take K, some K_i, o.d. $v \in V^K$.

$\mathrm{supp}(f_{K,v}) = \tilde{\Delta}_{K,v}$ is finite

$\Rightarrow \exists\, t_0 \geq 0 \ \forall \delta \in \tilde{\Delta}$ with $t_p(\delta) > t_0$: $\pi(f_K)\pi(\delta)v = 0$.

Take $\delta_0 \in \tilde{\Delta} \cap Z(L)$ with $t_p(\delta) > t_0$.

Then $\pi(\delta_0^{-1} K_+ \delta_0)\, v = \pi(\delta_0^{-1} K \delta_0)\, v$.

$\uparrow$

$\delta_0^{-1} K_L \delta_0^{+1} = K_L$

$\delta_0^{-1} K_- \delta_0^{+1} \subseteq K_-$, for $t_p(\delta_0)$ big enough.

and $K = K_+ K_L K_-$

$= \pi(\delta_0^{-1})\, \pi(K)\, \pi(\delta_0)\, v = 0.$

$\Rightarrow v \in V(\delta_0^{-1} K_+ \delta_0) \subseteq V(N)$. $\square$

<u>Fact:</u> Two induced reps. $i_{P_1,L_1}^{G}\sigma_1$ and $i_{P_2,L_2}^{G}\sigma_2$ have a subquotient in common iff σ_1 is conjugate to σ_2 under an element of G.

⇝ Reduces classification taske to:

- study subquotients of std parabolically induced irr. cuspidaly representations
- find all cuspidal irr. reps on all std levis.

We look at the second task:

Let $H \leq G$ closed $\sigma \in \mathcal{R}(H)$.

$$I_g(\sigma) = \mathrm{Hom}_{H \cap {}^gH}\left(\mathrm{Res}^{H}_{H\cap {}^gH}\sigma, \mathrm{Res}^{{}^gH}_{H\cap {}^gH}{}^g\sigma\right)$$

~~$I_g(\sigma)$~~ g <u>intertwines</u> σ iff $I_g(\sigma) \neq \emptyset$.

$I_G(\sigma) = \{g \in G \mid I_g(\sigma) \neq \emptyset\}$ g-intertwining space of σ.

<u>Mackey's ind. ~~theorem~~ criterion:</u> Let H be copen $\leq G$.

$$I_G(\sigma) = H \iff c\text{-}\mathrm{Ind}_H^G \sigma \text{ is irreducible.}$$

<u>Proof:</u> Problem 2. Sheet 6. □

Thm 63: Suppose $H \leq G$ ~~the open~~ open and compact mod centre. ~~Suppose $\pi := c\text{-Ind}_H^G \sigma$~~

Suppose $\sigma \in \mathcal{R}(H)$ is irreducible and $I_G(\sigma) = H$.

then ~~π is~~ $c\text{-Ind}_H^G \sigma$ is cuspidal. and irreducible.

XVII Classification of principal series representations for $GL_2(F) = G$

Goal: The following irreducibility criteria for induced representations $\mathrm{Ind}_B^G \chi$, $\chi \in \hat{T}$, $\chi = \chi_1 \otimes \chi_2$

Thm 64: Let $\chi \in \hat{T}$ and $X := \mathrm{Ind}_B^G \chi$. Then

(1) X irreducible $\Leftrightarrow$ $\chi_1 \chi_2^{-1} \notin \{1\!\!1_F, \|\cdot\|_F^2\}$

(2) Suppose that X is reducible. Then
- (a) $\mathrm{length}(X) = 2$
- (b) one decomposition factor of X is infinite dimensional and one is one-dimensional
- (c) X has a one-dim. subspace $\Leftrightarrow$ $\chi_1 = \chi_2$
- (d) — " — quo — " — $\Leftrightarrow$ $\chi_1 \chi_2^{-1} = \|\cdot\|_F^2$.

For the proof we take the following notation $(\pi_{\chi}, X) = \mathrm{Ind}_B^G \chi$,

$Y := \{ f \in X \mid f(1) = 0 \}$

$\Rightarrow$ We get an exact sequence of B-repr.

$0 \to Y \to X \to \mathbb{C} \to 0.$

Strategy to prove the irreducibility criteria:
- Analyse the length of X
- Classify the decomposition factors
- Where does the awkward condition $\chi_1(x)\chi_2(x)^{-1} = \|x\|_F^2$ come from?
  - $\chi_1 = \chi_2$ seems to be in the scheme of principal series of $GL_2(\mathbb{F})$
  - $\tilde{\pi} = \mathrm{Ind}_B^G(\delta_B \tilde{\chi})$ by duality

$\delta_B\left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\right) = \dfrac{\|t_1\|_F}{\|t_2\|_F}$

Then $(\delta_B \tilde{\chi})_1 = (\delta_B \tilde{\chi})_2$ is equivalent to

$\|x\|_F \, \chi_1(x)^{-1} = \frac{1}{\|x\|} \chi_2(x)^{-1}$.

$(\Leftrightarrow \quad \|x\|^2 = (\chi_1 \chi_2^{-1})(x))$

To analyze the factors we need to analyze Y.

<u>Lemma 65:</u> $\quad Y_N \cong \chi^\omega \otimes \delta_B$ with $\omega = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

<u>Proof:</u> Consider $Y \longrightarrow \mathcal{C}^\infty(T)$

$f \longmapsto f_N \qquad f_N(x) := \int_N f(x \omega n)\, dn$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\chi(x)} f_N(1)$

The integral is well defined, because $\exists_{N_0 \subseteq N}$ copen: $\operatorname{supp}(f) \subseteq B \omega N_0$ (GL(2), Lemma 9.2)

Action of T: $t \in T$. $(t.f)_N(x) = \chi(x)\, (t.f)_N(1)$

$$(t.f)_N(1) = \int_N f(\omega n t)\, dn = \int_N f(\omega t t^{-1} n t)\, dn$$

$$\underset{t^{-1}nt \leftrightarrow n}{=} \delta_B(t) \int_N f(\omega t n)\, dn = \delta_B(t) \int_N f({}^\omega t \cdot \omega n)\, dn$$

$$= \delta_B(t)\, \chi({}^\omega t)\, f_N(1) = (\delta_B \cdot {}^\omega\chi)(t) \cdot f_N(1).$$

$\Rightarrow$ We have a morphism $Y \xrightarrow{\Phi} {}^\omega\chi \otimes \delta_B$

$f \longmapsto f_N(1)$

- $\Phi(Y(N)) = \{0\}$, because dn is right inv. too.
- If $f_N(1) = 0$, then $\int_{N_0} f(\omega n)\, dn = 0$

for some $N_0 \leq N$ c-open s.d. $\operatorname{supp}(f) \subseteq BwN_0$

Thus $\left(\int_{N_0} (\pi_\chi(n) f) dn\right)(b)$

$= \int_{N_0} (\pi_\chi(n) f)(b)\, dn = \int_{N_0} f(\underbrace{bn}_{\in B})\, dn = 0$

$\forall b \in B$

and $\int_{N_0} (\pi_\chi(n) f)(b w n_0)\, dn$

$= \int_{N_0} f(b w n_0 n)\, dn \underset{b = t \cdot n_1}{\overset{\uparrow}{=}} \chi(t) \int_{N_0} f(w n_0 n)\, dn$

$\underset{\text{fellinv.}}{\overset{\uparrow}{=}} \chi(t) \int_{N_0} f(wn)\, dn = 0. \quad \forall b w n_0 \in BwN_0$

and $\int_{N_0} (\pi_\chi(n) f)(b w \tilde{n}_0)\, dn$

$= \int_{N_0} f(b w \underbrace{\tilde{n}_0 n}_{\notin N_0})\, dn \underset{b = t n_1}{\overset{\uparrow}{=}} \chi(t) \int_{N_0} f(w \underbrace{\tilde{n}_0 n}_{\notin N_0})\, dn$

$= 0$ for $b w \tilde{n}_0 \in BwN \setminus BwN_0$.

Thus $\pi_\chi(e_{N_0}) f = 0. \quad (G = B \,\dot{\cup}\, BwN)$

$\Rightarrow f \in Y(N_0) \subseteq Y(N)$.

Also $Y_N \cong {}^{w}\chi \otimes \delta_B$. $\square$

Thus $\text{length}(\text{Res}_B^G(X)) \leq 3$: Two one dimensional factors and one infinite dimensional factor $Y(N)$.

<u>Fact:</u> $Y(N)$ is an irred B - ~~rep~~ representation.

We have ~~a dichotomy~~ two cases for a reducible X.

- A) X has non-zero finite dim ~~direct~~ quotient
- B) X has a non-trivial zero finite dim. subrepr.

<u>Case B)</u> If X has a non-zero finite dim subrepr. X' then $\text{Res}_N^G X$ has a one dim – " – , because $\text{Res}_N^G X'$ has finite length by dimension reasons.

and every irred. smooth repr. of $N \cong (F, +)$ is one dimensional by the lemma of Schur. (N is abelian!)

The following lemma shows: X has a one-dim subrepr. and $x_1 = x_2$ in Case B).

<u>Lemma 66:</u> $x_1 = x_2 \iff \text{Res}_N^G X$ has a one-dim subrepr.

And in that case there is a one-dim subrepr. of X and $\text{Res}_N^G X$ has a <u>unique</u> one-dim subrepr. which is a G-representation.

Proof of lemma 66: "⇒" If $\chi_1 = \chi_2 = \phi$ then

$\underbrace{(\phi \circ \det)^{-1} \pi_X}$ $= \mathrm{Ind}_B^G \mathbb{1}_T$ and we can assume $\phi \equiv 1$.

We also write $\phi^{-1}\pi_X$

$\{f : G \to \mathbb{C} \mid \exists z \in \mathbb{C} : \forall_{g \in G} : f(g) = z\}$ is
a one-dim subrepresentation of X.

"⇐": ~~Suppose~~ Let $f \in X \setminus \{0\}$ and ~~...~~ $\psi \in \widehat{(F,+)} \cong \hat{N}$

s.t. N acts on $\mathbb{C} \cdot f$ as $(\pi_X(n) f)(g) = (\psi(n) f)(g)$

i.e. $f(gn) = \psi(n) f(g) \quad \forall_{g \in G} \forall_{n \in N}$.

Claim: $\mathrm{supp}(f) = G$: $\mathrm{Supp}(f)$ is a union of
elements of $B \backslash G / N = \{B, BwN\}$

$\mathrm{supp}(f) = B$ is impossible, because f is smooth.
If $\mathrm{supp}(f) = BwN$ then $\mathrm{supp}(f) \subseteq BwN$
and $\exists N_0 \leq N$ open: $\mathrm{supp}(f) \subseteq BwN_0$ ↯

$f \neq 0 \Rightarrow \mathrm{supp}(f) = G$. □ (Claim)

Claim $\Rightarrow f(1) \neq 0$. $\Rightarrow \mathbb{C} \cdot f \hookrightarrow X/Y \cong \mathbb{C}$

N acts trivially on X/Y. Thus it acts trivially on f.

$\ulcorner f(n) \underset{f \in X}{=} f(1)$ and $f(n) = \psi(n) f(1) \overset{f(1) \neq 0}{\Rightarrow} \psi(n) = 1 \lrcorner$

Thus $\psi = \mathbb{1}_N$.

We have $\omega\begin{pmatrix}1 & x\\ & 1\end{pmatrix} = \begin{pmatrix}1 & x^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}-x^{-1} & \\ 0 & x\end{pmatrix}\begin{pmatrix}1 & 0\\ x^{-1} & 1\end{pmatrix}$

for $x \in F^\times$

and for $\|x\| \gg 0$ we know that $\begin{pmatrix}1 & \\ x^{-1} & 1\end{pmatrix}$ fixes f (because f is smooth).

Thus for $\|x\| \gg 0$: $f(\omega) = f\left(\omega\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right)$

$$= f\left(\begin{pmatrix}1 & x^{-1}\\ & 1\end{pmatrix}\begin{pmatrix}-x^{-1} & \\ & x\end{pmatrix}\begin{pmatrix}1 & 0\\ x^{-1} & 1\end{pmatrix}\right) = f\left(\begin{pmatrix}1 & x^{-1}\\ & 1\end{pmatrix}\begin{pmatrix}-x^{-1} & \\ & x\end{pmatrix}\right)$$

$$\underset{f\in X}{=} f\left(\begin{pmatrix}-x^{-1} & \\ & x\end{pmatrix}\right) \underset{f\in X}{=} \chi_1(-1)\chi_1(x^{-1})\chi_2(x)\, f(1). \quad (*)$$

$\underset{f(1)\neq 0}{\Rightarrow}$ $\forall_{x, x'}$ $\|\cdot\| \gg 0$: $\chi_1(x)^{-1}\chi_2(x) = \chi_1(x')^{-1}\chi_2(x')$

$\Rightarrow$ $\forall_{x \in F^\times}$: $\chi_1(x)^{-1}\chi_2(x) = 1$ $\quad(\Leftrightarrow \chi_1(x) = \chi_2(x))$

$\Rightarrow$ $\chi_1 = \chi_2 =: \varphi$. $\quad\square\ (\Leftarrow)$

Exercise: From $(*)$ follows $f(g) = \varphi(\det(g))\, f(1)\ \forall_{g\in G}$.

Thus the one-dim N-subrepresentation is uniquely determined. $\square$

Proof of the Irreducibility criteria:

Suppose length$(X) \geq 2$. We know length$(X) \leq 3$.

Case 1: X has a non-zero finite dimensional subrepresentation.

$\Rightarrow$ $\mathrm{Res}^G_N X$ has a one-dim subrep L and by Lemma 66: $\chi_1 = \chi_2 =: \varphi$. and this L is in fact G-invariant and $L \cap Y = 0$ by $\mathrm{supp}(f) = G$ $\forall f \in L \setminus \{0\}$.

$\Rightarrow$ $Y \hookrightarrow X/L$ ~~[illegible]~~ in $\mathcal{R}(B)$

In $\mathcal{R}(B)$ has Y the factors Y_{N}, $Y(N)$ ~~and~~ /

and $\mathrm{Res}^G_B(X/L)$ has also 2 factors. $\Rightarrow$ $Y \hookrightarrow\!\!\!\!\rightarrow X/L$ (surjective!) in $\mathcal{R}(B)$

Case 1.1: $X' := X/L$ has a non-~~total~~ zero finite dimensional quotient X''.

Then this quotient has to be one-dimensional because it is a quotient of Y and $\mathrm{Res}^G_B X'' \cong {}^{\omega}\chi \cdot \delta_\beta$

Thus X'' is one-dim, i.e. $X'' \cong \psi \circ \det$

for some $\psi \in \widehat{F^\times}$. Thus on T we have

$$ {}^{\omega}\chi \delta_\beta = \psi \circ \det = \varphi \otimes \varphi. \qquad ↯ $$

$\Big[\; \varphi(x)\,\|x\| = \varphi(x) = \varphi(x) \cdot \frac{1}{\|x\|} \Rightarrow \|x\| \overset{!}{=} 1 \; \Big]$

Case 12: X' has no non-zero finite dimensional quotient.

Suppose length$(X')=2$.
Then X' has a one dim. subrepr. because the factors of X' restricted to B must be the factors of Y.

because length$(Y)=2$.
Thus in particular $\omega_\chi \delta_B \hookrightarrow Y$

N acts trivially on $\omega_\chi \delta_B$ and

$$Y \underset{\mathcal{R}(N)}{\simeq} C_c^\infty(N) \quad \text{via}$$

Exercise:
$$f \longmapsto \tilde{f}_N \qquad \tilde{f}_N(n) = f(\omega n).$$

(It has nothing to do with f_N from the beginning)

But the trivial representation is not a sub-representation of $C_c^\infty(N)$.
Thus $\omega_\chi \delta_B \hookrightarrow Y$ cannot happen.

We have a contradiction to length$(X')=2$.

Thus length$(X')=1$ and length$(X)=2$.

Summarize: We have shown: in Case 1: $\chi_1 = \chi_2$

and X has a one-dim subrepr. L and length$(X)=2$.

Case 2: X has a non-zero finite dim quotient

$\Longrightarrow \tilde{X}$ (contragredient of X) has a non-zero finite dim subrep.

$\tilde{X} = \mathrm{Ind}_B^G \delta_B \tilde{\chi}$. Apply Case 1 to $\tilde{X}$. □

$\not\Rightarrow$

Example. $0 \longrightarrow 1\!\!1_G \longrightarrow \mathrm{Ind}_B^G 1\!\!1_T \longrightarrow St_G \longrightarrow 0$

St_G is called the Steinberg representation of G.

We twist with $\varphi \circ \det$. $\varphi \in \widehat{F^\times}$

$$0 \longrightarrow \varphi \circ \det \longrightarrow \mathrm{Ind}_B^G(\varphi \otimes \varphi) \longrightarrow \varphi \cdot St_G \longrightarrow 0$$

$\varphi\, St_G$, $\varphi \in \widehat{F^\times}$ are called the special representations of G.

Thm 67: The irred. principal series representations of G are exactly:
- $\mathrm{Ind}_B^G \chi \qquad \chi \in \hat{T} \quad \chi_1 \chi_2^{-1} \notin \{1\!\!1_T, \|\cdot\|_F^{\pm}\}$
- $\varphi \circ \det \qquad \varphi \in \widehat{F^\times}$
- $\varphi \cdot St_G \qquad \varphi \in \widehat{F^\times}$

Proof: This follows from the irreducibility criteria and $\widetilde{St_G} \cong St_G$.

Exercise □

level 0 irreducible representations of $G = GL_2(F)$

$$K_1 = 1 + M_2(\mathfrak{p}_F) \subsetneq K_0 = GL_2(\mathcal{O}_F), \quad I_1 = \begin{pmatrix} 1+\mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F & 1+\mathfrak{p}_F \end{pmatrix}$$

<u>Theorem 68:</u> Let $(V, \Pi) \in \mathcal{R}(G)$ be an irred repr. s.t.

$V^{K_1} \neq 0$ (this is called <u>level 0</u>)

Then either: (a) $\exists\, \bar{\lambda} \in \mathcal{R}(GL_2(\mathbb{F}))$ cuspidal:

$$\lambda := \mathrm{Infl}^{K_0}_{K_0/K_1} \bar{\lambda} \subseteq \Pi|_{K_0} \quad (\text{Inflation})$$

(b) Π contains I_1

In case (a) λ can be extended to an irred repr. Λ of ZK_0 (using ω_Π) and $\Pi \cong c\text{-}\mathrm{Ind}^{G}_{ZK_0} \Lambda$ is cuspidal.

<u>Fact:</u> In fact in case (b) Π is not cuspidal. (more involved)

<u>Proof of Thm 68:</u> $V^{K_1} \in \mathcal{R}(K_0/K_1)$ is finite dimensional (Π is irred !) and semisimple (Maschke)

Let W be a composition factor of V^{K_1}

Then 2 cases:

$$1\!\!1_{(1\ \mathbb{F} \atop \ \ 1)} \hookrightarrow W \quad \text{or} \quad 1\!\!1_{(1\ \mathbb{F} \atop \ \ 1)} \not\hookrightarrow W$$

$\Updownarrow$ $\qquad\qquad\qquad\qquad$ $\Updownarrow$

W princ. series $\qquad\qquad$ W cuspidal.

We have to show: (i) For $W, W' \subseteq V^{K_1}$ irred:
W cuspidal $\Leftrightarrow$ W' cuspidal

(ii) Let λ be the inflation of W to K_0 and Λ be an extension of λ by ω_π to ZK_0.
We have to show
$$I_G(\Lambda) = ZK_0.$$
Thm of last lecture $\Rightarrow$ $c\text{-Ind}_{ZK_0}^{G} \Lambda$ irr. cusp.

and by
$$\mathrm{Hom}_G(c\text{-Ind}_{ZK_0}^{G} \Lambda, \pi) \cong \mathrm{Hom}_{ZK_0}(\Lambda, \mathrm{Res}_{ZK_0}^{G} \pi) \neq 0$$
we get $\pi \cong c\text{-Ind}_{ZK_0}^{G} \Lambda$.

We show (i) and (ii) in a second $\square$

Def: Let $\rho_i \in \mathcal{R}(H_i)$, $H_i \leq G$ closed, $i = 1, 2$.
$g \in G$ <u>intertwines</u> ρ_1 with ρ_2 if $\mathrm{Hom}_{H_1^g \cap H_2}(\rho_1^g, \rho_2) \neq 0$.

Remark 69: Let $\pi \in \mathcal{R}(G)$ be irred. $H_i \leq G$ $i = 1, 2$ open, $\rho_i \in \mathcal{R}(H_i)$ ≠ trivial. Then $\exists\, g \in G$: g intertwines ρ_1 with ρ_2.

Proof:
$$\mathrm{Hom}_G(c\text{-Ind}_{H_1}^{G} \rho_1, \pi) \cong \mathrm{Hom}_{H_1}(\rho_1, \pi) \neq 0$$
$$\Rightarrow c\text{-Ind}_{H_1}^{G} \rho_1 \twoheadrightarrow \pi$$

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_{H_2}^G \rho_2) \cong \mathrm{Hom}_{H_2}(\pi, \rho_2) \neq 0.$$

$$\Rightarrow 0 \neq \mathrm{Hom}_G(\text{c-}\mathrm{Ind}_{H_1}^G \rho_1, \mathrm{Ind}_{H_2}^G \rho_2)$$

$$\cong \mathrm{Hom}_{H_2}(\mathrm{Res}_{H_2}^G \text{c-}\mathrm{Ind}_{H_1}^G \rho_1, \rho_2)$$

$$\cong \mathrm{Hom}_{H_2}\Big(\bigoplus_{H_2\backslash G/H_1} \mathrm{Ind}_{H_2\cap {}^gH_1}^{H_2} \mathrm{Res}^{{}^gH_1}_{H_2\cap {}^gH_1} {}^g\rho_1, \rho_2\Big)$$

$$\cong \prod_{H_2\backslash G/H_1} \mathrm{Hom}_{H_2}(\mathrm{Ind}_{H_2\cap {}^gH_1}^{H_2} \mathrm{Res}^{{}^gH_1}_{H_2\cap {}^gH_1} {}^g\rho_1, \rho_2)$$

$$\cong \prod_{H_2\backslash G/H_1} \underbrace{\mathrm{Hom}_{H_2\cap {}^gH_1}({}^g\rho_1, \rho_2)}_{= \mathrm{Hom}_{H_2\cap H_1^{g^{-1}}}(\rho_1^{g^{-1}}, \rho_2)}$$

Thus $\exists g \in G$: g intertwines ρ_1 with ρ_2. $\square$

<u>Lemma 70:</u> Let $\overline{\lambda}_i$ be two irred. repr. of $GL_2(\mathbb{F})$ with inflations λ_i, s.t. $\overline{\lambda}_1$ is cuspidal.

Then: (1) ~~g~~ λ_1 and λ_2 intertwine $\Leftrightarrow \overline{\lambda}_1 \cong \overline{\lambda}_2$.

(2) $g \in G$ intertwines $\lambda_1 \Leftrightarrow g \in ZK_0$.

Proof of the lemma:

Prelude: If g intertwines λ_2 with λ_1 then every element of $K_0 g Z K_0$ intertwines λ_2 with λ_1. Thus one can take $g = \begin{pmatrix} \varpi^a & \\ & 1 \end{pmatrix}$ with $a \geq 0$ (Cartan decomposition)

Claim: $g \in Z K_0$.

Proof: Suppose not, say $a \geq 1$. Then

$$\begin{pmatrix} 1 & \mathfrak{p}_F \\ & 1 \end{pmatrix}^g = \begin{pmatrix} 1 & \mathfrak{p}_F^{1-a} \\ & 1 \end{pmatrix} \supseteq \begin{pmatrix} 1 & \mathcal{O}_F \\ & 1 \end{pmatrix}$$

From $\mathrm{Hom}_{K_0 \cap K_0^g}(\lambda_2^g, \lambda_1) \neq 0$ follows $1\!\!1_{\begin{pmatrix} 1 & \mathcal{O}_F \\ & 1 \end{pmatrix}} \hookrightarrow \lambda_1$ ↯ because λ_1 is cuspidal. □

(1) "⇒" If g intertwines λ_1 with λ_2, then $g \in K_0 Z$ and therefore $\lambda_1 \cong \lambda_1^g \cong \lambda_2$. Thus $\bar{\lambda}_1 \cong \bar{\lambda}_2$.

"⇐" ✓

(2) "only if" by the prelude.
"if" is clear. □

We now prove (i): $W, W' \subseteq V^{K_1}$ irred.
Remark 69 ⇒ W and W' intertwine $\overset{L70(1)}{\Rightarrow}$ $W \cong W'$.

(ii) If g intertwines λ then g intertwines λ_1 and thus $g \in K_0 Z$ by Lemma 70(2).

□

Approach to the other levels

Def 71: A unitary ring R is called left hereditary if all left ideals are projective
($\Leftrightarrow$ Every submodule of any free R-module is projective.)

Ex: $GL_n(F) \subseteq M_n(F) \supseteq \mathfrak{A}$ subring

$\mathfrak{A}$ is an hereditary order $\Leftrightarrow$

$\exists\, g \in GL_n(F)$: $g \begin{pmatrix} & \mathcal{O} \\ \mathfrak{p} & \end{pmatrix} g^{-1} = \mathfrak{A}$.

$\mathfrak{P}_{\mathfrak{A}} = \mathrm{Rad}(\mathfrak{A}) = g \begin{pmatrix} & \mathcal{O} \\ \mathfrak{p} & \end{pmatrix} g^{-1}$

$G := GL_2(F)$. $\quad \mathfrak{M} := M_2(\mathcal{O}_F)$

$\mathfrak{I} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}$.

A hereditary order $\mathfrak{A}$ of $GL_n(F)$ delivers a lattice chain, i.e. a map

$\mathbb{Z} \xrightarrow{L} \mathcal{O}_F$ lattices of V ($\cong \bigoplus_{i=1}^{n} \mathcal{O}_F$)

s.t. $L_i \supseteq L_j \quad \forall j \geq i$

and $\exists\, e_{\mathfrak{A}} \in \mathbb{N}\ \forall i \in \mathbb{Z}: L_{i+e_{\mathfrak{A}}} = \varpi_F L_i$, s.t. $\mathfrak{A} = \{ g \in GL_n(F) \mid \forall i: gL_i \subseteq L_i \}$

Ex: $\mathfrak{M}$: $L_i = \begin{pmatrix} \mathfrak{p}^i \\ \mathfrak{p}^i \end{pmatrix}$

$\mathfrak{I}$: $L_i = \begin{pmatrix} \mathfrak{p}^{\lfloor \frac{i}{2} \rfloor} \\ \mathfrak{p}^{\lfloor \frac{i+1}{2} \rfloor} \end{pmatrix} \qquad \lfloor x \rfloor := \max\{ z \in \mathbb{Z} \mid z \leq x \}$

Let $\mathfrak{A}$ be a hereditary order ~~now~~ with lattice chain L: $U_{\mathfrak{A}} = \mathfrak{A}^\times$

$$U_{\mathfrak{A}}^m := 1 + \{a \in M_2(F) \mid \forall_{z \in \mathbb{Z}} \, a L_z \subseteq L_{z+m}\} = 1 + \mathfrak{P}_{\mathfrak{A}}^m$$

Ex: $U_{\mathfrak{I}}^1 = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}$ $\quad U_{\mathfrak{I}}^2 = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{pmatrix}$.

$U_{\mathfrak{M}}^1 = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}$ $\quad U_{\mathfrak{M}}^m = \begin{pmatrix} 1+\mathfrak{p}^m & \mathfrak{p}^m \\ \mathfrak{p}^m & 1+\mathfrak{p}^m \end{pmatrix}$

Def 7.2: Let $\pi \in \mathcal{R}(G)$ be irreducible.

Let $S(\pi)$ be the set of pairs $(\mathfrak{A}, n)$, s.t. $1\!\!1_{U_{\mathfrak{A}}^{n+1}} \hookrightarrow \pi$.

The integer $\ell(\pi) = \min\left\{ n/e_{\mathfrak{A}} : (\mathfrak{A}, n) \in S(\pi) \right\}$

is called the normalized level of π.

Remark: $\ell(\pi) = 0 \iff$ For $U_{\mathfrak{M}}^1 = 1 + M_2(\mathfrak{p}_F)$ the trivial character is contained in π.

Characters on $U_{\mathfrak{A}}^{m+1}$ ~~[illegible]~~:

Take $\psi \in \hat{F}$ with $\psi|_{\mathfrak{p}_F} = 1\!\!1_{\mathfrak{p}_F}$

and $\psi|_{\mathfrak{o}_F} \not\equiv 1\!\!1_{\mathfrak{o}_F}$.

$\psi_A := \psi \circ \mathrm{tr}_{A/F}$ $\qquad\qquad A := M_2(F)$.

Prop 73: (parametrization of characters of $U_{\mathfrak{A}}^{m+1}$):

$\mathfrak{P}_{\mathfrak{A}} := \mathrm{Rad}(\mathfrak{A}) = \{a \in M_2(F) \mid \forall_{i\in\mathbb{Z}} aL_i \subseteq L_{i+1}\}$

⌜ Ex: $\mathrm{Rad}(\mathfrak{J}) := \begin{pmatrix} \mathfrak{o}+\mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o}+\mathfrak{p} \end{pmatrix}$

$\mathrm{Rad}(\mathfrak{M}) := M_2(\mathfrak{p})$ ⌟

For $2m+1 \geq n \geq m \geq 0$ there is a bijection

$$\mathfrak{P}_{\mathfrak{A}}^{-n} / \mathfrak{P}_{\mathfrak{A}}^{-m} \xrightarrow{\sim} \left(U_{\mathfrak{A}}^{1+m} / U_{\mathfrak{A}}^{1+n} \right)^{\wedge}$$

$$a + \mathfrak{P}_{\mathfrak{A}}^{-m} \longmapsto \psi_a \,;\ \psi_a(1+x) = \psi_A(ax).$$

We see ψ_a as a character on $U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}$.

We mainly need $m = n-1$.

Def 74: A stratum is a triple $(\mathfrak{A}, n, \alpha)$ with $n \geq 1$ and $\alpha \in \mathfrak{P}_{\mathfrak{A}}^{-n}$.

We say that $(\mathfrak{A}, n, \alpha)$ is contained in π if $\psi_\alpha|_{U_{\mathfrak{A}}^{n}} \hookrightarrow \pi$.

You should imagine a stratum as a coset $(\mathfrak{A}, n, \alpha) \mathrel{\hat{=}} \alpha + \mathfrak{P}_{\mathfrak{A}}^{1-n}$

Prop 75:
Given two strata $(\sigma_i, n_i, \alpha_i)$, $i=1,2$, $g \in G$.

Then: g intertwines $\psi_{\alpha_1}|U_{\sigma_1}^{n_1}$ with $\psi_{\alpha_2}|U_{\sigma_2}^{n_2}$

$$\Longleftrightarrow \quad g^{-1}(\alpha_1 + \mathfrak{p}_{\sigma_1}^{1-n_1})g \cap (\alpha_2 + \mathfrak{p}_{\sigma_2}^{1-n_2}) \neq \emptyset$$

Proof: For a lattice L of A we ~~have~~ define

$$L^{*,\psi_F} := \{a \in A \mid \psi_A(ax) = 1 \quad \forall x \in L\}$$

Exercise: $L_1^* \cap L_2^* = (L_1 + L_2)^*$ and $(L_1+L_2)^* = L_1^* \cap L_2^*$

and $(\mathfrak{p}_\sigma^{m})^* = \mathfrak{p}_\sigma^{1-m}$.

g intertwines $\psi_{\alpha_1}|U_{\sigma_1}^{n_1}$ with $\psi_{\alpha_2}|U_{\sigma_2}^{n_2}$

$$\Leftrightarrow \forall x \in g^{-1}\mathfrak{p}_{\sigma_1}^{n_1} g \cap \mathfrak{p}_{\sigma_2}^{n_2}: \psi_{\alpha_1}(1 + gxg^{-1}) = \psi_{\alpha_2}(x)$$

$$\psi_{\alpha_1}(1+gxg^{-1}) = \psi_A(\alpha_1 gxg^{-1}) = \psi_A(g^{-1}\alpha_1 gx), \qquad \psi_{\alpha_2}(x) = \psi_A(\alpha_2 x)$$

$$\Leftrightarrow \forall x \in g^{-1}\mathfrak{p}_{\sigma_1}^{n_1} g \cap \mathfrak{p}_{\sigma_2}^{n_2}: \psi_A((g^{-1}\alpha_1 g - \alpha_2)x) = 1$$

$$\Leftrightarrow \quad g^{-1}\alpha_1 g - \alpha_2 \in (g^{-1}\mathfrak{p}_{\sigma_1}^{n_1} g \cap \mathfrak{p}_{\sigma_2}^{n_2})^* = g^{-1}\mathfrak{p}_{\sigma_1}^{1-n_1} g + \mathfrak{p}_{\sigma_2}^{1-n_2}.$$

$$\Leftrightarrow \quad g^{-1}(\alpha_1 + \mathfrak{p}_{\sigma_1}^{1-n_1})g \cap (\alpha_2 + \mathfrak{p}_{\sigma_2}^{1-n_2}) \neq \emptyset. \quad \square$$

Def 76: 1) Two strata $(\mathfrak{A}_1, n_1, \alpha_1)$ and $(\mathfrak{A}_2, n_2, \alpha_2)$ are called equivalent if the cosets equal: $\alpha_1 + \mathfrak{P}_{\mathfrak{A}_1}^{1-n_1} = \alpha_2 + \mathfrak{P}_{\mathfrak{A}_2}^{1-n_2}$.

2) A stratum $(\mathfrak{A}, n, \alpha)$ is called fundamental if $\alpha + \mathfrak{P}_{\mathfrak{A}}^{-n}$ does not contain a nilpotent element. Otherwise non-fundamental.

Fact 77: The non-zero non-fundamental strata are conjugate to $(\mathfrak{M}, n, \varpi^{-n}\alpha)$ $\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

or $(\mathfrak{I}, 2n-1, \varpi^{-n}\alpha)$ $\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$n \geq 1$.

Thm 78: $\pi \in \mathcal{R}(G)$ irred $(\mathfrak{A}, n, \alpha) \subseteq \pi$.

Then ~~are eq~~: 1) $(\mathfrak{A}, n, \alpha)$ fund $\Leftrightarrow$ $\ell(\pi) = \frac{n}{e_{\mathfrak{A}}}$.

2) $\ell(\pi) > 0 \Leftrightarrow \pi$ contains a fundamental stratum.

Sketch: The reason for (1) is: [above "for": "⇐" in]

If $(\mathfrak{a}, n, \alpha)$ is non-fundamental one can reduce to the ~~sub~~ situation.

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathfrak{P}_{\mathfrak{M}}$ ~~and~~ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathfrak{P}_{\mathfrak{J}}^2$.

We have containments $\cap|$ $\mathfrak{P}_{\mathfrak{J}}$ and $\cap|$ $\mathfrak{P}_{\mathfrak{M}}$.

So thus one can find ~~$(\mathfrak{a}_1, n_1, \alpha_1)$~~: $\mathfrak{a}_1$:

$\alpha + \mathfrak{P}_{\mathfrak{a}}^{1-n} \subseteq \mathfrak{P}_{\mathfrak{a}_1}^{-n_1}$. s.t. $n_1/e_{\mathfrak{a}_1} < n/e_{\mathfrak{a}}$.

$\psi_\alpha \subseteq \pi \Rightarrow 1\!\!1_{U_{\mathfrak{a}_1}^{1+n_1}} \subseteq \psi_\alpha \subseteq \pi$.

$\Rightarrow \ell(\pi) \leq \frac{n_1}{e_{\mathfrak{a}_1}} < n/e_{\mathfrak{a}}$.

For "⇒" (1) the reason is the following lemma (12.9.2)

Lemma: Say $(\mathfrak{b}, m, \beta)$ is a second stratum. Suppose $(\mathfrak{a}, n, \alpha)$ is fund. and that both strata intertwine, then $n/e_{\mathfrak{a}} \leq m/e_{\mathfrak{b}}$ and "=" ⟺ $(\mathfrak{b}, m, \beta)$ is fundamental.

Once this lemma is proven we have "$\Rightarrow$"(1).

⌜ For $\ell(\Pi) > 0$: Suppose (σ, n, α) is fundamental ~~then~~ and $\ell(\pi) < \frac{n}{e_\sigma}$. Take $(b, m, \beta) \subseteq \Pi$ s.t. $\ell(\pi) = \frac{m}{e_b}$. $\underset{\text{lemma}}{\overset{\pi}{\Rightarrow}}$ $\lightning$ to $\frac{m}{e_b} \geq \frac{n}{e_\sigma}$.

because the strata must intertwine. $\lightning$

$\ell(\Pi) = 0$: Π does not contain a fundamental stratum: $U^1_{\mathfrak{U}_M} \subseteq \pi \Rightarrow$ $\pi \supseteq$ character of $U^1_{\mathfrak{U}_{\mathcal{J}}}$ which is trivial on $U^1_{\mathfrak{U}_M}$, say $(\mathcal{J}, 1, \beta)$.

$\beta = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$ $b \in \sigma$.

$(\mathcal{J}, 1, \beta)$ is not fundamental $\underset{\text{lem}}{\Rightarrow}$ Π does not contain a fundamental stratum. ⌟

$\square$

Classification of fundamental strata:

We do not need to consider all fundamental strata:

We can skip $(\mathfrak{J}, 2n, \alpha)$, because

$$U_{\mathfrak{M}}^{n+1} \subseteq U_{\mathfrak{J}}^{2n+1} \subseteq U_{\mathfrak{J}}^{2n} \subseteq U_{\mathfrak{M}}^{n}.$$

Let $\bar{f}_\alpha$ be the char. polynomial of $(\mathfrak{M}, n, \alpha)$, i.e.

$\bar{f}_\alpha$ is the " " of $\varpi^{-n}\alpha \bmod \mathfrak{p}_F$.

Intertwining fund. strata have the same char. polynomial.

Def.: · $(\mathfrak{A}, n, \alpha) = (\mathfrak{M}, n, \alpha)$ fundamental

We call it unramified fundamental if $\bar{f}_\alpha$ irred

split if $\bar{f}_\alpha$ has two diff roots

ess. scalar if $\bar{f}_\alpha$ has a double root.

· $(\mathfrak{J}, 2n-1, \alpha)$ fund. is called ramified simple.

Given an irred repr. π one can change the level in twisting with $\chi \in \widehat{F^\times}$.
One can always find a χ s.t. $\ell(\chi\pi)$ is smallest among al twists.

<u>Fact 79</u>: If $\ell(\pi) \leq \ell(\chi\pi)$ $\forall \chi$ then π does not contain an ess. scalar stratum.

Thus in this case we only have

$\pi \supseteq$
- split
- ram. simple
- unram. simple.

The two last cases α generates a field E ~~and gcd$(\nu_E(\alpha))$~~ which is ramified (unramified) in the ~~cases~~ ram. simple (unram. simple) case.
and $\nu_E(\alpha)$ odd ($\varpi_E^{-n}\alpha + \mathfrak{p}_E$ generates E).

~~We call~~

Strata and principal series

<u>Thm 80:</u> $\pi \in \mathcal{R}(G)$ irred s.t. $\pi \supseteq (\mathfrak{A}, n, \alpha)$ split fundamental. Then π is principal series.

~~Theorem~~

<u>Proof:</u> $\mathfrak{A} = \mathfrak{M}$ by conjugation.

$(\mathfrak{A}, n, \alpha)$ is split fundamental $\Longrightarrow$ $\overline{\varpi^{+n}\alpha} \in M_2(k_F)$ is conjugate to $\begin{pmatrix} \bar{a} & 0 \\ 0 & \bar{e} \end{pmatrix}$ with $\bar{a} \neq \bar{e}$. OE $\alpha = \varpi^{-n}\begin{pmatrix} a & 0 \\ 0 & e \end{pmatrix}$

We show that $\xi := \psi_a\big|_{U^n_{\mathfrak{A}}}$ is contained in V_N

$(\Leftarrow)$ V^{ξ} (isotypic component) $\trianglelefteq V(N)$

π irred. $\Rightarrow$ $\dim_{\mathbb{C}} V^{\xi} < \infty$. Assume $V^{\xi} \subseteq V(N)$.

$\overset{\Rightarrow}{\underset{\text{finite dim.}}{\Rightarrow}}$ $\exists\, i_1 \in \mathbb{N}$ $\quad N_i = \begin{pmatrix} 1 & p^i \\ & 1 \end{pmatrix}$ satisfies

$$\pi(N_{i_1})v = 0 \ \forall v \in V^{\xi}.$$

We take i_1 maximal with this property, which is impossible, because $\exists\, i_0 \gg 0$ s.t. $\pi(e_{N_{i_0}})v = v \ \forall_{v \in V^{\xi}}$.

The element $t = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}$ intertwines ξ, because it intertwines $\varpi^{-n}\begin{pmatrix} a & \\ & e \end{pmatrix} + \mathfrak{P}^{1-n}_{\mathfrak{A}}$, by $t^{-1}\alpha t = \alpha$.

We put $Y = U^n_{\mathfrak{M}} \cap t^{-1}U^n_{\mathfrak{M}} t = 1 + \begin{pmatrix} p^n & p^n \\ p^{n+1} & p^n \end{pmatrix}$

We want to show that $\exists\, v \in V^{\xi} : \pi(e_{N_{j_1}}) v \neq 0$.

<u>Strategy</u>: <u>Step 1</u>: For $v_2 := \pi(t^{-1}) v_1$ we have

$$\pi(e_{N_{j_1}}) v_2 = \frac{1}{q} \pi(t^{-1}) \pi(e_{t N_{j_1} t^{-1}}) v_1$$
$$= \frac{1}{q} \pi(t^{-1}) \pi(e_{N_{j_1+1}}) v_1 \neq 0$$

Modulus character

and $v_2 \in V^{\xi|_Y}$.

<u>Step 2</u>: Every irred. repr. of U_m^n containing $\xi|_Y$ is a character. Because such an irred representation has to trivial on U_m^{n+1}, because $U_m^{n+1} \trianglelefteq U_m^n$ and $\xi|_{U_m^{n+1}} = 1\!\!1_{U_m^{n+1}}$, and thus such an irreducible repr. is a character because $U_m^n / U_m^{n+1} \cong (M_2(\mathbb{F}), +) \cong \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$.

<u>Step 3</u>: $v_2 \in V^{\xi|_Y} = \bigoplus V^{\varphi}$ $\qquad v_2 = \sum_{\varphi} v_{\varphi}$

φ char. of U_m^n s.t. $\varphi|_Y = \xi|_Y$

$$\pi(e_{N_{j_1}}) v_2 \neq 0 \Rightarrow \exists\, \varphi : \pi(e_{N_{j_1}}) v_{\varphi} \neq 0.$$

<u>Step 4</u>: A character φ which contains $\xi|_Y$ is conj. to ξ by an element of $N_0 = \begin{pmatrix} 1 & \mathfrak{O}_F \\ & 1 \end{pmatrix}$.

Pf: $\varphi = \psi_\delta$ with $\delta = \begin{pmatrix} a & \delta \\ 0 & b \end{pmatrix} \varpi_F^{-n}$, $\delta \in \mathfrak{o}_F$.

Then $\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} (\delta + \mathfrak{p}^{1-n}_{\mathfrak{M}}) \begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix} + \mathfrak{p}^{1-n}_{\mathfrak{M}}$

$= \begin{pmatrix} a & \\ & b \end{pmatrix} + \mathfrak{p}^{1-n}_{\mathfrak{M}}$

for ~~$u = (\quad)$~~ $u = -\delta (b-a)^{-1}$.

⌈ $\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} a & \delta \\ & b \end{pmatrix} \begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix} = \begin{pmatrix} a & ub + \delta \\ & b \end{pmatrix} \begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix} = \begin{pmatrix} a & u(b-a) + \delta \\ & b \end{pmatrix}$

$\nu_F(b-a) = 0 \Rightarrow \exists u \in \mathfrak{o} : u(b-a) = -\delta$. ⌋

Thus $\exists x \in N_0 : \quad \varphi^x = \xi$ □

Step 5: We have $\pi(e_{N_{\delta_1}}) v_\varphi \neq 0$

$\Rightarrow$ (⇑ Step 4) $\exists x \in N_0 : \quad \varphi^x = \xi \quad , \quad v_3 := \pi(x^{-1}) v_\varphi$

$\Rightarrow \quad \pi(N_{\delta_1}) \pi(x^{-1}) v_\varphi = \pi(x^{-1}) \pi(e_{x N_{\delta_1} x^{-1}}) v_\varphi$

$= \pi(x^{-1}) \pi(e_{N_{\delta_1}}) v_\varphi \neq 0$ } because $v_3 \in V^\xi$

⌈ $\pi(g) v_3 = \pi(x^{-1}) \pi(x g x^{-1}) v_\varphi = \varphi(x g x^{-1}) \pi(x^{-1}) v_\varphi$

$= \varphi^x(g) v_3 = \xi(g) v_3$.

$\forall g \in U^n_{\mathfrak{M}}$. ⌋

□

Thm 81: Let $\chi \in \hat{T}$, $\chi = \chi_1 \otimes \chi_2$.

$X := \operatorname{Ind}_B^G \chi$. Then we have the following 3 cases:

(1) $n = \max(\ell(\chi_1), \ell(\chi_2)) > 0$ ~~then~~ and $\chi_1 \chi_2^{-1}|_{U_F^n} \neq 1|_{U_F^n}$ then X contains a split fundamental stratum.

(2) If $n = \ell(\chi_1) = \ell(\chi_2) \neq 0$ and $\chi_1 \chi_2^{-1}|_{U_F^n} = 1|_{U_F^n}$ then X contains an essentially scalar fund. stratum.

(3) If $\ell(\chi_1) = \ell(\chi_2) = 0$ then X contains $1|_{U_{\mathfrak{I}}^1}$.

Proof: (1) $\exists$: $\chi_i(1+x) = \psi(a_i x)$, $x \in \mathfrak{p}^n$, $a_i \in \mathfrak{p}^{-n}$

$a = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}$. Take $f \in \operatorname{Ind}_B^G(\chi)$ s.t.

$\operatorname{supp}(f) = B\begin{pmatrix} 1 & 0 \\ \mathfrak{p}^n & 1 \end{pmatrix}$, s.t. f is fixed by $\begin{pmatrix} 1 & 0 \\ \mathfrak{p}^n & 1 \end{pmatrix}$.

Then $\forall u \in U_{\mathfrak{M}}^n$: $u.f = \psi_a(u) f$, because:

$$(u.f)(t) = f(t u_+ u_0 u_-) = \chi(t)\chi(u_0) f(1) = \chi(t)\psi_a(u_0) f(1)$$
$$= \psi_a(u_0) f(t) = \psi_a(u) f(t).$$

$\Rightarrow \psi_a \subseteq X$ $\square$

Remark 8.2: $\ell(\varphi \cdot St) = \ell(\varphi)$, $\ell(\varphi \circ \det) = \ell(\varphi)$.

Exhaustion theorem ~~and positive level cuspidal irr. repres.~~

Theorem 8.3: Let $\pi \in \mathcal{R}(G)$ be irreducible, s.t. $\forall \varphi$: $\ell(\pi) \leq \ell(\pi\varphi)$

Case: $\ell(\pi) = 0$: π principal series $\Leftrightarrow$ $1\!\!1_{U^1_{\mathcal{J}}} \subseteq \pi$

π cuspidal $\Leftrightarrow$ $1\!\!1_{U^1_{\mathfrak{M}}} \subseteq \pi$ and

$\exists\, \tau \in \mathcal{R}(GL_2(\mathbb{F}))$ irr. cusp: $\pi \cong c\text{-}Ind_{B}^{G} \tau$.

Case: $\ell(\pi) > 0$: π principal series $\Leftrightarrow$ $\pi \supseteq$ split.

π cuspidal $\Leftrightarrow$ $\pi \supseteq$ simple stratum.

Proof: $\ell(\pi) = 0$. Last lecture.

$\ell(\pi) > 0$: If $\pi \supseteq$ split $\underset{\text{Thm.}}{\Rightarrow}$ $\pi_N \neq 0$, i.e. π is not cuspidal

$\Rightarrow$ π is principal series

If π is ~~not cuspidal~~ principal series

$\Rightarrow$ $\pi \subseteq Ind_B^G \chi$ for some $\chi \in \hat{T}$.

If $Ind_B^G \chi$ is irred. then $\pi \supseteq$ split, by the last Thm.

If $\operatorname{Ind}_B^G \chi$ is not irred.

$\Rightarrow \exists_\varphi\ \pi \cong \varphi\,\mathrm{St}$ or $\pi \cong \varphi \det$ ~~with~~

$\forall_\varphi\ \ell(\varphi\pi) \geq \ell(\pi)$

$\Rightarrow\ \ell(\pi) = \ell(\varphi) = 0.$ ⚡ □

Classification of cuspidal irred repr.

Part 1: Preparation: Let $(\mathfrak{a}_i, n_i, \alpha_i)$, $i=1,2$, be two simple strata.

Fact: 1) The intertwining of $(\mathfrak{a}, n, \alpha)$ is equal to $E^\times U_{\mathfrak{a}}^{\lfloor (n+1)/2 \rfloor}$, in particular is the intertwining equal to the normalizer of $(\mathfrak{a}, n, \alpha)$.

2) Suppose $g \in G$ intertwines ~~$(\mathfrak{a}_1, n_1, \alpha_1)$~~ $\psi_{\alpha_1}|U_{\mathfrak{a}}^{\lfloor \frac{n}{2}\rfloor+1}$ with $\psi_{\alpha_2}|U_{\mathfrak{a}}^{\lfloor \frac{n}{2}\rfloor+1}$ ~~$(\mathfrak{a}_2, n_2, \alpha_2)$~~, then $\exists g_0 \in G$: $g_0 \mathfrak{a}_1 g_0^{-1}$ and

$g_0 \psi_{\alpha_1} = \psi_{\alpha_2}$ on $U_{\mathfrak{a}_2}^{\lfloor \frac{n}{2}\rfloor+1}$. If $\mathfrak{a}_1 = \mathfrak{a}_2$ then g_0 can be found in $U_{\mathfrak{a}}$

Part 2: Construction. $(\mathfrak{a}, n, \alpha)$ simple.

$J_\alpha := E^\times U_{\mathfrak{a}}^{\lfloor \frac{n+1}{2} \rfloor} \supseteq Z$, compact mod Z.

Thm 84: Let Λ be an irred repr. of J_α s.t. $\Lambda \supseteq \psi_\alpha$ on $U_{\mathfrak{a}}^{\lfloor \frac{n}{2}\rfloor+1}$. Then

- Λ is a multiple of ψ_α
- $c\text{-}\operatorname{Ind}_{J_\alpha}^G \Lambda$ irred. and cuspidal.

Proof: Let $\varphi \subseteq \Lambda|_{U_\alpha^{\lfloor n/2 \rfloor + 1}}$, Λ irred.

$\Rightarrow \exists g \in J_\alpha$: $g \in I(\varphi|_{U_\alpha^{\lfloor n/2 \rfloor+1}}, \psi_\alpha|_{U_\alpha^{\lfloor n/2 \rfloor+1}})$

Every Elt of $I(\varphi, \psi)$ also conjugates φ to ψ_α

J_α normalizes $\psi_\alpha \Rightarrow \varphi = \psi_\alpha$.

Thus $\Lambda|_{U_\alpha^{\lfloor n/2 \rfloor+1}}$ is a multiple of ψ_α.

$c\text{-}Ind_B^G \Lambda$ is irred cuspidal, because

$I_G(\Lambda) = J_\alpha$. $\left[\, g \in I_G(\Lambda) \Rightarrow g \in I_G(\psi_\alpha) \Rightarrow g \in J_\alpha \,\right]$ □

Thm 85: Let $\underset{\pi_1}{\underset{\shortparallel}{c\text{-}Ind_{J_{\alpha_1}}^G \Lambda_1}} \cong \underset{\pi_2}{\underset{\shortparallel}{c\text{-}Ind_{J_{\alpha_2}}^G \Lambda_2}}$

then $n_1 = n_2$ and $\exists g_0 \in G$: $g_0 J_{\alpha_1} g_0^{-1} = J_{\alpha_2}$

and $\Lambda_2 = {}^{g_0}\Lambda_1$.

If $\alpha_1 = \alpha_2$ ~~we may~~ the g can be chosen in U_α.

Proof: $(\sigma_i, n_i, \alpha_i) \subseteq \pi_i$ simple

$\pi_1 \cong \pi_2 \Rightarrow$ $$n_1/e_{\sigma_1} = n_2/e_{\sigma_2} \quad (*)$$
$$\ell(\pi_1) = \ell(\pi_2).$$

Thus $n_1/e_{\sigma_1} \in \mathbb{Z} \Leftrightarrow n_2/e_{\sigma_2} \in \mathbb{Z}$

Thus σ_1 and σ_2 are conjugate to $\mathfrak{M}$

or σ_1 " " " " to $\mathfrak{J}$.

$\Rightarrow$ OE $\sigma_1 = \sigma_2$. $\Rightarrow n_1 = n_2$

$\psi_{\alpha_1}, \psi_{\alpha_2} \subseteq \pi_1 \Rightarrow \exists\, g \in G: g \in I(\psi_{\alpha_1}, \psi_{\alpha_2})$

[ψ_{α_1} is a character def. on $U_{\sigma}^{\lfloor \frac{n}{2} \rfloor + 1}$.]

$\Rightarrow$ g conjugates ψ_{α_1} to ψ_{α_2} ~~and $g \in$~~ by an element of U_σ.

Thus OE $\psi_{\alpha_1} = \psi_{\alpha_2}$. $\Lambda_1, \Lambda_2 \subseteq \pi_1$

$\Rightarrow I_G(\Lambda_1, \Lambda_2) \neq \emptyset$, because J_{α_i} ~~are~~ contain and are compact mod Z.

Now $I_G(\Lambda_1, \Lambda_2) \subseteq I(\psi_{\alpha_1}) = J_{\alpha_1} = J_{\alpha_2}$

Thus $\exists$ from $\mathcal{I}_G(\Lambda_1,\Lambda_2)\neq\emptyset$ follows
$\exists$ $g_0 \in \mathcal{I}_G(\Lambda_1,\Lambda_2)$

$\Lambda_1^{g_0}\cong\Lambda_2 \underset{\substack{\uparrow \\ g_0\in \mathcal{J}_{\mathfrak{A}_1}}}{\Rightarrow} \Lambda_1\cong\Lambda_2.$ □

<u>Def 86</u>: Cuspidal types: It is a tripel $(\mathfrak{A},\mathcal{J},\Lambda)$ s.t.

Either (1) $\mathcal{J}=ZGL_2(\mathcal{O})$, Λ extension of an inflation of a cuspidal irred of $GL_2(\mathbb{F})$.

or (2) $\exists$ simple stratum $(\mathfrak{A},n,\alpha)$: $\mathcal{J}=\mathcal{J}_\alpha \wedge \Lambda=\Lambda_\alpha$ irred $\supseteq \psi_\alpha$

or (3) $(\mathfrak{A},\mathcal{J},\Lambda)$ is a twist of (1) or (2), i.e. $\exists(\mathfrak{A}_0,\mathcal{J}_0,\Lambda_0)\in(1)\cup(2)$: $\exists x\in F^\times$: $\mathfrak{A}=\mathfrak{A}_0$, $\mathcal{J}=\mathcal{J}_\alpha \wedge$ $\Lambda=x\Lambda$.

<u>Classification Thm 87</u>:

$\forall$ π irred cusp: $\exists$ $(\mathfrak{A},\mathcal{J},\Lambda)$ cusp type: $\pi\cong c\text{-}\mathrm{Ind}_{\mathcal{J}}^{G}\Lambda.$

and from $c\text{-}\mathrm{Ind}_{J_1}^{G} \Lambda_1 \cong c\text{-}\mathrm{Ind}_{J_2}^{G} \Lambda_2$

follows $\exists_{g \in G} : g J_1 g^{-1}$ and ${}^{g}\Lambda_1 \cong \Lambda_2$.

V Representations of the Weil group

The Weil group

F n.a. local field $\quad G_F := \mathrm{Gal}(F^{sep}/F) \cong \varprojlim_{E/F \text{ finite galois}} \mathrm{Gal}(E/F)$

$F^{sep} \supseteq F_{tr} \supseteq F_{ur} \supseteq F \rightsquigarrow \{1\} \subseteq \mathrm{Gal}(F^{sep}/F_{tr}) \subseteq \mathrm{Gal}(F^{sep}/F_{ur}) \subseteq G_F$

where $\mathrm{Gal}(F^{sep}/F_{tr}) = P_F$ (wild inertia subgroup) and $\mathrm{Gal}(F^{sep}/F_{ur}) = I_F$ (inertia subgroup).

We consider the following ~~map~~ diagramm

$$G_F \longrightarrow \mathrm{Gal}(F_{ur}/F) \simeq \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}) \simeq \hat{\mathbb{Z}} \quad (\text{composite: } d)$$

$$\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}) \cong \varprojlim_n \mathrm{Gal}(\mathbb{F}_n/\mathbb{F}), \qquad \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$$

$$\left(\bar{\Phi}^{a_n}|_{\mathbb{F}_n}\right)_{n\in\mathbb{N}} \mapsto (\bar{a}_n)_{n\in\mathbb{N}}$$

and we define $W_F := d^{-1}(\mathbb{Z})$.

Thus we have the exact sequence

$$1 \longrightarrow I_F \longrightarrow W_F \xrightarrow{d} \mathbb{Z} \longrightarrow 1$$

$$\nu_F := d|_{W_F}.$$

The element Φ_F of $\mathrm{Gal}(F_{ur}/F)$ which corresponds to $1 \in \hat{\mathbb{Z}}$ is called ~~die~~ the <u>geometric</u> Frobenius substitution on F_{ur}.

(The one corresponding to $-1 \in \hat{\mathbb{Z}}$ is called the arithmetic Frob. substitution on F_{ur}.)

The topology on W_F is defined by ~~that~~

- I_F is open in W_F and
- the induced topology on I_F is the one induced from the Krull topology of G_F on I_F.

CAUTION: This is ~~different to the~~ finer than the induced topology on W_F!

We have a "norm" on W_F: $W_F \xrightarrow{\|\cdot\|} \mathbb{Z}$ $\|w\| := q^{-\nu_F(w)}$

An element of $\nu_F^{-1}(1)$ is called geometric Frobenius element of W_F.

The topics of this section are smooth representations of W_F and ~~Weil~~-Deligne representations of W_F.

Proposition 88: 1) A smooth irred. representation of G_F is finite dimensional

2) A smooth irred repr. of W_F is finite dim.

Proof: 1) clear, because G_F is compact.

2) $(V,\rho) \in \mathcal{R}(W_F)$ irred. Take $v \in V$.

ρ is smooth $\Rightarrow \exists\ E/F$ ~~non~~ galois, finite:

$v \in V^{J}$ $\quad J := Gal(\bar{F}^{sep}/E) \cap I_F \trianglelefteq G_F$

- I_F/J is finite because I_F is compact and J is open in I_F. A Frobenius element $\Phi \in W_F$ acts on I_F/J via conjugation.

$\Rightarrow \exists\ r \in \mathbb{N}$: Φ^r acts trivially on I_F/J

Thus $\rho(\Phi^r)$ commutes with $\rho(W_F)$, because for $\sigma \in I_F$

$$\rho(\Phi^r)\rho(\sigma) = \underbrace{\rho(\Phi^r\sigma\Phi^{-r})}_{\text{~~id~~}}\rho(\Phi^r) = \rho(\sigma)\rho(\Phi^r).$$

⌜ You can see ρ as a representation of $\mathcal{W}_F/\mathcal{J}$ ⌟

Schur $\Rightarrow$ $\rho(\Phi^r)$ acts by a scalar $\in \mathbb{C}$. $W := \mathrm{span}_{\mathbb{C}}\,\rho(I_F)v$

Thus $V = \sum_{j=0}^{\infty} \rho(\Phi^j)$ ~~$\rho(I)$~~ $W = \sum_{j=0}^{r-1} \rho(\Phi^j) W$

$\dim_{\mathbb{C}} W < \infty$, because $I_F/\mathcal{J}$ is finite.

$\Rightarrow \dim_{\mathbb{C}} V < \infty$ □

We give the relation to smooth representations of G_F.

<u>Prop 89</u>: 1) a) If $\rho \in \mathcal{R}(G_F)$ is irred. Then $\mathrm{Res}^{G_F}_{\mathcal{W}_F} \rho$ is irreducible.

b) If $\rho_1, \rho_2 \in \mathcal{R}(G_F)$ are irred then

$$\rho_1 \cong \rho_2 \iff \mathrm{Res}^{G_F}_{\mathcal{W}_F} \rho_1 \simeq \mathrm{Res}^{G_F}_{\mathcal{W}_F} \rho_2.$$

2) Let $\tau \in \mathcal{R}(\mathcal{W}_F)$ be irred. Then are equivalent:

1° $\tau(\mathcal{W}_F)$ is finite

2° $\exists\, \rho \in \mathcal{R}(G_F)$ irred : $\mathrm{Res}^{G_F}_{\mathcal{W}_F} \rho = \tau$.

3° $\det \tau$ ~~had~~ has finite order, i.e.

$\exists\, m \in \mathbb{N} : (\det \tau)^m = 1$.

Proof: a) Let ~~Gal(F~~ E/F be finite galois s.t.

$\mathrm{Gal}(\overline{F}^{sep}/E) \subseteq \ker(\rho)$.

To show: $\mathrm{Gal}(\overline{F}^{sep}/E) \cdot W_F = G_F$.

Pf: $f := f(E/F) \Rightarrow d(\mathrm{Gal}(\overline{F}^{sep}/E))$

$$\| $$

$$d(\mathrm{Gal}(\overline{F}^{sep}/F_f)) = f\hat{\mathbb{Z}}.$$

Take $\sigma \in G_F$. Then $d(\sigma) = z \in \hat{\mathbb{Z}}$. $\Phi \in W_F$ Frob elt.

$\hat{\mathbb{Z}}/f\hat{\mathbb{Z}} \cong \mathbb{Z}/f\mathbb{Z}$. $\Rightarrow \exists j \in \mathbb{Z}: d(\sigma\Phi^j) \in f\hat{\mathbb{Z}}$

~~⇒~~ Take $\tilde{\sigma} \in \mathrm{Gal}(\overline{F}^{sep}/E)$: $d(\tilde{\sigma}) = d(\sigma\Phi^j)$

$\Rightarrow \tilde{\sigma}^{-1}\sigma\Phi^j \in I_F$. □

Thus we have $W_F \cdot \ker(\rho) = G_F$ and

thus $\rho(W_F) = \rho(G_F)$ and ~~ρ is irred or~~

$\mathrm{Res}^{G_F}_{W_F}\rho$ is irreducible, because every ~~$\rho(W_F)$~~ W_F

subrep. is a ~~$\rho(G_F)$~~ G_F-subrep.

b) Because of ~~...~~ $W_F(\ker\rho_1 \cap \ker\rho_2) = G_F$

we have that every W_F morphism is a G_F

morphism.

2) $2^\circ \Rightarrow 1^\circ \to 3^\circ$ ✓

$3^\circ \Rightarrow 1^\circ$ Let $\det \tau$ have finite order on

Γ irred. Thus for a Frobenius elt. Φ $\exists \ell \in \mathbb{N}$:

Φ^ℓ acts as a scalar.

$\Rightarrow \tau(\Phi^{\ell m}) = id.$

Thus $\tau(W_F)$ is finite.

$1^\circ \Rightarrow 2^\circ$ $\ell := \operatorname{ord}(\tau(W_F))$. $w \in G_F$ has

the form $w = \sigma \Phi^\alpha$ $\sigma \in I_F$ and $\alpha \in \hat{\mathbb{Z}}$

We define $\rho(w) := \tau(\sigma)\tau(\Phi)^a$, where

$a \equiv \alpha \mod \ell\hat{\mathbb{Z}}$.

Exercise: We can choose Φ, s.t. ρ is a smooth representation of G_F.

(Thus irred, because τ is irreducible) □

A character χ of W_F is called unramified if

$\chi|_{I_F} = 1\!\!1_{I_F}$.

Remark: For every $\tau \in R(W_F)$ irred there exists an unramified character χ, s.t.

$\chi^{-1}\tau$ is the restriction of an irred $\rho \in R(G_F)$

Pf: $\exists \ell \in \mathbb{N}$: $\tau(\Phi^\ell)$ acts like a scalar, say λ

Take $\chi : W_F \longrightarrow \mathbb{C}^\times$ unramified s.t.

$\chi(\Phi)^\ell = \lambda$. $\chi^{-1}\tau$ has a finite image

Thm 90: Let τ be ~~an irred~~ a smooth finite dim representation of W_F. Let Φ be a Frob. elt. Then are equivalent:

1° τ is semisimple

2° $\tau(\Phi)$ — " —

3° $\forall \varphi \in W_F$: $\tau(\varphi)$ is semisimple

Pf: 1° ⇒ 3° w.l.o.g. let τ be irreducible. $\varphi \in W_F$.

$\exists_{\ell \in \mathbb{N}}$: $\tau(\varphi)^\ell$ acts ~~like a~~ as a scalar.

⇒ $\tau(\varphi)^\ell$ is semisimple ⇒ $\tau(\varphi)$ is semisimple.
(↑ Jordan-Normal form.)

3° ⇒ 2° ✓

2° ⇒ 1° τ is finite dimensional

⇒ $\exists \ell \in \mathbb{N}$: $\tau(\Phi)^\ell$ commutes with $\tau(I_F)$.

$\tau|_{I_F}$ is semisimple, because $I_F / I_F \cap \ker(\tau)$ is finite, because τ is finite dimensional.

$\tau(\Phi^\ell)$ is semisimple, i.e. diagonalizable because ~~$\mathbb{C}$~~ $\mathbb{C}$ is algebraically closed, and further ~~Φ~~ $\tau(I_F)$ commutes with $\tau(\Phi^\ell)$ ⇒ $\tau|\langle \Phi^\ell, I_F \rangle$ is semisimple

$$\langle \Phi, \mathcal{I}_F \rangle / \langle \Phi^\ell, \mathcal{I}_F \rangle \xrightarrow{\sim} \mathbb{Z} / \ell\mathbb{Z}.$$

Thus $\tau|_{\langle \Phi, \tilde{\mathcal{I}}_F \rangle}$ is semisimple. □

We have used the lemma:

<u>Lemma:</u> Let ρ be a ~~rep~~ complex representation of G and $H \leq G$ s.t. $(G:H) < \infty$. Then are equivalent:

1° ρ semisimple

2° $\mathrm{Res}^G_H \rho$ is semisimple

<u>Remark:</u> The same holds for induction:

σ a complex representation of H, $(G:H) < \infty$.

σ semisimple $\Leftrightarrow$ $\mathrm{Ind}^G_H \sigma$ semisimple.

<u>Pf of the lemma:</u> 2° ⟹ 1° (ρ, V) given $V' \subseteq V$ a subrepresentation. $\mathrm{Res}^G_H \rho$ semisimple

$\Rightarrow \exists\, W \leq V$: $V' \oplus W = V$ as H modules.
H-repr.

Let $p: V \longrightarrow V'$ the corresponding projection along W.

$v \overset{\tilde{p}}{\longmapsto} \sum_{\bar{g} \in H\backslash G} \rho(g)^{-1} p(\rho(g) v)$ is a projection and a G-morphism

$\Rightarrow \underbrace{\operatorname{im} \tilde{p}}_{V'} \oplus \ker \tilde{p} = V$

$1^\circ \Rightarrow 2^\circ$ We have already proven ($2^\circ \Rightarrow 1^\circ$) so we can assume w.l.o.g. that $H \trianglelefteq G$.
W.l.o.g. let (ρ, V) be irreducible.
$\Rightarrow$ $\operatorname{Res}^G_H \rho$ is of finite type and has an irreducible quotient W.
1^{st} Frobenius reciprocity $\Rightarrow$ $\rho \hookrightarrow \operatorname{Ind}^G_H W$

$$\operatorname{Ind}^G_H W = \bigoplus_{g \in H \backslash G} W^g$$

Thus $\operatorname{Res}^G_H \rho$ is irreducible. □

<u>Notation:</u> $\mathcal{G}^{ss}_n(F) \overset{\text{isom. classes}}{=}$ n-dim. smooth complex repr. of W_F.
$\mathcal{G}^{0}_n(F) \overset{\text{isom. classes}}{=}$ n-dim irred. smooth cplx representations.

Deligne - representations

<u>Def:</u> A —"— of W_F is a triple $(\rho, V, \mathcal{N})$
- (ρ, V) is a finite dim smooth repr. of W_F
- $\mathcal{N} \in \operatorname{End}_{\mathbb{C}} V$ is nilpotent s.t. $\forall x \in W_F$ $\rho(x) \mathcal{N} \rho(x^{-1}) = \|x\|_F \, \mathcal{N}$

A Deligne-repr. is called semisimple, if (ρ, V) is semisimple.

$G_n(F) =$ isom. classes of ~~Deligne~~ semisimple n-dim Deligne representations.

<u>Example</u>: $V = \mathbb{C}^n$, $N \in \mathrm{End}_{\mathbb{C}} V = M_n(\mathbb{C})$

$$N = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, \quad v_i := \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} - i$$

Define $\rho : W_F \to \mathrm{Aut}_{\mathbb{C}} V$ via

$$\rho(x) v_i = \|x\|^{(n-1)/2} \|x\|^i v_i$$

$\Rightarrow (\rho, \mathbb{C}^n, N)$ ~~is a semi~~ is a semisimple n-dim Deligne representation called $Sp(n)$.

Relation with ℓ-adic Representations.

~~ℓ prime number ≠ p~~, $G = GL_2(F)$.

Let C be a field of characteristic zero.

A representation (V, S) of over C is smooth if for all ~~sta~~ $v \in V$: $\mathrm{Stab}_G(v)$ is open.

Consider $C \cong \mathbb{C}$ (as rings)

Then we have the same classification results for $\mathcal{R}_C(G)$ as for $\mathcal{R}_{\mathbb{C}}(G)$ and an equivalence of categories $\mathcal{R}_C(G) \xrightarrow{\sim} \mathcal{R}_{\mathbb{C}}(G)$ except that we have no normalized induction, because $q^{1/2}$ ~~doesn't make sense in C.~~ has no clear meaning in C.

Deligne - Representations: Same definition

(ρ, V, n) n nilpotent, s.t. $\forall x \in W_F$: $\rho(x) n \rho(x)^{-1} = a(x) n$.

ρ smooth, finite dim.

$D\text{-}\mathrm{Rep}_C(W_F) =$ "category of Deligne-repr. of W_F over C"

We have $\exp n = 1 + \sum_{j=1}^{\infty} \frac{n^j}{j!}$ and for:

$u = 1 + n$ $\log u = \sum_{j=1} (-1)^{j-1} \frac{n^j}{j}$

s.t. $\exp \log u = u$ and $\log \exp n = n$

<u>Def 91:</u> Let H be a locally profinite group. $C = \overline{\mathbb{Q}}_\ell$

A repr. (σ, V) of H is called continuous, if $\sigma : H \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell} V$ is continuous.

<u>Rk:</u> Smooth repr. are continuous, but the converse is wrong in general.

We now consider W_F

Given a continuous repr. of W_F we can see a "big" part of it by the exponential.

We take a continuous group hom. $t : I_F \to \mathbb{Z}_\ell$ (unique up to mult with $\in \mathbb{Z}_\ell^\times$)

It satisfies $t(xyx^{-1}) = \|x\| \, t(y) \ \forall_{x \in W_F} \forall_{y \in I_F}$

<u>Thm 92:</u> Let (σ, V) be a finite dim continuous $\overline{\mathbb{Q}}_\ell$-repr. of W_F. Then $\exists! \, n_\sigma \in \mathrm{End}_{\overline{\mathbb{Q}}_\ell} V$ nilpotent:

$\exists H \leq I_F$ open: $\forall_{x \in H} \exp(t(x) n_\sigma) = \sigma(x)$

Thm 93.

This gives the following equivalence of categories:

$$\mathrm{Rep}^f_{\overline{\mathbb{Q}}_\ell}(W_F) = \text{"finite dim continuous repr. of } W_F \text{ over } \overline{\mathbb{Q}}_\ell\text{"}$$

$$\downarrow \wr$$

$$D-\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_F)$$

Sketch: (Construction)

$(\sigma, V) \in \mathrm{Rep}^f_{\overline{\mathbb{Q}}_p}(W_F)$. $\rightsquigarrow$ We get N_σ.

Claim: $\forall_{x \in W_F}: \sigma(x) N_\sigma \sigma(x)^{-1} = \|x\| N_\sigma$: For $y \in H \cap x^{-1}Hx$

$$\sigma(x)\sigma(y)\sigma(x)^{-1} = \exp(t(xyx^{-1}) N_\sigma)$$
$$= \exp(\|x\| \, t(y) N_\sigma)$$

and $\sigma(x)\sigma(y)\sigma(x)^{-1} = \exp(t(y)\, \sigma(x) N_\sigma \sigma(x)^{-1})$.

Uniqueness of N_σ $\Rightarrow$ Claim.

~~We define~~: Take a Frobenius elt Φ.

We define (σ_Φ, V) a representation of W_F over $\overline{\mathbb{Q}}_\ell$

via
$$\sigma_\Phi(\Phi^a y) := \sigma(\Phi^a y) \exp(-t(y) N_\sigma)$$
for $a \in \mathbb{Z}$ and $y \in I_F$.

By Thm 92 this is trivial on an open subgroup of I_F $\Rightarrow$ σ_Φ is smooth. $\square$ Sketch.

Rk: Isom. class does not depend on choice of t and Φ

Def 94: ~~A continuous~~

- A Deligne-repr. (σ, V, N) is called semisimple if (σ, V) is semisimple. (Here (σ, V) is smooth)
- A continuous finite dim. repr. (σ, V) of W_F is called Φ-semisimple if $(\sigma_\Phi, V, N_\sigma)$ is semisimple.

Prop 95: Let (σ, V) be a continuous ~~semisimple~~ repr. of W_F / $\bar{\mathbb{Q}}_\ell$. Then are equivalent.

1° (σ, V) is Φ semisimple

2° $\exists\ \Psi$ Frobenius elt. : $\sigma(\Psi)$ is semisimple

3° $\sigma(g)$ is semisimple $\forall g \in W_F \setminus I_F$.

Pf: 1°⇒2° ~~$(\sigma_\Phi, V, N_\sigma)$ $\sigma_\Phi($~~

1° ⇒ $(\sigma_\Phi, V, N_\sigma)$ is semisimple $\overset{\text{Def}}{\Rightarrow}$ (σ_Φ, V) is semisimple. From $\sigma_\Phi(\Phi) = \Phi$ and Thm 90 follows 2°

3° ⇒ 2° ✓

2° ⇒ 1° $\exists\ \Psi$ Frob. elt. : $\sigma(\Psi)$ is semisimple.

$\Rightarrow \sigma_\Psi(\Psi) = \sigma(\Psi)$ — " —

$(\sigma_\Phi, V, N_\sigma) \simeq (\sigma_\Psi, V, N_\sigma)$

$\Rightarrow$ $\sigma_\Phi(\Psi)$ is semisimple

$\Rightarrow$ $(\sigma_\Phi(\ldots), V)$ — " — $\Rightarrow$ (σ, V) is Φ-semisimple.
$\uparrow$
Thm 90

$2^\circ \Rightarrow 3^\circ$ $g \in W_F \setminus I_F$. To show $\sigma(g)$ is semisimple. ~~It is enough to consider a Frobenius element. Ψ.~~

If g is a Frobenius elt, then σ is g-semisimple because ($2^\circ \Rightarrow 1^\circ$) and thus $\sigma(g) = \sigma_g(g)$ is semisimple by Thm 90.

If $g = \Psi^a x$ $a \in \mathbb{Z}$, $x \in I_F$

Then $\langle \Psi^a, I_E \rangle = W_E$ for E/F unramified of degree $|a|$.

$\sigma(\Psi)$ is semisimple ~~by~~ $\Rightarrow \sigma(\Psi^a)$ is semisimple too. $\Rightarrow$ ~~$\sigma|_{W_F}$~~ $\mathrm{Res}_{E/F}\sigma(\Psi^a)$ is semisimple.

$\Rightarrow$ $(\mathrm{Res}_{E/F}\sigma)(g)$ is semisimple
$\uparrow$
~~$2^\circ \Rightarrow 1^\circ$~~
Case 1.

$\Rightarrow$ $\sigma(g)$ is semisimple. $\square$

Thm 96: Let $\ell \neq p$ be a prime. The sets of isom. classes of the following objects are in canonical bijection.

(i) n-dim $\overline{\mathbb{Q}}_\ell$-semisimple continuous repr. of W_F / $\overline{\mathbb{Q}}_\ell$

(ii) n-dim semisimple Deligne-repr. of $W_F / \overline{\mathbb{Q}}_\ell$

A choice of $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ give a bijection to

(iii) n-dim semisimple Deligne-repr. of $W_F / \mathbb{C}$.

VI L und ε – factors for irred. repr. of $GL_2(F) =: G$

ψ non trivial char. of $(F,+)$

To $\pi \in \mathcal{R}(G)^{\vee}_{\text{irr.}}$ we attach

1) an <u>L-function</u> $s \in \mathbb{C} \longmapsto L(\pi, s) \in \mathbb{C}$

2) a <u>local constant</u> $s \in \mathbb{C} \longmapsto \varepsilon(\pi, s, \psi) \in \mathbb{C}$

~~$\psi \in \hat{F} \setminus \{1\}$~~

We have for L the form $L(\pi, s) = f(q^{-s})$

for a polynomial $f \in \mathbb{C}[T]^{\deg \leq 2}$.

and $\varepsilon(\pi, s, \psi) = c(\pi, \psi)\, q^{-ms}$, $c(\pi, \psi) \in \mathbb{C}$ and $m \in \mathbb{Z}$.

L and ε generalize the theory of characters.

- In fact for π non-cusp. L and ε are described in terms of characters of $F^{\times}$.
- For π cuspidal : $L(\pi, s) = 1 \quad \forall_s \in \mathbb{C}$

  $\varepsilon(\pi, s, \psi)$ is given by a non-abel. Gauss sum

[For GL_1 : $\varepsilon(\chi, s, \psi)$ is described by an abelian ~~Gauß~~ Gauss sum.]

Converse Theorem : $\pi_{\cong}$ is determined by

$$\chi \longmapsto L(\chi\pi, *) \quad \text{and}$$

$$\chi \longmapsto \varepsilon(\chi\pi, *, \psi)$$

There is an absolute dichotomy

π

non-cusp.	cusp.
$\pi_{\cong}$ determined by $\chi \mapsto L(\chi\pi, *)$	$\pi_{\cong}$ determined by $\chi \mapsto \varepsilon(\chi, *, \psi)$.

VI 1. The construction for $GL_1(F)$

We fix $\psi \in \hat{F} \setminus \{1\}$. μ Haar measure on F.

We define for $\Phi \in C_c^\infty(F)$ the Fourier transform

$$\hat{\Phi}(x) := \int_F \Phi(y)\, \psi(xy)\, d\mu(y).$$

$\hat{\Phi}$ ~~ded~~ depends on ψ and μ.

Prop 97: (1) $\hat{\Phi} \in C_c^\infty(F)$ $\forall \Phi \in C_c^\infty(F)$

(2) $\exists\, c = c(\psi, \mu) > 0$: $\forall_{x \in F}$ $\hat{\hat{\Phi}}(x) = c\, \Phi(-x)$

(3) For ψ there is a unique Haar measure μ_ψ s.t. $c(\psi, \mu_\psi) = 1$. It satisfies

"self dual relative to ψ"

$$\mu_\psi(\mathcal{O}) = q^{l/2}, \quad l = l(\psi)$$

(4) For $a \in F^\times$: $\mu_{a\psi} = \|a\|^{\frac{1}{2}} \mu_\psi$. $\big(a\psi(x) := \psi(ax)\big)$

Proof: To show $\hat{\Phi} \in C_c^\infty(F)$. and (2)

~~$\hat{\Phi}(x) = \int_F \Phi(y)\, (x\psi)(y)\, d\mu(y)$. We have for $a \in F^\times$~~

~~$a\psi|_{\mathfrak{p}^j} = 1 \iff a\mathfrak{p}^j \subseteq \mathfrak{p}^l \iff a \in \mathfrak{p}^{l-j}$~~

~~$\Rightarrow l(a\psi) = -\nu_{\mathfrak{p}}(a) + l$.~~

$C_c^\infty(F)$ is spanned by $\mathbb{1}_{\mathfrak{p}^j}$ and ~~its~~ their translates $x \mapsto \mathbb{1}_{\mathfrak{p}^j}(x - a)$, $a \in F, j \in \mathbb{Z}$

Case 1: We consider $\Phi := \mathbb{1}_{\mathfrak{p}^j}$

If $x \in \mathfrak{p}^{\ell - j}$ then

$$\hat{\Phi}(x) = \int_F \Phi(y)\, \psi(xy)\, d\mu(y)$$

$$= \int_{\mathfrak{p}^j} d\mu(y) = \mu(\mathfrak{o}) \cdot \frac{1}{q^j}$$

If $x \notin \mathfrak{p}^{\ell - j}$, then

$$\hat{\Phi}(x) = \sum_{\xi \in \mathfrak{p}^j / \mathfrak{p}^{\ell - \nu_F(x)}} (x\psi)(y) = 0$$

$\uparrow$ $\psi \not\equiv \mathbb{1}$ on $\mathfrak{p}^j / \mathfrak{p}^{\ell - \nu_F(x)}$

by Schur orthogonality.

Thus $\hat{\Phi} = \mathbb{1}_{\mathfrak{p}^{\ell - j}} \cdot \mu(\mathfrak{o}) \cdot \frac{1}{q^j}$

$$\Rightarrow \quad \hat{\hat{\Phi}} = \Phi \cdot \mu(\mathfrak{o})^2 \frac{1}{q^j \cdot q^{\ell - j}} = \Phi\, (\mu(\mathfrak{o}))^2 \frac{1}{q^\ell}.$$

$$c = c(\psi, \mu) = \frac{\mu(\mathfrak{o})^2}{q^\ell}$$

Case 2: $\Phi_a := \mathbb{1}_{a+p^j}$.

$$\hat{\Phi}_a(x) = \int_F \Phi(y-a)\,\psi(xy)\,d\mu(y)$$
$$= \int_F \Phi(y)\,\psi(x(y+a))\,d\mu(y)$$
$$= \psi(ax)\,\hat{\Phi}(x) = ((a\psi)\cdot\hat{\Phi})(x).$$

$\Rightarrow \hat{\Phi}_a \in C_c^\infty(F)$

$$\hat{\hat{\Phi}}_a(x) = \int_F (a\psi)(y)\,\hat{\Phi}(y)\,\psi(xy)\,d\mu(y)$$
$$= \hat{\hat{\Phi}}(x+a) = c\cdot\Phi(-x-a)$$
$$= c\cdot\Phi_a(-x)$$

(3) Def of $c(\psi,\mu) \Rightarrow \forall_{b\in\mathbb{R}_{>0}} : c(\psi, b\mu) = b^2 c(\psi,\mu)$

We take $b>0$ s.t. $c(\psi, \underbrace{b\mu}_{=:\mu_\psi}) = 1$.

We get $\mu_\psi(\mathcal{O})^2 q^{-\ell(\psi)} = 1 \Rightarrow \mu_\psi(\mathcal{O}) = q^{\ell(\psi)/2}$

(4) Exercise.

□

μ_ψ is called the self dual Haar measure on F relative to ψ. For μ_ψ we have $\hat{\hat{\Phi}}(x) = \Phi(-x)$. $\forall \Phi \in C_c^\infty(F)$.

We define now the Z-function:

Let μ^* be a Haar measure on $F^\times$ and $\chi \in \widehat{F^\times}$ and $\varpi \in \mathfrak{p} \setminus \mathfrak{p}^2$. We define $Z(\Phi, \chi, X)$ ~~$\in \mathbb{C}((X))$~~:

$$Z(\Phi, \chi, X) := \sum_{m \in \mathbb{Z}} z_m X^m$$

$$z_m := z_m(\Phi, \chi) := \int \mathbb{1}_{\varpi^m \mathcal{U}_F} \Phi(x) \chi(x) \, d\mu^*(x)$$

We have $\varpi^m \mathcal{O}_F^\times = \mathfrak{p}^m \setminus \mathfrak{p}^{m+1}$ and thus $\mathbb{1}_{\varpi^m \mathcal{O}_F^\times} \Phi = 0$ for $m \ll 0$.

Therefore $Z(\Phi, \chi, X)$ is a Laurent series, i.e. $\in \mathbb{C}((X))$.

The Z functions lead to the L function of χ:

~~Step Set~~

Step 1: Consider the $\mathbb{C}[X, X^{-1}]$-module:

$$\mathcal{Z}(\chi) = \mathcal{Z}(\chi, X) := \{ Z(\Phi, \chi, X) \mid \Phi \in C_c^\infty(F) \}$$

This is a $\mathbb{C}[X, X^{-1}]$-module because

$$Z(\Phi(a^{-1}*), \chi, X) = \chi(a) \, X^{+\nu_F(a)} Z(\Phi, \chi, X)$$

because $z_m(\Phi(a^{-1}*), \chi) = \chi(a) \, z_{m-\nu_F(a)}(\Phi, \chi)$.

Step 2: This module is generated by one element.

Prop 98: Let $\chi \in \hat{F}^\times$. Then

$$Z(\chi, X) = P_\chi(X)^{-1} \mathbb{C}[X, X^{-1}]$$

where
$$P_\chi(X) = \begin{cases} 1 - \chi(\varpi) X, & \text{if } \chi \text{ is unramified, i.e. } \chi|_{\mathcal{O}^\times} = \mathbb{1}_{\mathcal{O}^\times} \\ 1, & \text{if } ~~\chi \text{ is ram}~~ \text{ otherwise.} \end{cases}$$

Proof: ~~If~~ We consider $Z(\Phi, \chi, X)$.

If $\Phi(0) = 0$ then $\Phi \in C_c^\infty(F^\times)$ and $Z(\Phi, \chi, X)$ is a Laurent polynomial. If one takes Φ to be $\mathbb{1}_V$ ~~or~~ for $V \subseteq ~~\mathcal{O}~~ \setminus \{0\}$ a small open neighbourhood of 1 ~~Then~~ ~~$Z(\Phi$~~ then $Z(\mathbb{1}_V, \chi, X) \in \mathbb{C}^\times$.

$$\Rightarrow \quad \{Z(\Phi, \chi, X) \mid \Phi \in C_c^\infty(F^\times)\} = \mathbb{C}[X, X^{-1}].$$

$$C_c^\infty(F) = \operatorname{span}_{\mathbb{C}}(\{\mathbb{1}_{\mathcal{O}}\} \cup C_c^\infty(F^\times))$$

$$Z(\mathbb{1}_{\mathcal{O}}, \chi, X) = \sum_{m \geq 0} \chi(\varpi^m) X^m \int_{\mathcal{O}^\times} \chi(x)\, d\mu^*(x)$$

$$\int_{\mathcal{O}^\times} \chi(x)\, d\mu^*(x) = \begin{cases} \mu^*(\mathcal{O}^\times), & \text{if } \chi|_{\mathcal{O}^\times} = \mathbb{1}_{\mathcal{O}^\times} \\ 0, & \text{if } \chi|_{\mathcal{O}^\times} \neq \mathbb{1}_{\mathcal{O}^\times}. \end{cases}$$

$$\Rightarrow \quad \mu^*(\mathcal{O}^\times)^{-1} Z(\mathbb{1}_{\mathcal{O}}, \chi, X) = \begin{cases} (1-\chi(\varpi)X)^{-1}, & \text{if } \chi \text{ is unram.} \\ 0, & \text{else.} \end{cases}$$

$$\Rightarrow \quad Z(\chi, X) = P_\chi(X)^{-1}\, \mathbb{C}[X, X^{-1}].$$

<u>Step 3:</u> $L(\chi, s) := P_\chi(q^{-s})^{-1} = \begin{cases} \dfrac{1}{1-\chi(\varpi)q^{-s}} \\ 1 \end{cases}$

Definition of the local constants: We take the Haar measure μ on F to be self dual relative to ψ, i.e. $\mu = \mu_\psi$.

<u>Thm 99:</u> Let χ be a char of $F^\times$.

$\exists!$ $c(\chi, \psi, X) \in \mathbb{C}(X)$, such that:

$$Z\left(\hat{\Phi}, \chi^{-1}, \frac{1}{qX}\right) = c(\chi, \psi, X)\, Z(\Phi, \chi, X)$$

$$\forall \Phi \in C_c^\infty(F).$$

<u>Proof:</u>

$$\Lambda := \left\{ \lambda : C_c^\infty(F) \to \mathbb{C}(X) \;\middle|\; \lambda(\Phi(a^{-1}x)) = \chi(a) X^{v_F(a)} \lambda(\Phi)\ \forall \Phi \in C_c^\infty(F)\ \forall a \in F^\times \right\}$$

$(\Phi \overset{\lambda_0}{\longmapsto} z(\Phi, \chi, X)) \in \Lambda$ and non-zero

because $P_\chi(X)$ is non-zero.

Exercise: $\lambda_1 : \Phi \longmapsto z(\hat{\Phi}, \hat{\chi}, \frac{1}{q}X)$

$\in \Lambda$.

It is enough to prove that Λ is 1-dimensional as a $\mathbb{C}(X)$ vector space.

~~Pf~~ We take $n \geq 0$, s.t. $U_\sigma^n \subseteq \ker \chi$.

<u>Claim</u>: The map $\Lambda \longrightarrow \mathbb{C}(X)$

$\lambda \longmapsto \lambda(\mathbb{1}_{U_\sigma^n})$

is injective. (Then it is bijective)

<u>Pf of the Claim</u>: If ~~$\lambda(\mathbb{1}$~~ $\lambda(\mathbb{1}_{U_\sigma^n}) = 0$

then by the definition of Λ we have for $k \geq n$

and $a \in U_\sigma^n$ $\lambda(\mathbb{1}_{U_\sigma^k}(a^{-1}*)) = \lambda(\mathbb{1}_{aU_\sigma^k})$

$\qquad \| $

$\chi(a)\, X^{\nu_F(a)} \lambda(\mathbb{1}_{U_\sigma^k}) = \lambda(\mathbb{1}_{U_\sigma^k})$

$$\Rightarrow \lambda(\mathbb{1}_{U^k_\sigma}) = \frac{1}{|U^n_\sigma / U^k_\sigma|} \cdot \lambda(\mathbb{1}_{U^n_\sigma}) = 0$$

The ~~indicator~~ characteristic functions $\mathbb{1}_{U^k_\sigma}$ $k \geq n$ span $C_c^\infty(F^\times)$. Thus λ is zero on $C_c^\infty(F^\times)$ and thus $\lambda(\Phi)$ only depends on $\Phi(0)$.

$$\Rightarrow \lambda(a\Phi) = \lambda(\Phi) \quad \forall a \in F^\times \ \forall \Phi \in C_c^\infty(F)$$

Take a with $v_F(a) \neq 0$. $\Rightarrow$ λ is zero.

□ Claim.

This finishes the Theorem □

We define:

$$\zeta(\Phi, \chi, s) = Z(\Phi, \chi, q^{-s}) = \int_{F^\times} \Phi(x)\chi(x)\|x\|^s \, d\mu^\times(x)$$

$$L(\chi, s) = P_\chi(q^{-s})$$

$$\gamma(\chi, s, \psi) = c(\chi, \psi, q^{-s})$$

$$\varepsilon(\chi, s, \psi) := \gamma(\chi, s, \psi) \frac{L(\chi, s)}{L(\tilde{\chi}, 1-s)}$$

VI 2. Construction for $GL_2(F) =: G$

$A = M_2(F)$. $\quad \psi_A = \psi \circ \mathrm{tr}_{A/F}$.

$$\Phi \longmapsto \hat{\Phi} \qquad \hat{\Phi}(x) = \int_A \Phi(x)\, \psi_A(xy)\, d\mu(y)$$

Fourier transform of Φ. This gives a map $C_c^\infty(A) \to C_c^\infty(A)$.

There is again a unique Haar measure μ_ψ^A such that $\hat{\hat{\Phi}}(x) = \Phi(-x)$. μ_ψ^A is called the ~~Ha~~ self-dual relative to ψ. We have $\mu_\psi^A(\mathcal{M}) = q^{2l}$ $\quad \mu_{a\psi}^A = \|a\|^2 \mu_\psi^A$.

Let $(\pi, V) \in \mathcal{R}(G)$ be irreducible.

$C(\pi) :=$ $\mathbb{C}$-v.s. of matrix coeff of π.

We consider ~~f~~ integrals:

$$\zeta(\Phi, f, s) = \int_G \Phi(x) f(x) \|\det(x)\|^s d\mu^*(x).$$

Fact 7.01: $\exists s_0 \in \mathbb{R}$: ~~$\zeta(\Phi, f, s)$ converges~~
$\zeta(\Phi, f, s)$ converges absolutely and uniformly ~~on~~ in vertical strips in $\mathrm{Re}(s) > s_0$.

$$\tilde{Z}(\pi) = Z(\pi, \chi) := \{ \zeta(\Phi, f, s+\tfrac{1}{2}) \mid \Phi \in C_c^\infty(A), f \in C(\pi) \}$$

is a $\mathbb{C}[q^s, q^{-s}]$ module, ~~generated by a~~ ~~$P_\pi(X)$ $P_\pi(q^{-s})^{-1}$ which o.~~
and there is a unique polynomial $P_\pi(X) \in \mathbb{C}[X]$ satisfying $P_\pi(0)=1$, such that $P_\pi(q^{-s})^{-1}$ generates $Z(\pi)$ over $\mathbb{C}[q^{-s}; q^s]$.

<u>We define</u> ~~$\zeta(\hat{\Phi}$~~ $L(\pi, s) := P_\pi(q^{-s})^{-1}$.

<u>We come to the local constant:</u>

We have
$$C(\pi) \longrightarrow C(\tilde{\pi})$$
$$f \longmapsto \tilde{f} \qquad \tilde{f}(g) = f(g^{-1})$$

<u>Thm 100:</u> There is a unique rational function $\gamma(\pi, s, \psi) \in \mathbb{C}(q^{-s})$ such that

$\forall \Phi \in C_c^\infty(A)\ \forall f \in C(\pi)$:

$$\zeta(\hat{\Phi}, \tilde{f}, \tfrac{3}{2}-s) = \gamma(\pi, s, \psi)\, \zeta(\Phi, f, \tfrac{1}{2}+s)$$

We define $\quad \varepsilon(\pi, s, \psi) := \gamma(\pi, s, \psi) \dfrac{L(\pi, s)}{L(\tilde{\pi}, 1-s)}.$

<u>Prop 101</u> If Π is cuspidal irr. then $L(\Pi,s)=1$.

<u>Proof:</u> To ~~proof~~ prove the existence of P_Π the authors use again the construction

$$\mathcal{Z}(\pi) := \{ Z(\Phi, f, q^{-1/2} X) \mid \Phi \in C_c^\infty(A), f \in C(\Pi) \}$$

where $Z(\Phi, f, X) := \sum_{m \in \mathbb{Z}} z_m(\Phi, f) X^m$

and $z_m(\Phi, f) := \int_G \Phi 1\!\!1_{G_m}(x) \, f(x) \, d^\times x$

$$G_m := \{ g \in G \mid \nu_F(\det(g)) = m \}$$

and the equivalent assertion to Fact 101 is $\mathcal{Z}(\Pi) = P_\Pi(X)^{-1} \mathbb{C}[X, X^{-1}]$

The claim is that in the cuspidal case we have $\mathcal{Z}(\Pi) = \mathbb{C}[X, X^{-1}]$, i.e. $P_\Pi(X) = 1$.

<u>Step 1:</u> It is enough to consider any $f_0 \in C(\Pi) \setminus \{0\}$ to generate $\mathcal{Z}(\Pi)$ as a $\mathbb{C}[X, X^{-1}]$-module, i.e. $\mathcal{Z}(\Pi) = \{ Z(\Phi, f_0, q^{1/2} X) \mid \Phi \in C_c^\infty(A) \}$

This is because $C(\pi)$ is irreducible as a $G\times G$ module and

$$Z((g,h)\Phi, (g,h)f, X) = X^{\nu_F(\det gh^{-1})} Z(\Phi, f, X).$$

Step 2: Take a cuspidal type $\subseteq \pi$.

$(\mathfrak{A}, \Lambda, J)$

$$\Theta := \mathrm{Ind}_J^{\mathfrak{A}} \Lambda$$

Fact: Θ is irreducible, because J is compact mod and contains the center ~~and~~ of G and Λ is irred.

V^{Θ} is the isotypic component of V

Take $v \in V^{\Theta}$ and $\tilde{v} \in \tilde{V}^{\tilde{\Theta}}$.

$$f_0(g) := \tilde{v}(\pi(g)v), \quad f_0 \in C(\pi).$$

Take $v, \tilde{v}$, s.t. $f_0(1) = 1$.

Exercise $Z(\Phi, f_0, X) = Z(e_\Theta * \Phi * e_\Theta, f_0, X)$

for $\Phi \in C_c^\infty(A)$. $(*)$

We need the following lemma.

Lemma: (1) $Z(\mathbb{1}_{\mathfrak{a}}, f_0, X) = 0$

(2) Let $\Phi \in C_c^\infty(A)$ and $e_\Theta * \Phi * e_\Theta = \Phi$

Then $\Phi \in C_c^\infty(G)$ and $\operatorname{supp}(\Phi) \subseteq$ normalizer of $\mathfrak{a} = \mathcal{N}(\mathfrak{a}) = K_{\mathfrak{a}}$.

(3) $\forall \Phi \in C_c^\infty(A)$: $Z(\Phi, f_0, X) \in \mathbb{C}[X, X^{-1}]$

for all $\Phi \in C_c^\infty(A)$.

Pf: (1) $\Phi = \mathbb{1}_{\mathfrak{a}}$ invariant und action of $U_{\mathfrak{a}} \times U_{\mathfrak{a}}$.

$\Rightarrow \Phi \in C(\pi)^{U_{\mathfrak{a}} \times U_{\mathfrak{a}}}$

$\Rightarrow$ $e_\Theta * \Phi * e_\Theta = 0 \underset{(*)}{\Rightarrow} Z(\mathbb{1}_{\mathfrak{a}}, f_0, X) = 0$.

$\uparrow$

$\Theta \otimes \Theta \neq \mathbb{1}_{U_{\mathfrak{a}} \times U_{\mathfrak{a}}}$

(2) $e_\Theta * \Phi * e_\Theta = \Phi \Rightarrow \operatorname{supp} \Phi \cap G$

$\subseteq$ ~~supp~~ $I_G(\Theta) \subseteq K_{\mathfrak{a}}$.

To show $\operatorname{supp}\Phi \subseteq G$. If it contains a singular matrix, then a neighbourhood and thus a diagonal matrix $\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}$ with $\|a_1\| \neq \|a_2\|$ ↯ because $\operatorname{supp}\Phi \cap G \subseteq \mathcal{N}(\mathfrak{a})$.

From this follows that $\operatorname{supp}\Phi \subseteq$ finite union of $\Pi^r \mathcal{U}_{\mathfrak{a}}$ where $\Pi \in \mathcal{N}(\mathfrak{a})$:

$\Pi\,\mathfrak{a} = \mathfrak{p}_{\mathfrak{a}} = \operatorname{Rad}(\mathfrak{a})$.

$\Rightarrow Z(\Phi, f_0, X) \in \mathbb{C}[X, X^{-1}]$.

(3) follows from (2) and (*) □

From (3) follows Prop 102. □

VI 3. L-functions and local constant on the Galois side

1-dim W_F-repr.: Local class field theory

$\Rightarrow$ There is a top. isomorphism

$\bar{a}_F : W_F^{ab} \xrightarrow{\sim} F^\times$ and inflation $a_F : W_F \twoheadrightarrow F^\times$

s.t. $a_F(\tilde{I}_F) = \sigma_F^\times$

and $a_F(P_F) = U^1_{\sigma_F}$

and $a_F(x)$ is a geometric Frobenius $\Leftrightarrow$

$a_F(x)$ is a uniformiser of F.

a_F = "Artin reciprocity map"

For $\chi \in \hat{F}^\times$ we define $L(\chi \circ a_F, s) = L(\chi, s)$

$\varepsilon(\chi \circ a_F, s, \psi) = \varepsilon(\chi, s, \psi)$

Extending the definition to $\mathcal{G}_n^0(F)$: $\sigma \in \mathcal{G}_n^0(F)$

$n \geq 2$: $L(\sigma, s) := 1$.

For $\sigma = \sigma_1 \oplus \sigma_2 \in \mathcal{G}_n^{ss}(F)$: $L(\sigma_1 \oplus \sigma_2, s) = L(\sigma_1, s) L(\sigma_2, s)$

The definition of the local constant more difficult.

$$\mathcal{G}^{ss}(F) = \bigcup_{n \geq 1} \mathcal{G}_n^{ss}(F)$$

For a finite ext. $E/F \subseteq \bar{F}/F$ we define $\psi_E := \psi_F \circ \mathrm{tr}_{E/F}$.

<u>Thm 103:</u> Let $\psi \in \hat{F}$, $\psi \neq 1$. There is a unique family $(\varepsilon(*, s, \psi_E))_{E/F \text{ finite}}$ of functions

$$\mathcal{G}^{ss}(E) \longrightarrow \mathbb{C}[q^s, q^{-s}]^\times$$

$$\rho \longmapsto \varepsilon(\rho, s, \psi_E)$$

such that

- $\forall_{\chi \in \hat{E}^\times}$: $\varepsilon(\chi, s, \psi_E) = \varepsilon(\chi \circ \sigma_E, s, \psi_E)$
- $\forall_{\rho_1, \rho_2 \in \mathcal{G}^{ss}(E)}$: $\varepsilon(\rho_1 \oplus \rho_2, s, \psi_E)$
$$= \varepsilon(\rho_1, s, \psi_E)\, \varepsilon(\rho_2, s, \psi_E)$$
- $\forall_{\rho \in \mathcal{G}^{ss}(E)}$ and $E \supseteq K \supseteq F$, then
$$\frac{\varepsilon(\mathrm{Ind}_{E/K}\, \rho, s, \psi_K)}{\varepsilon(\rho, s, \psi_E)} = \frac{\varepsilon(\mathrm{Ind}_{E/K} 1\!\!1_{W_E}, s, \psi_K)}{\varepsilon(1\!\!1_E, s, \psi_E)}.$$

VI.4. Calculation of local constants in the 1-dim. case

Lemma 104: Let ψ be an additive character of F of level 1 and χ be an unramified char. of $\hat{F}^\times$. Then $\varepsilon(\chi, s, \psi) = q^{s-\frac{1}{2}} \chi(\varpi_F)^{-1}$

Proof: We take $\Phi = \mathbb{1}_{\mathcal{O}}$

Last lecture $\hat{\Phi} = \mathbb{1}_{\mathfrak{p}^{\ell-\sigma}} \mu_\psi(\mathcal{O}) q^{-0}$

$$= \mathbb{1}_{\mathfrak{p}} q^{\frac{1}{2}}$$

$(\mu_\psi(\mathcal{O}) = q^{\frac{\ell}{2}} \wedge \ell = 1)$

"L-function in X"

$$\varepsilon(\chi, X, \psi) = C(\chi, X, \psi) \frac{Z(\mathbb{1}_{\mathcal{O}}, \chi, X)}{Z\left(\mathbb{1}_{\mathcal{O}}, \tilde{\chi}, \frac{1}{qX}\right)}$$

$$= \frac{Z\left(\hat{\mathbb{1}}_{\mathcal{O}}, \tilde{\chi}, \frac{1}{qX}\right)}{Z(\mathbb{1}_{\mathcal{O}}, \chi, X)} \cdot \frac{Z(\mathbb{1}_{\mathcal{O}}, \chi, X)}{Z\left(\mathbb{1}_{\mathcal{O}}, \tilde{\chi}, \frac{1}{qX}\right)}$$

$$= \frac{Z(\hat{\mathbb{1}}_{\sigma}, \tilde{\chi}, \frac{1}{q\gamma})}{Z(\mathbb{1}_{\sigma}, \tilde{\chi}, \frac{1}{q\gamma})}$$

$$Z(\hat{\mathbb{1}}_{\sigma}, \tilde{\chi}, \frac{1}{q\gamma}) = q^{\frac{1}{2}} Z(\mathbb{1}_{\mathfrak{p}}, \tilde{\chi}, \frac{1}{q\gamma})$$

$$= q^{\frac{1}{2}} \sum_{m=1}^{\infty} \tilde{\chi}(\varpi_F)^m \frac{1}{(q\gamma)^m}$$

$$= q^{\frac{1}{2}} \frac{1}{\chi(\varpi_F) q\gamma} \sum_{m=0}^{\infty} \cdots = \frac{q^{\frac{1}{2}}}{\chi(\varpi_F) q\gamma} Z(\mathbb{1}_{\sigma}, \tilde{\chi}, \frac{1}{q\gamma})$$

$$\Rightarrow \varepsilon(\chi, q^{-s}, \psi) = \chi(\varpi_F)^{-1} q^{s-\frac{1}{2}}. \qquad \square$$

<u>Theorem 105:</u> Suppose $\chi \in \hat{F}^\times$ has level $n \geq 0$ and is <u>not</u> unramified. $\psi \in \hat{F}$ with level 1. Then

$$\varepsilon(\chi, s, \psi) = q^{n(\frac{1}{2} - s)} \, \tau(\chi, \psi) / q^{(n+1)/2}$$

where $\tau(\chi, \psi) := \sum_{x \in \sigma^\times / U^{n+1}(\sigma)} \hat{\chi}(\alpha x) \, \psi(\alpha x)$

for any $\alpha \in \mathfrak{p}^{-n}$ ~~s.t. $\chi(1+x) = \psi(\alpha x)$~~

~~on $\mathfrak{p}^n$~~ s.t. $v_F(\alpha) = -n$.

$\tau(\chi, \psi)$ is called <u>Gauß sum of χ relative to ψ.</u>

<u>Proof</u>: $\Phi = 1\!\!1_{U^{n+1}(\mathcal{O})}$

$$\zeta(\Phi, \chi, s) = \int_{F^\times} \Phi(x)\chi(x)\|x\|^s d\mu^*(x)$$

$$= \int_{U^{n+1}(\mathcal{O})} d\mu^*(x) = \mu^*(U^{n+1}(\mathcal{O})).$$

An easy calculation shows $\hat{\Phi}(y) = 1\!\!1_{\mathfrak{p}^{-n}} q^{-n-1} \cdot q^{\frac{1}{2}} \cdot \psi(y)$

⌜$\psi(y)$ comes from a substitution $x-1 \to x$.⌟

$$\Rightarrow \zeta(\hat{\Phi}, \chi^{-1}, s) = q^{\frac{1}{2}-n-1} \int_{\mathfrak{p}^{-n} - \{0\}} \psi(y)\chi(y)^{-1}\|y\|^s d^* y$$

$$= \sum_{m \geq -n} z_m q^{-ms} \cdot q^{\frac{1}{2}-n-1}$$

with $z_m = \int_{\mathfrak{p}^m - \mathfrak{p}^{m+1}} \psi(y)\chi(y)^{-1} d^* y$

The coeff for $\underline{m=-n}$ is

$$z_{-n} = \int_{\mathcal{O}^\times} \psi(\alpha x)\, \chi^{-1}(\alpha x)\, d^\times x$$

for $\alpha \in \mathfrak{p}^{-n} \setminus \mathfrak{p}^{1-n}$.

$$= \sum_{\mathcal{O}^\times / U^{n+1}(\mathcal{O})} \chi^{-1}(\alpha x)\, \psi(\alpha x) \cdot \mu^\times(U^{n+1}(\mathcal{O})).$$

The coeff for $m > -n$: For $\beta \in F$ with $\nu_{\mathfrak{p}}(\beta) = m$ we have $z_m = \int_{\mathcal{O}^\times} \psi(\beta x)\, \chi(\beta x)^{-1} d\mu^\times(x)$.

We show for $v \in U^n(\mathcal{O})$: $z_m = \chi(v)^{-1} z_m$. (*)

From this follows $z_m = 0$, because one can take v s.t. $\chi(v) \neq 1$, by $l(\chi) = n$.

$$z_m \underset{\substack{\uparrow \\ \text{Haar} \\ \text{measure}}}{=} \int_{\mathcal{O}^\times} \psi(\beta x v)\, \chi(\beta x v)^{-1} d\mu^\times(x)$$

$$\underset{\uparrow}{=} \int_{\mathcal{O}^\times} \psi(\beta x)\, \chi(\beta x v)^{-1} d\mu^\times(x)$$

$(m+n>0 \Rightarrow \psi(\beta x v) = \psi(\beta x) \cdot \psi(\beta x(v-1)) = \psi(\beta x))$

$$= \chi(v)^{-1} z_m$$

Thus $\zeta(\hat{\Phi}, \tilde{\chi}, s) = q^{\frac{1}{2}-n-1} \mu^*(U^{n+1}(\sigma))$

$$\cdot \sum_{\sigma^* / U^{n+1}(\sigma)} \chi(\alpha x)^{-1} \psi(\alpha x) \cdot q^{ns}$$

$\Rightarrow$
$$\varepsilon(\chi, s, \psi) = \zeta(\hat{\Phi}, \check{\chi}, 1-s) \cdot \zeta(\Phi, \chi, s)^{-1}$$
$$= \tau(\chi, \psi)\, q^{\frac{1}{2}-n-1+n(1-s)}$$
$$= q^{n(\frac{1}{2}-s)}\, \tau(\chi, \psi) / q^{(n+1)/2}$$

$\square$

IV 5. Calculation of local factors for principal series repr.

Theorem E 1: (26.1. $GL_2(2)$) (E for extra)

Let $\chi = \chi_1 \otimes \chi_2 \in \hat{T}$ and π be a comp. factor of $i_B^G \chi$ and $\psi \in \hat{F}$, $\psi \neq 1$. Then

$$L(\pi, s) = L(\chi_1, s)\, L(\chi_2, s)$$

$$\varepsilon(\pi, s, \psi) = \varepsilon(\chi_1, s, \psi)\, \varepsilon(\chi_2, s, \psi)$$

except for the case $\pi = \phi \cdot St_G$.

in this case is

$$L(\pi, s) = L(\phi, s + \tfrac{1}{2})$$

$$\varepsilon(\pi, s, \psi) = -\varepsilon(\phi, s, \psi).$$

IV 6. ~~Uniqueness theorems~~ The Converse theorem

Converse Theorem E2: (27.1. GL(2)) Let $\psi \in \hat{F}$, $\psi \neq 1$, $\pi_1, \pi_2 \in \mathcal{R}(GL_2(F))$ irred.. Suppose

$$L(\chi\pi_1, s) = L(\chi\pi_2, s) \text{ and } \varepsilon(\chi\pi_1, s, \psi) = \varepsilon(\chi\pi_2, s, \psi)$$

for all χ of $F^\times$. Then $\pi_1 \cong \pi_2$.

Sketch of strategy of the proof:

- π is ~~principal~~ cuspidal $\Leftrightarrow$ $L(\pi\phi, s) = 1$ $\forall \phi \in \widehat{F^\times}$.

- π principal series / π cuspidal. (very difficult)

The principal series case follows essentially from E.1., Say for example that $\pi \cong \iota_B^G(\chi_1 \otimes \chi_2)$ and $L(\pi, s) \neq 1$. If $L(\pi, s)$ has degree 2 ~~then~~ (the degree of the polynomial) then χ_1 and χ_2 are unramified determined by $L(\pi, s) = L(\chi_1, s) L(\chi_2, s)$

~~Say $L(\pi,s) = L(\theta,s)$ has degree one~~

- Say π is not special and $L(\pi,s)$ has degree one $\Rightarrow$ w.l.o.g. χ_2 is ramified, so $L(\pi,s) = L(\chi_1,s)$. Then we get χ_2 back because $\exists$ ϕ ramified: $L(\phi\pi,s) \neq 1$ (e.g. χ_2^{-1})

  $\Rightarrow$ $\exists$ θ unramified: $L(\theta,s) = L(\phi\pi,s)$.

  Then $\chi_2 = \phi^{-1}\theta$.

~~If $\pi = \theta\, St_G$ is special then $\forall$ ϕ ramified: $L(\phi\pi,s) = 1$, because $L(\phi\pi,s) = L(\phi\theta|\cdot|^{+\frac{1}{2}},s)$~~

- Say π is special. $L(\pi,s) = L(\theta,s)$. thus $\pi = \theta|\cdot|^{\frac{1}{2}} St_G$.

  It distinguishes from the case before because $L(\phi\pi,s) = L(\phi\theta,s) = 1$ $\forall$ $\phi \in \widehat{F^\times}$ ramified. $\square$

VII The Langlands Correspondence

$\mathcal{A}_2(F) :=$ "set of equivalence classes of smooth irred representations of $G = GL_2(F)$"

$\mathcal{G}_2(F) =$ "set of equivalence classes of 2-dim semisimple Deligne representations"

Recall: $a_F : \mathcal{W}_F \twoheadrightarrow F^\times$, s.t. $\bar{a}_F : \mathcal{W}_F^{ab} \xrightarrow{\sim} F^\times$ is a topological isomorphism of groups.

$\Rightarrow \widehat{\mathcal{W}_F} \cong \widehat{F^\times}$ via a_F

Thm 106 (Langlands Correspondence) Let $\psi \in \hat{F}$, $\psi \neq 1$.

There exists a unique map

$$LC : \mathcal{G}_2(F) \to \mathcal{A}_2(F)$$

such that for all $\chi \in \widehat{F^\times}$ we have $\forall \underline{\rho} \in \mathcal{G}_2(F)$

$$L(\chi LC(\rho), s) = L(\chi \rho, s)$$

and $\varepsilon(\chi LC(\rho), s, \psi) = \varepsilon(\chi \rho, s, \psi)$.

The map LC is called the Langlands Correspondence for G.

From now on we assume $p \neq 2$ for simplicity

Recall: on W_F

$$\mathcal{G}_2(F) = \mathcal{G}_2^1(F) \;\dot\cup\; \mathcal{G}_2^0(F)$$

- $\mathcal{G}_2(F)$: 2-dim semis. Deligne-repr.
- $\mathcal{G}_2^1(F)$: 2-dim semis. Deligne repr. (ρ, V, N) with red. ρ
- $\mathcal{G}_2^0(F)$: 2-dim irred smooth repr.

on $G = GL_2(F)$

$$\mathcal{A}_2(F) = \mathcal{A}_2^1(F) \;\dot\cup\; \mathcal{A}_2^0(F)$$

- $\mathcal{A}_2(F)$: irred.
- $\mathcal{A}_2^1(F)$: principal series irred
- $\mathcal{A}_2^0(F)$: Cusp irred

VII 1. The principal series part

. We construct

$$LC : \mathcal{G}^1_2(F) \longrightarrow \mathcal{A}^1_2(F).$$

Take $(\rho, V, N) \in \mathcal{G}^1_2(F)$.

ρ is semisimple $\Rightarrow$ $\rho \cong \chi_1 \oplus \chi_2$

$\chi := \chi_1 \otimes \chi_2 \in \hat{T}$.

⌜Recall ι^{uG}_B is the normalized parabolic induction

$$\iota^{uG}_B \chi = \operatorname{Ind}^G_B \delta_B^{\frac{1}{2}} \chi$$ ⌟

Def. $LC((\rho, V, N)) := \iota^{uG}_B \chi$ if $\underbrace{\chi_2^{-1}\chi_1 \neq \|x\|^{\pm 1}}_{(**)}$

i.e. if $\iota^{uG}_B \chi$ is irreducible.

$(**) \Rightarrow N = 0$, because ~~ker(N) is a sub~~ of the equation for N.

Suppose now $\chi_2^{-1}\chi_1 = \|x\|^{\pm 1}$, i.e. $\iota^{uG}_B \chi$ is not irreducible.

$\Rightarrow \exists \phi \in \widehat{F^\times}$: $\chi_1 = \phi \, \| \, \|^{-1/2}$ and

$\chi_2 = \phi \, \| \, \|^{1/2}$

$\exists$ two Deligne repr. in $\mathcal{G}_2^1(F)$ with

$\rho \cong \chi_1 \oplus \chi_2$. One with $N \neq 0$ and

one with $N = 0$.

$$LC(\rho, V, 0) := \phi \circ \det.$$

$$LC(\rho, V, \underset{\neq 0}{N}) := \phi \, St_G. \qquad \square$$

Strategy: • One introduces the set $\mathbb{P}_2(F)$ eq. class of admissible pairs, i.e. of certain pairs $(E/F, \xi)$ of a quadratic field ext. and $\xi \in \hat{E}^\times$ ~~(which is supp not to~~, s.t. ξ is not a restriction of a character of W_F.

• One constructs

$$(E/F, \xi) \longmapsto \pi_\xi$$

$$\mathbb{P}_2(F) \xrightarrow{\sim} \mathcal{A}_2^0(F)$$

$\cong$ cl. irred cusp. 2-dim repr.

and $\mathbb{P}_2(F) \xrightarrow{\sim} \mathcal{G}_2^0(F)$

$\cong$ classes smooth irred 2-dim repr. of W_F

$$(E/F, \xi) \longmapsto \mathrm{Ind}_{E/F}\, \xi =: \rho_\xi$$

$\Rightarrow$ One get a map $\mathcal{G}_2^0(F) \longrightarrow \mathcal{A}_2^0(F)$

$$\rho_\xi \longmapsto \pi_\xi$$

CAUTION: This map is not LC $\omega_{\pi_\xi} \neq \det \rho_\xi$

So one modifies ξ by a level 0 character Δ_ξ of $E^\times$

$$\mathcal{G}_2^0(F) \xrightarrow{\sim} \mathbb{P}_2(F) \xrightarrow{\sim} \mathbb{P}_2(F) \xrightarrow{\sim} \mathcal{A}_2^0(F)$$

$$\varrho_\xi \longmapsto \xi \longmapsto \xi\Delta_\xi \longmapsto \pi_{\xi\cdot\Delta_\xi}$$

This gives the LC.

So we have to explain:

- The construction of π_ξ
- — " — of Δ_ξ.

VII 2.1. Admissible pairs and π_ξ

Def 107: A pair $(E/F, \xi)$ of a quadratic extension E/F and $\xi \in \hat{E}^\times$, s.t.

- ξ does not factorize through $N_{E/F}$
- if $\xi|_{U^1(\sigma_E)}$ " " , then E/F is unramified.

Rk 108: The first property says that for $\overset{id}{\underset{\times}{\sigma}} \in Gal(E/F)$ we have $\xi^\sigma \neq \xi$, i.e. $\rho_\xi := Ind_{E/F}\, \xi$ is irreducible.

Rk 109: $(E/F, \xi)$ is called minimal if $\xi|_{U^{l(\xi)}(\sigma)}$ does not factorizes through the determinant.

$\mathbb{P}_2(F) :=$ "set of ~~equi~~ isomorphism classes of admissible pairs".

$$\begin{array}{ccc} E_1 & \xrightarrow{\sim} & E_2 \\ | & & | \\ F & \overset{id}{=} & F \end{array}$$

Construction of π_ξ: <u>Case 1:</u> $l(\xi) = 0$

$$\Rightarrow \xi|_{U^1(\sigma_E)} = 1\!\!1_{U^1(\sigma_E)}$$

thus factors through $N_{E/F}$.

$\Rightarrow$ E/F is unramified by definition of admissible pairs. $[\mathbb{E}:\mathbb{F}]=2$.

We have to construct a cuspidal type.

- $\mathfrak{p}_E^{\mathbb{Z}}$ is a lattice chain in F^2

$\rightsquigarrow$ we get a hereditary order $\mathfrak{A}$

E/F is unramified $\Rightarrow$ ϖ_F is uniformizer for $\mathfrak{p}_E^{\mathbb{Z}}$

$\Rightarrow$ $\varpi_F \mathfrak{A} = \mathfrak{P}_{\mathfrak{A}} = \mathrm{Rad}(\mathfrak{A})$. $\Rightarrow$ $\mathfrak{A}$ is a maximal order.

We conjugate $\mathfrak{A}$ to $\mathfrak{M} = M_2(\mathfrak{o}_F)$.

ξ reduces to $\bar{\xi}$ on $\mathbb{E}$.

Theory of $GL_2(\mathbb{F})$-representation and $\bar{\xi}^\sigma \neq \bar{\xi}$ for the Gal(E/F)-generator σ $\Rightarrow$ $\exists$ irred cusp. repres. $\pi_{\bar{\xi}}$ of $GL_2(\mathbb{F})$: $\pi_{\bar{\xi}} = \mathrm{Ind}_{ZN}^{G} \bar{\xi}_{\bar{\psi}} - \mathrm{Ind}_{\mathbb{E}}^{G} \bar{\xi}$

$\rightsquigarrow$ Inflate to $\mathfrak{M} = GL_2(\mathfrak{o}_F)$

$\rightsquigarrow$ Extend to Λ on $F^\times GL_2(\mathfrak{o}_F)$ s.t. $\Lambda|_{F^\times}$ is a multiple of ξ.

$\rightsquigarrow$ $(\mathfrak{M}, \mathcal{K}(\mathfrak{M}), \Lambda)$ is a cuspidal type.

Define $\pi_\xi := c\text{-}\mathrm{Ind}_{\mathcal{K}(\mathfrak{M})}^{G} \Lambda$

Case $(E/F, \xi)$ is minimal and $\ell(\xi) = n \geq 1$

We define $\psi_E := \psi \circ \mathrm{Tr}_{E/F}$ and $\psi_A := \psi \circ \mathrm{tr}_{A/F}$.

We take $\alpha \in \mathfrak{p}_E^{-n}$: $\xi(1+x) = \psi_E(\alpha x)$,

$x \in \mathfrak{p}_E^{\lfloor n/2 \rfloor + 1}$.

We find a simple stratum as follows.

- $E \hookrightarrow A$
- $\mathfrak{A}$ the unique hereditary order, s.t. $E^\times \subseteq \mathcal{K}(\mathfrak{A})$

$(\mathfrak{A}, n, \alpha)$ is a simple stratum, because α is minimal.

If $n = 2m+1$ is odd, one takes a character Λ on $E^\times U_{\mathfrak{A}}^{m+1}$, s.t. $\Lambda|_{U_{\mathfrak{A}}^{m+1}} = \psi_\alpha$ and $\Lambda|_{E^\times} = \xi$

If $n = 2m$ is even: There $\exists!$ irred repr. Λ of J_α ($J_\alpha := E^\times U_{\mathfrak{A}}^m$) s.t. $\cdot\ \Lambda|_{U_E^1 U_{\mathfrak{A}}^m}$ is irred. and on $U_{\mathfrak{A}}^{m+1}$ a multiple of ψ_α. and

- $\Lambda|_{E^\times}$ is a multiple of ξ on $F^\times$ and $U_{\mathfrak{o}_E}^1$
- and an extra cond. on the roots of unity in $\mathfrak{o}_E^\times - \mathfrak{o}_F^\times$.

i.e. $\operatorname{tr}\Lambda(x) = -\xi(x)$ $\forall\, x \in \mu_E \setminus \mu_F$ (μ_E root of unity in $E^\times$ of order prime to p)

Define: $\pi_\xi := \text{c-Ind}_{J_\alpha}^{G} \Lambda$.

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Case 3 $(E/F, \xi)$ admissible pair arbitrary.

$\Rightarrow \exists\, \phi \in F^\times$: $\phi_E \xi$ is minimal

$$\pi_\xi := \phi^{-1} \pi_{\phi_E \xi}$$

In all cases we have $\omega_{\pi_\xi} = \xi|F^\times$.

<u>Remark:</u> the condition comes from extending from J_α^1 to $J_\alpha^1 \sigma_E^\times$

$$\underset{\psi_\alpha}{U_\alpha^{m+1}} \subseteq \underbrace{U_\alpha^{m+1} U^1_{\sigma_E}}_{H_\alpha^1,\ \psi_\alpha \xi} \subseteq \underbrace{U_\alpha^{m} U^1_{\sigma_E}}_{J_\alpha^1,\ \eta \text{ (unique)}}$$

$$\subseteq U_\alpha^m \sigma_E^\times \subseteq J_\alpha$$

$\uparrow$

One cannot just take ξ to extend, because conjugation by $\sigma_E^\times$ does not ~~fix η~~ act trivial on η, but ${}^{\gamma}\eta \cong \eta$ for $\gamma \in \sigma_E^\times$.

The character Δ_{ρ} $(\in \widehat{E^\times})$

We fix $\psi \in \widehat{F}$, ψ of level 1. (This is just for simplicity.)

We need to construct LC, s.t.

$$\forall \chi \in \widehat{F^\times}: \left(\varepsilon(\chi\rho, s, \psi) = \varepsilon(\chi L(\rho), s, \psi) \right. \\ \left. \forall \rho \in \mathcal{G}_2^{\circ}(F) \right) \quad (*)$$

A short observation: Say ρ and π ($\in \mathcal{A}_2^{\circ}(F)$) satisfy this property $(*)$. Then the two following facts show $\det(\rho) = \omega_\pi$

⌈ $\det(\rho): W_F^{ab} (\simeq F^\times) \longrightarrow \mathbb{C}^\times$ ⌋

a) Stability theorem (GL(2), 25.7): Let $\pi \in \mathcal{R}(G)$ be irred. cuspidal, $\chi \in \widehat{F^\times}$ of level $m > 2\ell(\pi)$. ~~Let~~ Let $c \in \mathfrak{p}_F^{-m}$:

$\chi(1+x) = \psi(cx)$ $\forall x \in \mathfrak{p}^{\lfloor \frac{m}{2} \rfloor + 1}$. Then

$$\varepsilon(\chi\pi, s, \psi) = \omega_\pi(c)^{-1} \, \cancel{\varepsilon(\pi, s, \psi)} \, \varepsilon(\chi \circ \det, s, \psi).$$

⌈ One dim. version: $\Theta \in \widehat{F^\times}$, ~~$\chi \in \widehat{F^\times}$, $c \in \mathfrak{p}$~~, $\ell(\Theta) \geq 0$

(χ, c, m) as above, s.t. $2\ell(\Theta) < \ell(\chi) = m$.

Then $\varepsilon(\chi\theta, S, \psi) = \theta(c)^{-1}\, \varepsilon(\chi, S, \psi)$ $\lrcorner$

(This follows from $\tau(\theta\chi, \psi) = \theta(c)^{-1}\tau(\chi, \psi)$ for the Gaußsums)

b) Prop.: $\rho \in \mathcal{G}^{ss}(F)$. $\exists\, n_\rho \in \mathbb{N}: \forall_{\chi \in \hat{F^\times}} : \ell(\chi) \geq n_\rho$:

$$\varepsilon(\chi\rho, S, \psi) = \det(\rho(c))^{-1}\, \varepsilon(\chi, S, \psi)^{\dim \rho}$$

i.e. in the case of $\rho \in \mathcal{G}_2(F)$ we get:

$$\varepsilon(\chi\rho, S, \psi) = \det(\rho(c))^{-1}\, \varepsilon(\chi, S, \psi)^2$$

$$= \det(\rho(c))^{-1}\, \|c\|^{-1/2}\, \varepsilon(\chi\|\cdot\|^{-1/2}, S, \psi) \cdot \|c\|^{1/2}\, \varepsilon(\chi\|\cdot\|^{1/2}, S, \psi)$$

$$= \det(\rho(c))^{-1}\, \varepsilon(\chi\|\cdot\|^{-1/2}, S, \psi)\, \varepsilon(\chi\|\cdot\|^{1/2}, S, \psi)$$

$$= \det(\rho(c))^{-1}\, \varepsilon(\chi \circ \det, S, \psi).$$

Thus from (*) for ρ and π follows

$$\det(\rho) = \omega_\pi$$

Let (E, ξ) be an admissible pair.

We have $\omega_{\pi_\xi} = \xi|_{F^\times}$ and

$\det(\rho_\xi) = \varkappa_{E/F} \cdot \xi|_{F^\times}$, where

$\varkappa_{E/F}$ is the non-trivial character of $E^\times$ which is trivial on $N_{E/F}(E^\times)$.

The reason is

$$\begin{array}{ccc} W_F^{ab} & \xrightarrow[\sim]{a_F} & F^\times \\ {\scriptstyle \mathrm{ver}_{E/F}} \downarrow & \circlearrowleft & \downarrow \mathrm{incl.} \\ W_E^{ab} & \xrightarrow[\sim]{a_E} & E^\times \end{array}$$

⊣

LL constant

So we ~~need to~~ modify ξ ~~P_ξ~~ ~~at~~ to $\xi\Delta_\xi$, where Δ_ξ is a character of $E^\times$ of level zero.

<u>Case 1</u>: E/F unramified: Then $\varkappa_{E/F}$ ~~has~~ is unramified, because $\mathcal{O}_F^\times \subseteq N_{E/F}(E^\times)$.) Then we just take $\Delta_\xi := \varkappa_{E/F}$.

<u>Case 2</u>: If E/F is ramified, then $\varkappa_{E/F}$ is a non-trivial character of $\mathcal{O}_F^\times / N_{E/F}(\mathcal{O}_E^\times)$.

The construction of Δ_ξ is more involved, because we also need to ensure the equality of the local constants at the end.

Take $\varpi_E \in E$ a uniformizer of E. Then

$$\mathfrak{o}_E = \mu_E\, U_E^1 \underset{\uparrow}{=} \mu_F\, U_E^1$$

E/F ramified.

Thus $\forall x \in E^\times$: $\quad x\, \varpi_E^{-\nu_E(x)} \equiv \zeta(x, \varpi_E)$ mod U_E^1

for some $\zeta(x, \varpi_E) \in \mu_F$.

(GL(2), 34.4)

<u>Two step definition of Δ_ξ in the ramified case</u>

(1) $(E/F, \xi)$ minimal and ramified.

$n := \ell(\xi)$ and $\alpha \in \mathfrak{p}_E^{-n}$, i.e. $\xi(1+x) = \psi_E(\alpha x)$ on $1+\mathfrak{p}_E^n$. Then $\exists!\, \Delta_\xi \in \hat{E}^\times$:

$$\Delta|_{U_E^1} = 1\!\!1, \quad \Delta|_{F^\times} = \varkappa_{E/F}$$

$$\Delta(\varpi_E) = \varkappa_{E/F}(\zeta(\alpha, \varpi_E))\, \lambda_{E/F}(\psi)^n$$

Δ_ξ is independent of the choices of φ and α.

(2) $(E/F, \xi)$ ~~and~~ ramified.

$\xi = \xi' \cdot \chi_E$ for a minimal $(E/F, \xi')$

and $\chi \in \widehat{F^\times}$

Define $\Delta_\xi := \Delta'_\xi$.

Definition LC ~~for~~ on $\mathcal{G}^0_2(F)$:

$$LC(\rho_\xi) := \pi_{\xi \Delta_\xi}.$$

[Include in the Langlands constant

$$\lambda_{E/F}(\psi) := \frac{\varepsilon(\operatorname{Ind}_{E/F} 1\!\!1_{W_E}, s, \psi)}{\varepsilon(1\!\!1_{W_E}, s, \psi_E)}$$

[$\psi_E := \psi \circ \operatorname{Tr}_{E/F}$]

is called Langlands constant.

One of the properties of ε is:

$$\varepsilon(\operatorname{Ind}_{E/F} \rho, s, \psi)\, \varepsilon(\rho, s, \psi_E) = \lambda_{E/F}(\psi)$$

$\forall\, \rho \in \mathcal{G}^{ss}_{*}(E)$.

Using the functional equation

$$\varepsilon(\rho, s, \psi)\, \varepsilon(\tilde{\rho}, 1-s, \psi) = \det \rho(-1)$$

we get by $\operatorname{Ind}_{E/F} \widetilde{1\!\!1}_{W_E} = \operatorname{Ind}_{E/F} 1\!\!1$:

$\operatorname{Ind}_{E/F} 1\!\!1_{W_E}$

$$\lambda_{E/F}(\psi)^2 = \frac{\det \rho(-1)}{\rho(-1)} = \kappa_{E/F}(-1).$$

Thus $\lambda_{E/F}$ has order $|\,4$.

Prop A1: We have
$$\lambda_{E/F}(\psi) = \begin{cases} -1, & E/F \text{ unram.} \\ \tau(\kappa_{E/F}, \psi)/q^{1/2}, & E/F \text{ ramified.} \end{cases}$$

Proof: $\mathrm{Ind}_{E/F} \mathbb{1}_{W_E} = \mathbb{1}_{W_F} \oplus \kappa_{E/F}$.

$$\Rightarrow \lambda_{E/F}(\psi) = \frac{\varepsilon(\mathbb{1}_{W_F}, s, \psi)\, \varepsilon(\kappa_{E/F}, s, \psi)}{\varepsilon(\mathbb{1}_{W_E}, s, \psi)}$$

$$\underset{\text{Thm 104}}{=} \varepsilon(\kappa_{E/F}, s, \psi) \begin{cases} q^{s-\frac{1}{2}}/q^{2(s-\frac{1}{2})}, & E/F \text{ unram.} \\ 1, & E/F \text{ ramified.} \end{cases}$$

$$\underset{\text{Thm 104 and 105}}{=} \begin{cases} q^{-(s-\frac{1}{2})} \cdot q^{s-\frac{1}{2}} \cdot \underbrace{\kappa_{E/F}(\varpi_F)^{-1}}_{=-1}, & E/F \text{ unram.} \\ \tau(\kappa_{E/F}, \psi)/q^{1/2}, & E/F \text{ ramified.} \end{cases}$$

$\square$

B- Calculations on Δ_ξ

Independence of α and ψ:

$\psi' := u\psi \quad u \in \mathcal{O}_F^\times \rightsquigarrow \alpha' = u^{-1}\alpha$

~~$u\psi(x) =$~~

$u\psi(t) := \psi(tu)$

$$\Delta_\xi(\varpi_E) = \varkappa_{E/F}(\xi(u^{-1}\alpha, \varpi_E))\, \lambda_{E/F}(u\psi)^n$$

$$= \varkappa_{E/F}(u)^{-1}\, \varkappa_{E/F}(\xi(\alpha, \varpi_E))\, \varkappa_{E/F}(u)^{-n}$$

$$\lambda_{E/F}(\psi)$$

$$= \varkappa_{E/F}(\xi(\alpha, \varpi_E))\, \lambda_{E/F}(\psi).$$

↑

$2 \mid n+1$, because

$(E/F, \xi)$ is minimal and ramified.

⇝ Gives the desired independence.

~~Considering~~ Independence of choice of ϖ_E, ~~we~~

$$\Delta(u\varpi_E) = \Delta(u)\,\Delta(\varpi_E) = \Delta(u_0 \cdot v)\;\Delta(\varpi_E)$$

$u_0 \in U_E^1$, $v \in \mu_F$

$$= \varkappa_{E/F}(v)\, \varkappa_{E/F}(\xi(\alpha, \varpi_E))\, \lambda_{E/F}(\psi)^n$$

~~$= \varkappa_{E/F}(\xi(v\alpha, \varpi_E))\, \lambda_{E/F}(\psi)^n$~~

~~$= \varkappa_{E/F}(\xi(u\alpha, \varpi_E))\, \lambda_{E/F}(\psi)^n$~~

~~$\uparrow\; v\alpha = u_0 v\alpha\, u_E^1$~~

$$= \mathcal{U}_{E/F}(\nu)\ \varkappa_{E/F}(\nu^{-n}\ \xi(\alpha, u\varpi_E))$$ B-2

$$\lambda_{E/F}(\psi)^n$$

$$\underset{2 \nmid n}{\overset{\uparrow}{=}} \varkappa_{E/F}(\xi(\alpha, u\varpi_E))\ \lambda_{EF}(\psi)$$

Evidence. On $\mathcal{U}_E^1 \cdot F^\times$ ✓ We take ϖ_E skew-symm.

We define for $v \cdot t\varpi_E^{i} \in E^\times = \mathcal{U}_E^1 F^\times \varpi_E^{\mathbb{Z}}$

$$\Delta_\xi(u t\varpi_E) = \varkappa_{E/F}(t) \cdot \Delta_\xi(\varpi_E)^i.$$

For ~~consiste~~ welldef. : $\Delta_\xi(\varpi_E)^2 = \Delta_\xi(\varpi_E^2)$, for ϖ_E skew-symmetric.

$$\Delta_\xi(\varpi_E^2) = \Delta_\xi(-N_{E/F}(\varpi_E))$$

$$= \varkappa_{E/F}(-1) = \Delta_\xi(\varpi_E)^2.$$

□

A glimpse why this choice of Δ_ξ is correct.

Say we are in the ramified case

$$\varepsilon(\xi, \tfrac{1}{2}, \psi_E) = \xi(\alpha)\, \psi_E(\alpha)$$

(by analyzing the formula for $\tau(\xi, \psi_E)$ further, because n is odd)

$\pi = LC(\xi)$. We have an analogous formula

$$\varepsilon(\pi, \tfrac{1}{2}, \psi) = \tilde{\xi}(\alpha)\, \psi_A(\alpha)\, \tilde{\Delta}_\xi(\alpha)$$

To show $\tilde{\Delta}_\xi(\alpha) = \lambda_{E/F}(\psi)$. ~~Take ζ, s.t. $\alpha \varpi_E^n = \zeta$~~

$$\Delta_\xi(\alpha) \underset{\uparrow}{=} \varkappa_{E/F}(\zeta)\, \Delta(\varpi^{-n})$$

$$\alpha \varpi_E^n \equiv \zeta \in \mu_F$$

$$\Rightarrow \varkappa_{E/F}(\zeta)\, \Delta(\varpi^{-n})\, \lambda_{E/F}(\psi)$$

$$\underset{\uparrow}{=} \varkappa_{E/F}(\zeta)^{1-n} \left(\lambda_{E/F}(\psi)^n\right)^{-n} \lambda_{E/F}(\psi)$$

Def of $\Delta_\xi(\varpi_E)$

$$\underset{\substack{\uparrow \\ 2 \mid 1-n}}{=} 1 \cdot \lambda_{E/F}(\psi)^{1-n^2} \underset{\substack{\uparrow \\ 4 \mid 1-n^2}}{=} 1. \qquad \square$$

C- transfer $\tau_{E/F}$.

G a group. $H \leq G$

$$G/D(G) \xrightarrow{\mathrm{Ver}_{G/H}} H/D(H)$$

$$g\,D(G) \longmapsto \prod_{x \in G/H} h_{g,x}\, D(H)$$

where $t : G/H \to G$ is a section

and $h_{g,x}$ is defined via

$$g\,t(x) = t(gx)\, h_{g,x}.$$

$\mathrm{Ver}_{G/H}$ is a group homomorphism.

For E/F finite sep. and the artin ~~an~~ rec. map we have

$$\begin{array}{ccc} W_F^{ab} & \xrightarrow[\sim]{a_F} & F^\times \\ {\scriptstyle \mathrm{Ver}_{E/F}} \downarrow & \circlearrowleft & \cap \\ W_E^{ab} & \xrightarrow[\sim]{a_F} & E^\times \end{array}$$

c^{-2} Then we get as an exercise.

For $f_0, f_1 \in \mathrm{Ind}_{E/F}\,\xi$ and $x \in W_F$ s.t. $x^2 \in W_E$:

$$\rho(x) = \begin{pmatrix} 0 & 1 \\ \xi(\sigma_E(x^2)) & 0 \end{pmatrix}$$

$$\Rightarrow \det(\rho(x)) = -\xi(\sigma_E(x^2))$$

On the other hand

$$\xi(\sigma_F(x)) = \xi(\sigma_E(\mathrm{Ver}_{E/F}(\bar{x})))$$

$$= \xi(\sigma_E(x^2)) \neq -\xi(\sigma_E(x^2)).$$

$\uparrow$ $\mathrm{Ver}_{E/F}(\bar{x}) = \bar{x}^2$ because $x^2 \in W_E$

$$\Rightarrow \det\rho \cdot (\xi \circ \sigma_F)^{-1} \neq \text{trivial}, \text{ thus } = \mathcal{U}_{E/F}.$$