SEMISIMPLE CHARACTERS FOR INNER FORMS II: QUATERNIONIC FORMS OF p-ADIC CLASSICAL GROUPS (p ODD)

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Abstract. In this article we consider the set G of rational points of a quaternionic form of a symplectic or an orthogonal group defined over a non-archimedian local field of odd residue characteristic. We construct all full self-dual semisimple characters for G and we classify their intertwining classes using endo-parameters. We compute the set of intertwiners between self-dual semisimple characters, and prove an intertwining and conjugacy theorem. Finally we count all G-intertwining classes of self-dual semisimple characters which lift to the same $\tilde{G}$-intertwining class of a semisimple character for the ambient general linear group $\tilde{G}$ for G. MSC2010 [11E57] [11E95] [20G05] [22E50]

1. Introduction

Let G be the point set of a reductive group over a non-Archimedean local field F. The smooth representation theory of G has been a subject of ongoing research over the last 50 years, motivated in particular by the local Langlands correspondence. A major problem is the classification of irreducible representations which can be partitioned by an invariant known as the depth. The irreducible representations of depth zero essentially arrive purely from the representation theory of finite reductive groups so are (relatively) well understood (see Morris [17], Moy-Prasad [18]). On the other hand, positive depth representations, where the arithmetic information of F reside, are more challenging. Among many others let me mention the work of Adler [1], Yu and Kim [29] [14], Kaletha [13] for tame cases and Bushnell-Kutzko [9], Sechérre–Stevens [20] and Kurinczuk–Skodlerack–Stevens [15] for general linear and classical groups, the latter if F has odd residue characteristic.

Their main tool to approach the arithmetic information of F in the positive depth case is the concept of simple and semisimple characters. They are certain characters on compact open subgroups of pronilpotent parahoric subgroups, so far they have been constructed in [9] and [24] for $\mathrm{GL}_n(F)$ and p-adic classical groups for odd p, in [19] and [23] for inner forms of general linear groups and for $G_2(F)$ in [2]. Now let $\mathbb{G}$ be a quaternionic form of a symplectic or an orthogonal group defined over F and assume that F has odd residue characteristic p. We write G for the set of its F-rational points. The purpose of this paper is to provide the theory of semisimple characters for G and to classify and count their intertwining classes in introducing endo-parameters for G. For the second we needed a new notion of Witt group for split quaternion algebras with a unitary anti-involution. We come to that later. For endo-parameters in the non-quaternionic case (for example for $\mathbb{G}(L)$ where $L|F$ is a quadratic unramified extension of F) see [15].

This paper has many applications for forthcoming articles in the future. Not only that it provides the step for the construction and the rigidity of the classification of supercuspidal representations of G, see [28] and [15] for $\mathbb{G}(L)$, it also has applications on the decomposition of the category of smooth representations of G. Further, Endo-classes have been successively used to approach an explicit Jacquet–Langlands correspondence for general linear groups, see Sechérre–Stevens [21] and Dotto [12], and so there is the hope that endo-parameters for classical groups enable a similar approach. Semisimple characters play also an important role in the explicit description of the Local Langlands correspondence on the level of wild inertia. See for example [8] for the first ramification theorem for general linear groups from Bushnell and Henniart and more recently the work of Blondel, Henniart and Stevens [3] for the role of semisimple characters for an explicit description of the Local Langlands correspondence for symplectic groups. To extend the results to G it is important to distinguish and count the different intertwining classes of self-dual semisimple characters of G. Let us also mention that the construction of these characters should

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conjecturally imply that the category of smooth representations of G on vector spaces over a field of positive characteristic different from p is Noetherian, as proven in [11] for the case of $\mathbb{G}(L)$ by Dat.

We now outline the main results: Let F be a non-Archimedean local field with odd residual characteristic and (D,ρ) be a non-split quaternion algebra over F with an orthogonal anti-involution and (V,h) be an ϵ-hermitian form with respect to ρ on a finite dimensional D-vector space V. We consider the set $G=\mathrm{U}(h)$ of isometries of h in the ambient general linear group $\tilde{G}$. A semisimple character of $\tilde{G}$ is a character on a compact open subgroup of $\tilde{G}$ which is constructed from a datum $\Delta:=[\Lambda,n,r,\beta]$, called semisimple stratum, such that the following hold.

- β is an element of $\mathrm{End}_D(V)$ generating a product $E=\prod_{i\in I}E_i$ of fields over F. It also gives a direct sum decomposition $V=\oplus_{i\in I}V^i$ into $E_i\otimes_F D$-modules.
- Λ is an o_D-lattice sequence in V, i.e. a point of the Bruhat-Tits building $B(\tilde{G})$ of $\tilde{G}$ with rational barycentric coordinates, which is in the image of the embedding
$$\prod_{i\in I}B(\tilde{G}_i)\to B(\tilde{G})$$
where $\tilde{G}_i=\mathrm{Aut}_{E_i\otimes_F D}(V^i)$. In particular Λ splits into $\oplus_{i\in I}\Lambda^i$.
- The integers $n>r$ are non-negative and indicate on which "level" the characters should be defined (r) and should be trivial (n).
- Some more conditions which ensure that the intertwining of the stratum is the centralizer of β in $\tilde{G}$ modulo the pro-p-radical of the stabilizer of Λ in $\tilde{G}$.

We attach to Δ a set of semisimple characters and denote this set by $\mathrm{C}(\Delta)$. All the characters in $\mathrm{C}(\Delta)$ have the same domain which we call $H(\Delta)$. To define self-dual semisimple characters we suppose additionally Λ to be a point in the building of $(\prod_i\tilde{G}_i)\cap G$ and β to be an element of the Lie-algebra of G. Now the self-dual semisimple characters for G (w.r.t. Δ) are the restrictions of elements of $\mathrm{C}(\Delta)$ to $H_-(\Delta)=H(\Delta)\cap G$. We denote the set of them by $\mathrm{C}_-(\Delta)$. The most important self-dual semisimple characters are the ones for $r=0$. We call them full.

The first steps in the study of self-dual semisimple characters give the following results:

- A nice intertwining formula for $\theta_-\in\mathrm{C}_-(\Delta)$, i.e. the set of elements in G which intertwine θ_- is of the form $UG_\beta U$ where U is an open prop-p-subgroup of G and G_β is the centralizer of β in G, see Theorem 6.12.
- Intertwining is an equivalence relation on the set of full self-dual semisimple characters for G, see Corollary 6.16.
- We have an intertwining and conjugacy theorem, see Theorem 6.17, which we now explain.

Given two full semisimple characters $\theta_-\in\mathrm{C}_-(\Delta)$ and $\theta'_-\in\mathrm{C}_-(\Delta')$ which intertwine by some element of G, they possess a bijection $\zeta:I\to I'$ between the index sets such that there is an element of $G\cap\prod_i\mathrm{End}_D(V^i,V^{\zeta(i)})$ which intertwines θ_- with θ'_-. Further the intertwining gives a bijection $\bar{\zeta}$ between the residue algebras of $F[\beta]$ and $F[\beta']$. Now Theorem 6.17 states

Theorem 1.1 (Main Theorem I, see 6.17). Suppose there is an element $t\in G$ such that $t\Lambda^i$ is equal to $\Lambda^{\zeta(i)}$, for all $i\in I$, and that the conjugation with t induces $\bar{\zeta}$. Then there is an element $g\in G$ such that $g.\theta_-=\theta'_-$.

In the second part we parametrize the intertwining classes of self-dual semisimple characters using endo-parameters. The stratum Δ comes along with an action of σ, the adjoint involution of h, on the index set which leads to a disjoint union $I=I_0\cup I_+\cup I_-$ where I_0 is the set of σ fixed points and I_+ is a section through the σ-orbits of length 2. The idea is to break up θ_- in elementary self-dual pieces, i.e. in self-dual semisimple characters where the index set is just one σ-orbit, $\theta_{i,-}=\theta_-|_{H(\Delta)\cap\tilde{G}_i}$, for $i\in I_0$, and $\theta_{i,-}=\theta_-|_{H(\Delta)\cap\tilde{G}_{\pm i}}$, for $i\in I_+$. To every elementary character we attach an endo-class, see after Definition 7.1. We denote the set of all full elementary endo-classes by $\mathcal{E}_-$. Further we can attach to any $h_i=h|_{V^i}$, $i\in I_0$, an ϵ-hermitian $\sigma_{E_i}\otimes\rho$-form $\tilde{h}_{\beta_i}$ which corresponds to some element t_i of the Witt group $W_\epsilon(\sigma_{E_i}\otimes\rho)$ which we call Witt tower. A major complication was to find the right description of $W_\epsilon(\sigma_{E_i}\otimes\rho)$ (see the remark after 7.3 for its definition), in particular of its elements. We have chosen the following way: t_i is interpreted as a map from the set of primitive $\sigma_{E_i}\otimes\rho$-orthogonal idempotents of $E_i \otimes D$ to the Witt group $W_\epsilon(\sigma_{E_i})$ such that the values are related to each other in a canonical manner, see 7.6. This was necessary because it was not enough to just take any isomorphism from $W_\epsilon(\sigma_{E_i} \otimes \rho)$ to $W_\epsilon(\sigma_{E_i})$ coming from an equivalence between the categories $\mathcal{H}_{\sigma_{E_i}\otimes\rho,\varepsilon}$ and $\mathcal{H}_{\sigma_{E_i},\varepsilon}$ of hermitian forms because we do not want to fix such an equivalence for every possible simple stratum. Now, we introduce an equivalence relation on the set of such pairs (γ, t), see section 7.4, and we call the equivalence classes (ρ,ϵ)-Witt types and the set of Witt types is denoted by $\mathcal{W}_{\rho,\epsilon}$. An endo-parameter is a map of finite support

$$f_- = (f_1, f_2) : \mathcal{E}_- \to \mathbb{N}_0 \times \mathcal{W}_{\rho,\epsilon}$$

such that the value $f_1(c_-)$ essentially plays the role of a Witt index and $f_2(c_-)$ is a Witt type which occurs in c_- (note that in c_- can occur several Witt types). These endo-parameters classify intertwining classes of self-dual semisimple characters, see Theorem 7.17.

With the endo-parameters in hand we calculate the number of G-intertwining classes of self-dual semisimple characters whose semisimple lifts are in the same $\tilde{G}$-intertwining class:

Theorem 1.2 (**Main Theorem II, 8.3**)**.** Let θ be σ-fixed semisimple character of $\tilde{G}$. Then the number of G-intertwining classes of σ-fixed semisimple characters in the $\tilde{G}$-intertwining class of θ is equal $2^{\#I_0}$ if there is no null block restriction for θ and $2^{\#I_0-1}$ if θ has a null block restriction.

We now give details within the structure of the paper. G is the group of F-rational points of an F-form $\mathbb{G}$ of an orthogonal or a symplectic group. Let L be a quadratic unramified extension of F. After writing $\mathbb{G}(L)$ as a group of isometries of a symplectic or an orthogonal bilinear form h_L in terms of h in 2.2 we prove in 2.9 that G is in fact the group of F-rational points of the identity component $\mathbb{G}^0$ of $\mathbb{G}$, which has the advantage that we do not have to distinguish between intertwining classes of characters under the orthogonal and the special orthogonal group, contrary to the case of $\mathbb{G}(L)$ in [15]. We describe the Witt group $W_\epsilon(\rho)$ in 2.3 in terms of D which has its application in the counting of intertwining classes of self-dual semisimple characters in Section 8

In Section 3 Stevens' cohomology argument [25] in fact generalizes to certain triple cosets, more precisely we show that if Γ is an l-group ($l \neq p$) acting on $\tilde{G}$ and U_1, U_2 are pro-p-subgroups and H a subgroup of $\tilde{G}$, all supposed to be Γ-invariant, then we obtain for the Γ-fixed point sets

$$(U_1 g H U_2)^\Gamma = U_1^\Gamma (gH)^\Gamma U_2^\Gamma,$$

for $g \in \tilde{G}$ if gH is Γ-invariant and a certain intersection condition is satisfied. The main purpose is that it enables a very neat proof of the intertwining formulas for pairs of semisimple characters for G in 6.12.

In Section 4 and 5 we develop the theory of semisimple strata for G which we call self-dual semisimple strata. It leads to Section 6 where we develop the theory of self-dual semisimple characters. The approach here is a bit different to the one in [24] because we do not use a translation principle for self-dual semisimple characters here. In fact we use that the action of a 2-group on a set of odd cardinality has a fixed point, and we derive a diagonalization theorem:

Theorem 1.3 (see 6.10)**.** Suppose $\theta_- \in \mathrm{C}(\Delta)$ and $\theta'_- \in \mathrm{C}(\Delta')$ are endo-equivalent (see 6.4) and suppose that $\Delta \oplus \Delta'$ is self-dual. Then, there are a self-dual semisimple stratum Δ'', such that $\Delta \oplus \Delta''$ is a self-dual semisimple stratum, and $\tilde{\theta}_- \in \mathrm{C}_-(\Delta \oplus \Delta'')$ such that the restrictions of $\tilde{\theta}_-$ to $H_-(\Delta)$ and $H_-(\Delta'')$ are θ_- and θ'_-, respectively.

The main point is that the Lie algebra elements β and β'' of Δ and Δ'', respectively, have the same characteristic polynomial. This diagonalization theorem and the following Skolem–Noether theorem imply the Main Theorem I.

Theorem 1.4 (Skolem–Noether, see 4.13)**.** Let β and β' be two elements of the Lie algebra of G which generate field extensions over F and assume that β and β' have the same characteristic polynomial. Suppose further that there are two strata $[\Lambda, n, n-1, \beta]$ and $[\Lambda', n, n-1, \beta']$, both not equivalent to a null-stratum, which are intertwined by some element of G. Then β and β' are conjugate by an element of G.

In Section 7 we generalize the theory of endo-classes and endo-parameters to quaternionic forms. Endo-classes where firstly introduced by Bushnell and Kutzko [10] in the theory of covers for $\mathrm{GL}_n(F)$ and then studied in many occasions, see [5], [15] and [23]. Another application is that endo-classes offer a tool to prove that two full semisimple characters intertwine in moving to a nicer (V,h). More precisely, we consider pss-characters. A pss-character Θ_- is a map, defined on a set $\mathfrak{E}_-$ (depending on Θ_-) of pairs (Δ,h) where Δ is a self-dual stratum for the signed hermitian form h, and such that $\Theta(\Delta,h)$ is an element of $\mathrm{C}_-(\Delta)$, and such that the values of Θ_- are related to each other via "transfer". For simplicity we assume that the values are full. Two full pss-characters Θ_- and Θ'_- (with domain $\mathfrak{E}_-$) are called *endo-equivalent* if there are arguments $(\Delta,h)\in\mathfrak{E}_-$ and $(\Delta',h')\in\mathfrak{E}'_-$ such that $h=h'$ and $\Theta_-(\Delta,h)$ intertwines with $\Theta_-(\Delta',h)$ by an element of $\mathrm{U}(h)$. Given two endo-equivalent pss-characters Θ_- and Θ'_- the Theorem 7.14 gives a necessary and sufficient condition on a pair $((\Delta^1,h^1),(\Delta^2,h^2))\in\mathfrak{E}_-\times\mathfrak{E}'_-$ such that the values $\Theta_-(\Delta^1,h^1)$ and $\Theta_-(\Delta'^2,h^2)$ intertwine by an element of $\mathrm{U}(h)$.

In Section 8 we prove the second main theorem.

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2. Quaternionic Forms of p-adic classical groups

2.1. **Fixing notation.** At first, this article is the second in a series of articles where the first one is [23]. There will be only one difference in the notation, see Remark 2.1. Let F be a non-archimedean local field of odd residual characteristic. We use the usual notation $o_F,\mathfrak{p}_F,\kappa_F$ and ν_F for the valuation ring, the valuation ideal, the residue field and the normalized valuation of F, the image of ν_F being $\mathbb{Z}$, and we use similar notation for other non-archimedean local skewfields. Further we fix a non-split quaternion algebra D with centre F together with an orthogonal anti-involution ρ on D, i.e. an F-linear automorphism of D which satisfies $\rho(xy)=\rho(y)\rho(x)$. We can choose ρ such that there is an unramified field extension $L|F$ and a uniformizer π_D in D both point-wise fixed by ρ such that π_D normalizes L. We denote the non-trivial automorphism of $L|F$ by τ. The square of π_D is a uniformizer of F and we denote it by π_F.

We further fix a finite dimensional non-zero right-D vector space V and an ϵ-hermitian form h on V, $\epsilon\in\{-1,1\}$, i.e. a $\mathbb{Z}$-bilinear form such that:

$$h(vx,wy)=\epsilon\rho(x)\rho(h(w,v))y,$$

for all $x,y\in D$ and $v,w\in V$. We denote by G the set

$$\mathrm{U}(h)=\{g\in\mathrm{Aut}_D(V)|\ h(gv,gw)=h(v,w)\text{ for all }v,w\in V\}$$

of isometries of h. We write σ_h for the adjoint anti-involution of h, $\tilde{G}$ for the ambient general linear group $\mathrm{Aut}_D(V)$ and A for the ring of D-linear endomorphisms of V.

Remark 2.1. There will be only one difference in the notation between [23] and this article. Precisely $\tilde{G}$ denotes the group $\mathrm{GL}_D(V)$ and G denotes the classical group in question.

2.2. **L-rational points.** The group G is the set of F-rational points of an F-form of a symplectic or an orthogonal algebraic group $\mathbb{G}$. In this section we will describe the set $\mathbb{G}(L)$ by an hermitian L-form on V. The set of L-rational points of $\mathbb{G}$ is given by the anti-involution $\sigma_h\otimes_F\mathrm{id}$ on $\mathrm{End}_D(V)\otimes_F L$, and the latter L-algebra is canonically isomorphic to $\mathrm{End}_L(V)$ via

$$\Phi:\mathrm{End}_D(V)\otimes_F L\to\mathrm{End}_L(V),\ f\otimes_F l\mapsto(v\mapsto f(vl)).$$

The group of L-rational points of $\mathbb{G}$ is algebraically isomorphic to

$$\{g\in\mathrm{End}_L(V)|\ \Phi((\sigma_h\otimes_F\mathrm{id})(\Phi^{-1}(g)))g=1\}.$$

We identify this set with $\mathbb{G}(L)$. By [6] there is a unique ϵ-hermitian L-form h_L on V such that

$$\mathrm{tr}_{L|F}\circ h_L=\mathrm{trd}_{D|F}\circ h.$$

Proposition 2.2. $\mathbb{G}(L)$ is equal to $\mathrm{U}(h_L)$.

Proof. We need to show that σ_{h_L} is equal to the push forward of $\sigma_h \otimes_F \mathrm{id}$ via Φ. Take $v, w \in V, f \in \mathrm{End}_D(V)$ and $l \in L$. Then,

$$\begin{aligned}
\mathrm{tr}_{L|F}(h_L(\Phi(f \otimes_F l)v, w)) &= \mathrm{tr}_{L|F}(h_L(f(v)l, w)) \\
&= \mathrm{tr}_{L|F}(h_L(f(v), wl)) \\
&= \mathrm{trd}_{D|F}(h(f(v), wl)) \\
&= \mathrm{trd}_{D|F}(h(v, \sigma_h(f)(wl))) \\
&= \mathrm{tr}_{L|F}(h_L(v, \Phi((\sigma_h \otimes_F \mathrm{id})(f \otimes l))(w)))
\end{aligned}$$

□

2.3. **Witt group of a local non-split division algebra.** Let $\tilde{D}$ be a central division algebra over F of finite degree, with anti-involution $\tilde{\rho}$, such that $(\tilde{D}, \tilde{\rho})$ is orthogonal or unitary. Then the Witt group of $\tilde{D}$ with respect to ϵ and $\tilde{\rho}$ is the set of equivalence classes of ϵ-hermitian forms

$$\tilde{h} : \tilde{V} \times \tilde{V} \to (\tilde{D}, \tilde{\rho})$$

on finite dimensional $\tilde{D}$-vector spaces $\tilde{V}$, where two forms are equivalent, if they have isomorphic anisotropic components in there Witt-decomposition. We write $\tilde{h}_{\equiv}$ for the classes. We write $W_{\epsilon,\tilde{\rho}}(\tilde{D})$ for the Witt group. We call this Witt group

- *orthogonal* if $\epsilon = 1$ and $\tilde{\rho}|_F = \mathrm{id}_F$,
- *symplectic* if $\epsilon = -1$ and $\tilde{\rho}|_F = \mathrm{id}_F$ and
- *unitary* if $\tilde{\rho}|_F \neq \mathrm{id}_F$.

If we have a Gram matrix M for $\tilde{h}$ we also write $\langle M \rangle$ for the form with Gram matrix M. If f is a symmetric or skew-symmetric element of $\mathrm{End}_{\tilde{D}}(\tilde{V})$ then we write $\tilde{h}^f$ for the form which maps $(\tilde{v}, \tilde{w}) \in \tilde{V}^2$ to $\tilde{h}(\tilde{v}, f(\tilde{w}))$ and we call $\tilde{h}^f$ the *twist* of $\tilde{h}$ by f.

We refer to section 6 of [24] for the description of the Witt group in the case where $\tilde{D}$ is abelian. We write C_n for the cyclic group of order n. In the case $(\tilde{D}, \tilde{\rho}) = (D, \rho)$, see 2.1, we have the following result for the two Witt groups. Recall, that D is not abelian.

Proposition 2.3. (i) The orthogonal Witt group of (D, ρ) is isomorphic to an elementary 2-group of order 8. A set of generators is given by

$$\langle 1 \rangle_{\equiv},\ \langle \pi_D \rangle_{\equiv},\ \langle \alpha \rangle_{\equiv},$$

where α is a non-square unit of L. We could take α skew-symmetric with respect to τ if and only if -1 is a square in F.

(ii) The symplectic Witt group is a cyclic group of order two. The non-hyperbolic class is $\langle \pi_D l_s \rangle_{\equiv}$, where l_s is a non-zero element of L which is skew-symmetric with respect to τ.

At first we need the next lemmas.

Lemma 2.4. For every symmetric element $d \in D^\times$ and field extension $F'|F$ which is fixed point-wise by ρ such that $F[d]|F$ is isomorphic to $F'|F$ there is an element g of $D^\times$ such that $gF[d]\rho(g) = F'$. In fact if $d \notin F$ every element which conjugates the first field extension to the second works.

Proof. In the case $d \in F$ we take $g = 1$. So we only need to consider the case $d \notin F$, i.e. where $F[d]|F$ has degree 2. By Skolem-Noether there is an element g conjugating $F[d]|F$ to $F'|F$. The element gdg^{-1} is symmetric by assumption. So, $\rho(g)g$ centralizes $F[d]$ and is thus an element of $F[d]$ and we obtain that $gF[d]g^{-1}$ is equal to $gF[d]\rho(g)$. □

Lemma 2.5. For every symmetric or skew-symmetric element $d \in D^\times$ and $x \in F^\times$ the Witt classes $\langle d \rangle_{\equiv}$ and $\langle dx \rangle_{\equiv}$ are equal.

Proof. We have to find an element y of D such that $\rho(y)dy$ is equal to dx. For the first case let us assume that d is $\pi_D l_s$, π_D or in $L^\times$. Using elements $y \in L$ we can obtain all xd for $x \in F^\times$ with $2|\nu_F(x)$. If we use $y \in L^{\times}\pi_D$ we obtain every xd for $x \in F^{\times}$ with odd valuation. This also solves the case $d \in F[\pi_D]$, because

$$F[\pi_D] = F[\pi_D]^{\times(2)}(F \cup F\pi_D).$$

A general symmetric d generates a field extension which is generated by an element d' of $Fl_s + F\pi_D$. If d' has an odd D-valuation then $F[d]$ is isomorphic to $F[\pi_D]$ and if d' has an even D-valuation then $F[d]|F$ is unramified. Thus Lemma 2.4 finishes the proof. □

We say that two elements d and d' of D are congruent to each other modulo ν_D if $d - d'$ is an element of $\mathfrak{p}_D d$. This is an equivalence relation, and two non-zero elements d and d' are congruent modulo ν_D if and only if dd'^{-1} is an element of $1 + \mathfrak{p}_D$.

Lemma 2.6. Let a_i, $i = 1,2,3,4$ be an F-splitting basis of D and suppose that two non-zero elements $d = \sum_i x_i a_i$ and $d' = \sum_i y_i a_i$ satisfy $dd'^{-1} \in 1 + \mathfrak{p}_D$, $x_i, y_i \in F$. Then, if $\nu_D(x_1 a_1) = \nu_D(d)$ then $x_1 y_1^{-1} \in 1 + \mathfrak{p}_F$.

Proof. We have $\nu_D(d - d') > \nu_D(d)$ and therefore $\nu(x_i a_i - y_i a_i) > \nu_D(d)$ for all i because (a_i) is a splitting basis for D. Thus, $\nu_D(x_1 a_1 - y_1 a_1) > \nu_D(x_1 a_1)$ if $\nu_D(x_1 a_1) = \nu_D(d)$, i.e. $\nu_F(x_1 - y_1) > \nu_F(x_1)$. □

Lemma 2.7. Suppose $d, d' \in D^{\times}$ are two symmetric or skew-symmetric elements of D which are congruent to each other modulo ν_D. Then

(i) If d and d' are not elements of $F^{\times}(1 + \mathfrak{p}_D)$ then $F[d]|F$ and $F[d']|F$ are isomorphic and there is a $g \in D^{\times}$ which conjugates the first field extension to the second such that $gdg^{-1}d'^{-1} \in 1 + \mathfrak{p}_{F[d']}$.
(ii) The Witt classes $\langle d \rangle_{\equiv}$ and $\langle d' \rangle_{\equiv}$ are equal.

Proof. The residue characteristic is odd, so either both are skew-symmetric or both are symmetric.

Case 1: Let us first consider the case where d and d' commute. It is worth to remark that this is already implied if both elements are skew-symmetric, because ρ is orthogonal. Now, then d and d' are elements of a ρ-invariant field F', and we have $d = ud'$ for some ρ-symmetric element u of $1 + \mathfrak{p}_{F'}$, using the congruence of d to d'. The residue characteristic is odd and thus u is a square of a ρ-symmetric element v of $1 + \mathfrak{p}_{F'}$ and thus $\langle d \rangle_{\equiv}$ is equal to $\langle d' \rangle_{\equiv}$. This proves (ii) in this case, and in (i) we can take $g = 1$.

Case 2: Suppose now that d and d' are symmetric and do not commute. We have two sub-cases.

Case 2.1: At first let us assume that d is an element of $F^{\times}(1+\mathfrak{p}_D)$, say $dx^{-1} \in (1+\mathfrak{p}_D)$, for some $x \in F^{\times}$. The elements d and x commute and therefore $\langle d \rangle_{\equiv}$ is equal to $\langle x \rangle_{\equiv}$, by Case 1, and similar we have $\langle x \rangle_{\equiv} = \langle d' \rangle_{\equiv}$. So, the Witt classes of $\langle d \rangle$ and $\langle d' \rangle$ are the same. It proves (ii). The statement (i) is empty in this case.

Case 2.2: Let us assume that d is not an element of $F^{\times}(1 + \mathfrak{p}_D)$. d' is congruent to d and thus it is not an element of $F^{\times}(1 + \mathfrak{p}_D)$ either.

(i) We apply Lemma 2.6 on

$$d = x_1 + x_2 l_s + x_3 \pi_D, \; d' = y_1 + y_2 l_s + y_3 \pi_D$$

to obtain $d - x_1$ and $d' - y_1$ are congruent modulo ν_D. Indeed, either $\nu_D(d) = \nu_D(x_1)$ and thus $x_1 - y_1$ is an element of $\mathfrak{p}_D x_1$, by Lemma 2.6, or $\nu_D(d) < \nu_D(x_1)$ and all four elements $d, d', d - x_1$ and $d' - y_1$ are congruent to each other modulo ν_D. Nevertheless, the difference of $d - x_1$ with $d' - y_1$ must be an element of $\mathfrak{p}_D(d - x_1)$, because

$$\mathfrak{p}_D d = \mathfrak{p}_D(d - x_1) \supseteq \mathfrak{p}_D x_1,$$

by $\nu_D(d) = \nu_D(d - x_1) \leq \nu_D(x_1)$. Thus, we can assume for the proof of (i) that d and d' are elements of $Fl_s + F\pi_D$, by the following argument: Suppose (i) is proven for $d - x_1$, $d' - y_1$, then

- if $d, d', d - x_1, d' - y_1$ are all congruent then

$$gdg^{-1} \equiv g(d - x_1)g^{-1} \equiv d' - y_1 \equiv d' \bmod d\mathfrak{p}_D$$

- if $x_1 \equiv y_1 \bmod d\mathfrak{p}_D$ then

$$gdg^{-1} = g(d - x_1)g^{-1} + x_1 \equiv d' - y_1 + x_1 \equiv d' \bmod d\mathfrak{p}_D.$$

So let us finish the case where x_1 and y_1 vanish. The squares d^2 and d'^2 are congruent elements of F, i.e. there is a one-unit v of F such that $v^2 d^2 = d'^2$, because the residue characteristic is odd. Both, vd and d' are not elements of F, and therefore vd is conjugate to d' by an element of $D^\times$ by Skolem-Noether.

(ii) We know that there is an element g satisfying the first assertion. The element g is congruent to an element of L or to an element of $L\pi_D$. Thus $\rho(g)g$, an element of $F[d]$, is congruent to an element of the form $\pi_F^i x^2$ with some unit x of L. We claim that $\rho(g)g$ is of the form ay^2 for some $a \in F^\times$ and $y \in o_{F[d]}^\times$. If $F[d]|F$ is ramified then $\rho(g)g$ is congruent to an element of F because $\nu_{F[d]}(\rho(g)g)$ is even and Hensel's Lemma implies the claim. If $F[d]|F$ is unramified then

$$\pi_F^{-i}\rho(g)g = \rho(\pi_D^{-i}g)\pi_D^{-i}g.$$

So, its residue class is a square and hence the claim. Thus we get

$$\langle d\rangle_\equiv = \langle gd\rho(g)\rangle_\equiv = \langle gdy^2g^{-1}\rangle_\equiv = \langle gyg^{-1}gdg^{-1}gyg^{-1}\rangle_\equiv.$$

The latter term is $\langle gdg^{-1}\rangle_\equiv$, because gyg^{-1} is ρ-symmetric, and in the second equality we used Lemma 2.5. Now, gdg^{-1} and d' are ρ-symmetric, congruent and commute, and hence we get (ii) by Case 1.

□

Remark 2.8. The proof of Lemma 2.7 shows that if two ρ-symmetric elements d and d' are conjugate by an element of $D^\times$ then they define the same Witt class.

Proof of Proposition 2.3. Using [7, 1.14] every signed hermitian space (h, V) over D has a Witt basis, i.e. a basis which has a Gram matrix with two diagonal blocks:

(i) An anti-diagonal matrix having 1 and ϵ in the anti-diagonal (block $(1,1)$),
(ii) A diagonal matrix which is the Gram matrix of an anisotropic subspace of V (block $(2,2)$).
(iii) The blocks $(2,1)$ and $(1,2)$ have only zero entries.

Thus to classify all equivalence classes of signed hermitian forms, one only needs to classify the possible diagonal blocks $(2,2)$, and for them only the diagonal entries. The F-vector space of skew-symmetric elements of D is $F\pi_D l_s$. Thus, from Lemma 2.5 follows that there is only one non-trivial Witt class in the symplectic case. The F-vector space of symmetric elements of D is $L + \pi_D F$. By Lemma 2.7 and Lemma 2.5, the class $\langle d\rangle_\equiv$ is $\langle \pi_D\rangle_\equiv$ or $\langle 1\rangle_\equiv$ or $\langle \alpha\rangle_\equiv$. They are pairwise different because of valuation reasons and because α cannot be of the form $\rho(x)x$ because the residue class of a unit of the form $\rho(x)x$ is a square. □

2.4. $G \subseteq \mathrm{SL}_D(V)$. An element g of G satisfies $g\sigma_h(g) = 1$, and $\rho|_F = \mathrm{id}$ leaves for $\mathrm{Nrd}_{A|F}(g)$ only 1 or -1. It is remarkable that in fact all elements of G have reduced norm 1. So there is not a distinction between orthogonal and special orthogonal intertwining classes of characters later on, which is contrary to the treatment of $\mathbb{G}(L)$ in [15]. We are going to prove this fact in this section.

Proposition 2.9. Every element of G has reduced norm 1.

Proof. We prove the assertion by induction on $m = \dim_D V$. We only need to consider orthogonal groups, because in the symplectic case the group $\mathbb{G}(L)$ is a split symplectic group where every element has determinant 1. Thus let us assume that $\mathbb{G}$ is a quaternionic form of an orthogonal group.

Induction start ($m = 1$): Let g be an element of G. We take the canonical isomorphism from D to $\mathrm{End}_D D$. Then $F[g]$ is a field in D, which is invariant under the action of σ_h. If $\sigma_h(g) = g$ then $g^2 = 1$, because $g \in G$ and thus $g \in \{1, -1\}$, i.e. the reduced norm of g would be 1. If $\sigma_h(g) \neq g$ then $\sigma_h|_{F[g]}$ is the Galois generator of $F[g]|F$. Thus $\mathrm{Nrd}(g) = N_{F[g]|F}(g) = \sigma_h(g)g = 1$.

Induction step ($m > 1$): We consider $F[g]$ generated by an element g of G . The minimal polynomial $\mu_{g,F}$ of g over F has a prime factorization

$$\mu_{g,F} = P_1^{\nu_1} \dots P_l^{\nu_l},$$

which gives a decomposition:

$$F[g] \cong F[X]/P_1^{\nu_1} \dots F[X]/P_l^{\nu_l}$$

The factors are permuted by σ_h. For every orbit of this action we get a σ_h-fixed idempotent. In the case of at least two orbits we conclude that g is an element of the product of at least two quaternionic forms of classical groups and the induction hypothesis implies that the reduced norm of the restrictions of g to the factors is 1. Thus we are left with the case of one orbit.

Case $\mu_{g,F} = P_1^{\nu_1} P_2^{\nu_2}$ and σ_h flips $F[X]/P_1^{\nu_1}$ and $F[X]/P_2^{\nu_2}$: The primitive idempotents 1^1 and 1^2 of $F[g]$ give rise to decompositions $V^1 + V^2$ of V and $g_1 + g_2$ of g. From $g \in G$ follows $g_1^{-1} = \sigma_h(g_2)$. Thus

$$1^1 P_2(g_1^{-1})^{\nu_2} = 1^1 P_2(\sigma_h(g_2))^{\nu_2} = \sigma_h(1^2 P_2(g_2)^{\nu_2}) = \sigma_h(1^2 0) = 0.$$

We put $b_0 = P_2(0)$. Note that b_0 is non-zero. Thus, the irreducible polynomials $\frac{1}{b_0} P_2(X^{-1}) X^{\deg P_2}$ and P_1 coincide. Further, the reduced characteristic polynomials of g and g^{-1} coincide by $\sigma_h(g) = g^{-1}$, we call the polynomial χ, and by linear algebra $X^{md}\chi(X^{-1})$ and χ are equal up to multiplication by an element of $F^\times$. Thus P_1 and P_2 have the same multiplicity in χ, and therefore $\chi(0) = 1$, taking $a_0 = P_1(0) = \frac{1}{b_0}$ into account.

Case $\mu_{g,F} = P^\nu$: As in the case above we obtain that $X^{\deg P} P(X^{-1}) \frac{1}{P(0)}$ and P coincide. We split P in an algebraic closure of F:

$$P = (X - \lambda_1) \dots (X - \lambda_l).$$

The inverse map of $F^\times$ permutes the roots of P. Thus $P(0) = 1$ if every root of P satisfies $\lambda^{-1} \neq \lambda$. If the latter is not the case then P has a root λ which is 1 or -1, and in this case $P = (X - \lambda)$ because it is irreducible. Then χ is an even power of $(X - \lambda)$ and thus $\chi(0) = 1$. □

3. Stevens' cohomology argument on double cosets

Let p and l be different primes and let Γ be an l-group acting continuously on some topological Hausdorff group Q. Kurinczuk and Stevens proved the following result in [16, 2.7]: We denote by S^Γ the set of Γ-fixed points of S for any subset S of Q.

Theorem 3.1 ([16] 2.7(ii)(b))**.** Suppose U_1 and U_2 are two Γ-stable pro-p-subgroups of Q. Let H be a further Γ-stable subgroup of Q such that for every $h \in H$ the following identity holds:

$$(U_1 h U_2) \cap H = (U_1 \cap H) h (U_2 \cap H). \tag{3.2}$$

Then, we have the double coset decomposition:

$$(U_1 H U_2)^\Gamma = U_1^\Gamma H^\Gamma U_2^\Gamma.$$

In this section we are going to generalize this result in allowing Γ-stable cosets of H instead of H.

Proposition 3.3. Suppose U_1 and U_2 are two Γ-stable pro-p-subgroups of Q and H is a further Γ-stable subgroup of Q. Suppose gH is a Γ-stable coset of H in Q such that for every $h \in H$ the following identity holds:

$$(U_1 g h U_2) \cap gH = (U_1 \cap gHg^{-1}) gh (U_2 \cap H). \tag{3.4}$$

Then, we have:

$$(U_1 g H U_2)^\Gamma = U_1^\Gamma (gH)^\Gamma U_2^\Gamma.$$

Remark 3.5. The condition (3.4) for U_1, g, H, U_2 is equivalent to (3.2) for $g^{-1}U_1 g, H, U_2$. So it is enough to establish (3.2) for a big class of triples U_1', H', U_2'.

The proof of Proposition 3.3 is literally the same, but we repeat the argument where minor changes occur. At first we need the following two statements from [16].

Lemma 3.6 ([16] 2.7(i))**.** Suppose that U_1 and U_2 are two subgroups of Q such that the (non-abelian) cohomology $H^1(\Gamma, gU_1g^{-1} \cap U_1)$ is trivial. Then we have for any Γ-fixed element g of Q the identity $(U_1 g U_2)^\Gamma = U_1^\Gamma g U_2^\Gamma$.

Lemma 3.7 ([16] 2.7(ii)(a))**.** For any two Γ-stable pro-p-subgroups U_1 and U_2 of Q and any element g of Q the following assertions are equivalent:

(i) $(U_1gU_2)^\Gamma \neq \emptyset$
(ii) U_1gU_2 is Γ-stable.

Proof of Proposition 3.3. There is nothing to prove for the inclusion $\supseteq$, so we continue with the other inclusion. Let x be an element of $(U_1gHU_2)^\Gamma$. Then there is an element h of H and there are elements $u_1 \in U_1$, $u_2 \in U_2$ such that $x = u_1ghu_2$. We have to show that we could have taken h such that gh is Γ-fixed, because Lemma 3.6 would then imply that x is an element of $U_1^\Gamma(gH)^\Gamma U_2^\Gamma$. Now, U_1ghU_2 and gH are Γ-stable, so, by (3.2) and Lemma 3.7, there is a Γ-fixed point in $(U_1 \cap gHg^{-1})gh(U_2 \cap H)$, which we could have chosen instead of gh in the product decomposition of x. This finishes the proof. □

4. Strata for G

In this section we generalize the notion of self-duality for strata from p-adic classical groups, see [27] and [24], to its quaternionic forms. We generalize the usual statements about strata, as for example about intertwining formulas and a Skolem–Noether result, and recall endo–classes of semisimple strata. We take all definitions and results from [23].

4.1. First definitions.

We refer to [27] and [24] for the non-quaternionic case. We use all notation and definitions in [23]. For example the definitions for strata (pure, simple, semi-pure, semisimple) can be found in section 4.

A stratum is denoted as a quadruple $[\Lambda, n, r, \beta]$ and if we write Δ' for a stratum, then the entries appear with a superscript $'$, i.e. $[\Lambda', n', r', \beta']$, and similar with subscripts: $\Delta_c := [\Lambda^c, n_c, r_c, \beta_c]$ (The superscript on Λ is not a typo.) We write E for $F[\beta]$ and similar E', E_c etc.. And we write $\mathrm{C}_?(!)$ for the centralizer of ! in ?.

If Δ is a semi-pure stratum it has an associated splitting which we denote by $V = \oplus_{i\in I} V^i$ coming from the decomposition of E into a product of fields, and we call the corresponding idempotents 1^i. Split means that Δ is the direct sum of its restrictions $\Delta|_{V^i}$ which is called the ith block of Δ.

Given a full o_F-module M in V we define the dual for M with respect to h via

$$M^\# = \{v \in V \mid h(v, M) \subseteq \mathfrak{p}_D\}.$$

The form h defines an involution $\#$ on the set of lattice sequences for V, in defining $\Lambda_i^\#$ as $(\Lambda_{-i})^\#$. This depends on the choice of h. A lattice sequence Λ is called self-dual if $\Lambda^\#$ and Λ differ by a translation, i.e. there is an integer k such that $\Lambda - k$, which is defined as $(\Lambda_{i+k})_{i\in\mathbb{Z}}$, coincides with $\Lambda^\#$. We also get an action of $\#$ on the set of strata for V:

$$\Delta^\# := [\Lambda^\#, n, r, -\sigma_h(\beta)].$$

In our notation the latter says: $n^\# = n$, $r^\# = r$ and $\beta^\# = -\sigma_h(\beta)$.

Definition 4.1. A stratum Δ is called *self-dual* if $\Delta^\#$ and Δ only differ by a translation of Λ, i.e. if Λ is self-dual and $\sigma_h(\beta) = -\beta$. Further if Δ is a self-dual semi-pure stratum, then σ induces an involution on the index set I, which we also call σ. The action of $\{1, \sigma\}$ decomposes the index set I into a set of fixed points I_0 and the set I_{+-} of elements which have an orbit of length 2. We usually choose a section $I_+ \subseteq I_{+-}$ through all orbits of length 2. Given a union of σ-orbits $J \subseteq I$ we denote the restriction of h to $V^J := \oplus_{i\in J} V^i$ by h_J. We also write $h_{i,\dots,j}$ instead of $h_{\{i,\dots,j\}}$. We call Δ *skew* if $I = I_0$.

4.2. Diagonalization for self-dual strata.

One of the first important properties for strata is the diagonalization proposition for self-dual simple strata. Let us first state the diagonalization proposition for $\mathrm{GL}_D(V)$.

Proposition 4.2 ([23] Theorem 4.30)**.** Let V^i, $i = 1, \dots, l$, be sub-D-vector spaces of V whose direct sum is V. Let Δ be a stratum which splits under $\oplus_j V^j$ into a direct sum of pure strata $\Delta|_{V^j}$. Suppose further that Δ is equivalent to a simple strata. Then, there is a simple stratum which is equivalent to Δ and split by $\oplus_i V^i$.

We want to prove:

Proposition 4.3 (see [24] 6.16 for the case over F)**.** Let $V = \bigoplus_{j\in J} V^j$ a decomposition of V such that the projections $1^j : V \to V^j$ are permuted by σ_h. Let Δ be a self-dual stratum which is split under (V^j) such that the restrictions $\Delta|_{V^j}$ and Δ are equivalent to a simple stratum. Then there is a self-dual simple stratum which is split by (V^j) and equivalent to Δ.

For the proof we need three further technical lemmas:

Lemma 4.4 ([23] 4.28)**.** Let Δ be a stratum. Then Δ is equivalent to a simple stratum if and only if $\mathrm{Res}_F(\Delta)$ is equivalent to a simple stratum.

Lemma 4.5 ([23] 4.21)**.** Let Δ be a pure stratum. Then Δ is a simple stratum if and only if $\mathrm{Res}_F(\Delta)$ is a simple stratum.

Lemma 4.6 ([26] 1.9)**.** Let $[\Lambda_F, n, r, \gamma_t]$ be a sequence of equivalent simple strata in $\mathrm{End}_F(V)$ such that γ_t converges to some γ in $\mathrm{End}_F(V)$. Then the stratum $[\Lambda_F, n, r, \gamma]$ is simple.

Here we consider $h_F := \mathrm{trd}_{D|F} \circ h$.

Proof of Proposition 4.3. This proof uses the strategy of [26, 1.10], where the author shows the existence of a sequence $\gamma_t,\ t \in \mathbb{N}_0$, satisfying:

(i) $[\Lambda_F, n, r, \gamma_t]$ is simple and equivalent to $\mathrm{Res}_F(\Delta)$ for all $t \in \mathbb{N}_0$,
(ii) $\gamma_t + \sigma_{h_F}(\gamma_t) \in \mathfrak{a}_{\Lambda_F, -r+t}$ for all $t \in \mathbb{N}_0$.
(iii) $\gamma_t - \gamma_{t+1} \in \mathfrak{a}_{\Lambda_F, -r+t}$ for all $t \in \mathbb{N}_0$.

We show that we could have chosen γ_t in $\prod_{i=1}^l \mathrm{End}_D V^i$. This follows from Proposition 4.2 for γ_0. So suppose that $\gamma_0, \ldots, \gamma_t$ are elements of $\prod_{i=1}^l \mathrm{End}_D V^i$ which satisfy (i)-(iii). σ induces an involution on J. Let J_0 be the set of σ-fixed points in J. We denote the stratum $[\Lambda, n, r-t-1, \frac{\gamma_t - \sigma_h(\gamma_t)}{2}]$ by $\tilde{\Delta}$. Let s be a tame corestriction with respect to γ. The derived stratum $\partial_{\gamma_t}(\mathrm{Res}_F(\tilde{\Delta}))$ is equivalent to a stratum $[\Lambda_F, r-t, r-t-1, \delta]$ for some $\delta \in F[\gamma_t]$, by the proof of [26, 1.10]. Now, $\tilde{\beta}$ and δ commute with the projections 1^j and thus $\partial_{1^j\gamma_t}(\mathrm{Res}_F(\tilde{\Delta}|_{V^j}))$ is equivalent to a simple stratum. Therefore the strata $\mathrm{Res}_F(\tilde{\Delta})$ and $\mathrm{Res}_F(\tilde{\Delta}|_{V^j})$ are equivalent to simple strata, by [20, 2.13]. Thus $\tilde{\Delta}$ and its restrictions are equivalent to simple strata, by Lemma 4.4. Now, we can find by Proposition 4.2 a simple stratum $\tilde{\tilde{\Delta}}$ split under (V^j) and equivalent to $\tilde{\Delta}$. We put $\gamma_{t+1} := \tilde{\tilde{\beta}}$.

The sequence $(\gamma_t)_{t\in\mathbb{N}_0}$ converges and we denote the limit by γ. Then $[\Lambda_F, n, r, \gamma]$ is simple, by Lemma 4.6, and therefore $[\Lambda, n, r, \gamma]$ is simple, by Lemma 4.5. This finishes the proof. □

One consequence is:

Corollary 4.7. Let Δ be a semisimple stratum such that $\#\Delta$ is equivalent to Δ. Then, Δ is equivalent to a self-dual semisimple stratum.

Proof. By Corollary [23, 4.37] and Corollary [15, A.4] we can assume that the idempotents of the associate splitting of Δ are permuted by the action of σ_h. We apply Proposition 4.3 on Δ_i, for $i \in I_0$, and we replace β_i by $-\sigma_h(\beta_{\sigma(i)})$ for $i \in I_-$. □

4.3. **Intersection formulas.** For the next sections we need some special cases of (3.2). For this we need to recall that on attaches to a semisimple stratum two further pro-p-subgroups of $\tilde{G}$: $S(\Delta)$ and $M(\Delta) := 1 + \mathfrak{m}(\Delta)$, see [23, before 4.25, after 5.14] for the definitions of $S(\Delta)$ and $\mathfrak{m}(\Delta)$. We need the technique of † construction for strata, see [23, 4.6], to attach to a stratum Δ a stratum $\Delta^\dagger$ where the lattice sequence is principal.

Proposition 4.8. Let Δ_1 and Δ_2 be two semisimple strata which share the associated splitting and the parameters r and n. Let H be one of the following groups:

- $\prod_{i\in I_1} \mathrm{Aut}_D V^i$ or
- $C_A(\beta_1)^\times$, if $\beta_1 = \beta_2$.

Let h be an element of H. Then we have the following intersection formulas.

(i) $(M(\Delta_2)hM(\Delta_1)) \cap H = (M(\Delta_2) \cap H)h(M(\Delta_1) \cap H)$,
(ii) $(S(\Delta_2)hS(\Delta_1)) \cap H = (S(\Delta_2) \cap H)h(S(\Delta_1) \cap H)$,

Proof. We only show (i) because the proof of the second equation is similar. We start with the case that H is equal to $\prod_{i\in I_1} \mathrm{Aut}_D V^i$. Consider the group $\Gamma := \{\pm 1\}^{\# I_1}$ acting on $\tilde{G}$ by conjugation induced by the decomposition of V. The Γ-fixed point set is H and Γ is contained in $P(\Lambda^1) \cap P(\Lambda^2)$. Lemma 3.6 implies (ii).

Let us now assume $\beta_1 = \beta_2 =: \beta$ and that H is $C_A(\beta)^\times$. Without loss of generality we can assume that both lattice sequences have the same F-period. We make the †-construction for $\Delta_1 \otimes L$ and $\Delta_2 \otimes L$, and we get $(\Delta_1 \otimes L)^\dagger$ and $(\Delta_2 \otimes L)^\dagger$ which are in fact equal to $\Delta_1^\dagger \otimes L$ and $\Delta_2^\dagger \otimes L$, respectively. Both latter strata are semisimple and the lattice sequences on the blocks are principal lattice chains. Thus there is an invertible element g of $C := C_{\mathrm{End}_L(V^\dagger)}(\beta^\dagger)$ which sends $\Lambda^{1,\dagger}$ to $\Lambda^{2,\dagger} =: \Lambda_L$. Formula (i) is true for the triple $(\Delta_2 \otimes L)^\dagger, (\Delta_2 \otimes L)^\dagger$, $C^\times$, and for every element $c \in C^\times$ by [28, 2.7]:

$$\begin{aligned}(M(\Delta_2 \otimes L)cM(\Delta_2 \otimes L)) \cap C^\times &\subseteq P^{-k_0-r}(\Lambda_L)cP^{-k_0-r}(\Lambda_L) \cap C^\times \\ &\overset{[28,\ 2.7]}{=} (P^{-k_0-r}(\Lambda_L) \cap C^\times)c(P^{-k_0-r}(\Lambda_L) \cap C^\times) \\ &= (M(\Delta_2 \otimes L) \cap C^\times)c(M(\Delta_2 \otimes L) \cap C^\times).\end{aligned}$$

We conjugate back with g to obtain (i) for $(\Delta_2 \otimes L)^\dagger, (\Delta_1 \otimes L)^\dagger$, $C^\times$ and every $c \in C^\times$. In particular the formula is valid for $\mathrm{diag}(b_i|\ i \in I_1)$ with $b_i \in C_{\mathrm{Aut}_L(V)}(\beta)^\times$, $i \in I_1$. Using a limit-argument we obtain for all $h \in C_{\mathrm{Aut}_L(V)}(\beta)^\times$ that $M(\Delta_2 \otimes L)hM(\Delta_1 \otimes L)) \cap C_{\mathrm{Aut}_L(V)}(\beta)^\times$ is contained in

$$(M(\Delta_2 \otimes L) \cap C_{\mathrm{Aut}_L(V)}(\beta)^\times)h(M(\Delta_1 \otimes L) \cap C_{\mathrm{Aut}_L(V)}(\beta)^\times).$$

This is in fact an equality. We consider $h \in C_A(\beta)^\times$ and take τ-fixed points. Then 3.6 implies the assertion. ☐

4.4. **Matching and intertwining.** The intersection formulas of the last section allow us to determine the G-intertwining of two self-dual semisimple strata with the same β. Two semisimple strata with same parameters which intertwine by an element of $\tilde{G}$ possess a matching, i.e. they have a unique one-to-one correspondence between the block structures, in particular with coinciding dimensions. We prove in the self-dual case that this correspondence between the σ-fixed blocks is given by isometries.

In this section we suppose that Δ and Δ' are two semisimple strata which satisfy $e(\Lambda|F) = e(\Lambda'|F)$, $n = n'$ and $r = r'$.

At first we recall the matching and the intertwining formulas for semisimple strata.

Theorem 4.9 ([23] 4.41 (with 4.32))**.** (i) Suppose $I(\Delta, \Delta') \neq \emptyset$. Then there is a unique bijection $\zeta :\ I \to I'$ such that the set $I(\Delta, \Delta') \cap \prod_{i\in I} \mathrm{Hom}_D(V^i, V^{\zeta(i)})$ is not empty.
(ii) Let $\zeta; I \to I'$ be a bijective map. Then the following assertions are equivalent:
  (a) The set $I(\Delta, \Delta') \cap \prod_{i\in I} \mathrm{Hom}_D(V^i, V^{\zeta(i)})$ is not empty.
  (b) For all $i \in I$ the D-dimensions of V^i and $V^{\zeta(i)}$ coincide and the direct sum $\Delta_i \oplus \Delta'_{\zeta(i)}$ is equivalent to a simple stratum.

The map ζ of Theorem 4.9(i) is called the matching of (Δ, Δ'), and we denote it also as $\zeta_{\Delta.\Delta'}$.

Theorem 4.10 ([23] 4.36)**.** Suppose $I(\Delta, \Delta') \neq \emptyset$ with matching ζ. Then the following holds.

(i) $I(\Delta, \Delta') = M(\Delta')(I(\Delta, \Delta') \cap \prod_{i\in I} \mathrm{Hom}_D(V^i, V^{\zeta(i)}))M(\Delta)$.
(ii) Suppose there is an element $\tilde{g}$ of $\tilde{G}$ such that $\tilde{g}\beta\tilde{g}^{-1} = \beta'$, then we have

$$I(\Delta, \Delta') = M(\Delta')\tilde{g}C_A(\beta)^\times M(\Delta).$$

Proof. The second assertion follows from [23, 4.36] and the first assertion follows from the second by diagonalization, Proposition 4.2, and the fact that $M(*)$ only depends on the equivalence class of the semisimple stratum. ☐

We now can come to the self-dual case:

Theorem 4.11 (see [24] 6.22 for the case over F)**.** Suppose Δ and Δ' are self-dual and are intertwined by an element of G with matching ζ. Then the following holds.

(i) The map ζ is σ-equivariant, and

$$I_G(\Delta,\Delta') = (G\cap M(\Delta'))(I_G(\Delta,\Delta')\cap\prod_{i\in I}\mathrm{Hom}_D(V^i,V^{\zeta(i)}))(G\cap M(\Delta)).$$

(ii) Suppose there is an element $\tilde{g}$ of $\tilde{G}$ such that $\tilde{g}\beta\tilde{g}^{-1}=\beta'$, then we have

$$I_G(\Delta,\Delta') = (G\cap M(\Delta'))(G\cap \tilde{g}C_A(\beta)^{\times})(G\cap M(\Delta)).$$

Proof. This Theorem follows from Theorem 4.10, Proposition 4.8 and Proposition 3.3 for the group $\Gamma = \{\sigma,1\}$, noting that $I(\Delta,\Delta')\cap\prod_{i\in I}\mathrm{Hom}_D(V^i,V^{\zeta(i)})$ is σ-invariant by the uniqueness of ζ. The latter also implies that σ and ζ commute. □

The last Theorem has now consequences for the matching.

Corollary 4.12 (see [15] 9.5 for the case over F)**.** Suppose Δ and Δ' are self-dual and intertwined by an element of G with matching ζ. Then $I_G(\Delta,\Delta')\cap\prod_{i\in I}\mathrm{Hom}_D(V^i,V^{\zeta(i)})$ is non-empty and h_J is isometric to $h_{\zeta(J)}$ for all σ-invariant subsets J of I.

Proof. The non-emptiness of $I_G(\Delta,\Delta')\cap\prod_{i\in I}\mathrm{Hom}_D(V^i,V^{\zeta(i)})$ follows from 4.11(i). Every element of this intersection restricts to an isometry from h_J to $h_{\zeta(J)}$. □

Let us underline how remarkable short the proof of the last Corollary is. By similarity it also provides a short proof of [24, Conjecture 10.3].

4.5. **Skolem-Noether.** We prove a version of Skolem–Noether for G.

Theorem 4.13 (see [24] 5.2 for the case over F)**.** Let Δ and Δ' be two pure strata such that $e(\Lambda|F) = e(\Lambda'|F)$ and $n = n' > r \geq r'$. Suppose that there is an element of G which intertwines Δ with Δ' and that β and β' have the same minimal polynomial. Then the element β is conjugate to β' by an element of G.

Proof. We only consider pure strata, so without loss of generality we can assume $n = r+1 = r'+1$. We can find an element $\tilde{g}\in\tilde{G}$ such that $\tilde{g}\beta\tilde{g}^{-1}=\beta'$. The proof consists of two steps.

Step 1 Let us at first assume that both strata are simple. The set $G\cap\tilde{g}B$ is non-empty by Theorem 4.11 because $I_G(\Delta,\Delta')$ is non-empty. This finishes Step 1.

Step 2 There are simple strata $\tilde{\Delta}$ and $\tilde{\Delta}'$ equivalent to Δ and Δ', respectively. By Proposition 4.3 we can assume that $\tilde{\beta}$ and $\tilde{\beta}'$ have the same minimal polynomial. Let e_w be the wild ramification index of $\tilde{E}|F$. There is an element $\gamma\in\tilde{E}$ which is congruent to $\tilde{\beta}^{e_w}$ and generates the maximal tamely ramified sub-extension $\tilde{E}_{tr}$ in $\tilde{E}$. By [15, 2.15] there is an F-linear field monomorphism $\phi:\tilde{E}_{tr}\to E$ such that $\phi(\gamma)$ is congruent to β^{e_w} in E. We take γ' to be the image of γ under the isomorphism $\tilde{E}\cong\tilde{E}'$ which sends $\tilde{\beta}$ to $\tilde{\beta}'$. Again by [15, 2.15] we get an F-linear field monomorphism $\phi':\tilde{E}'_{tr}\to E'$ such that $\phi'(\gamma')$ is congruent to β'^{e_w}. The elements $\phi(\gamma)$ and $\phi'(\gamma')$ are conjugate by an element of G by Step 1 (because they are minimal), and we can assume that $\phi(\gamma)$ and $\phi'(\gamma')$ coincide and we denote $\phi(\gamma)$ by t. If $E|F[t]$ and $E'|F[t]$ are isomorphic by a map which sends β to β', then β is conjugate to β' by an element of $C_G(F[t])$ by [24, 5.1]. Let ψ be the F-linear isomorphism from E to E' which sends β to β'. Then

$$\psi(t)\equiv\psi(\beta^{e_w})=\beta'^{e_w}\equiv t.$$

Thus, $\psi(x)$ is congruent to x in E' for all $x\in F[t]$ because t is minimal over F. We obtain at first that ψ is the identity on the maximal unramified sub-extension $F[t]_{ur}$ of $F[t]|F$. Let π_t be a uniformizer of $F[t]$ whose $e(F[t]|F)$th power is an element of $F[t]_{ur}$. We just denote the ramification index of $F[t]|F$ by e_t. From the congruence $\psi(\pi_t)\equiv\pi_t$ we obtain an element $y\in$

$1+\mathfrak{p}_{E'}$ such that $y\pi_t=\psi(\pi_t)$. Thus $y^{e_t}=1$. Therefore y and 1 are e_tth root of unity which are congruent in E'. Thus $y=1$ because e_t is not divisible by the residue characteristic of F. We have established that ψ is the identity on $F[t]$ which finishes the proof.

□

This has an immediate analogue consequence for semisimple strata.

Corollary 4.14. Suppose Δ and Δ' are self-dual semisimple strata with $r=r'$, the same F-period and a bijection $\zeta: I\to I'$ such that β_i and $\beta'_{\zeta(i)}$ have the same minimal polynomial over F. Suppose further there is an element of $G\cap\prod_{i\in I}\mathrm{Hom}(V^i,V^{\zeta(i)})$ which intertwines $[\Lambda^i,n_i,n_i-1,\beta_i]$ with $[\Lambda'^{\zeta(i)},n'_{\zeta(i)},n'_{\zeta(i)}-1,\beta'_{\zeta(i)}]$ for all $i\in I$ with $\beta_i\neq 0$. Then β is conjugate to β' by an element of G.

Proof. We take at first an index $i\in I_0$ with $\beta_i\neq 0$. Then we get $n_i=n'_{\zeta(i)}$ from [24, 6.9], and we get by 4.13 an isometry from h_i to $h_{\zeta(i)}$ which conjugates β_i to $\beta'_{\zeta(i)}$. For $i\in I_+$ we can take any D-linear isomorphism g_i from V^i to $V^{\zeta(i)}$ which conjugates β_i to $\beta'_{\zeta(i)}$, and we obtain an element $\sigma_h(g_i)\in\mathrm{Hom}(V^{\zeta(\sigma(i))},V^{\sigma(i)})$ which conjugates $\beta'_{\zeta(\sigma(i))}$ to $\beta_{\sigma(i)}$. This finishes the proof. □

4.6. **Conjugate semisimple strata.** In this section we prove an "intertwining implies conjugacy"-kind of result for self-dual semisimple strata. It will lead directly to the analogue result for self-dual semisimple characters. In this section we fix two intertwining semisimple strata Δ and Δ' and we assume $n=n'$, $r=r'$ and $e(\Lambda|F)=e(\Lambda'|F)$.

Intertwining does not imply conjugacy in general, even if one would assume that $\tilde{g}\Lambda^i$ is equal to $\Lambda^{\zeta(i)}$ for some $\tilde{g}\in\tilde{G}$ because one has to keep track of the embedding type. This is the reason why we have introduced the map $\bar{\zeta}$ in [23, 4.46]. It is the map from κ_E to $\kappa_{E'}$ given by the diagram

$$\kappa_E\hookrightarrow\mathfrak{a}_0/\mathfrak{a}_1\to(g\mathfrak{a}_0g^{-1}+\mathfrak{a}'_0)/(g\mathfrak{a}_1g^{-1}+\mathfrak{a}'_1)\leftarrow\mathfrak{a}'_0/\mathfrak{a}'_1\hookleftarrow\kappa_{E'},\ g\in I(\Delta,\Delta'). \tag{4.15}$$

This map does not depend on the choice of g. We call the pair $(\zeta,\bar{\zeta})$ the matching pair. In fact we do not need the whole map for the conjugacy. The restriction to κ_{E_D} is enough. The algebra E_D is the product of the unramified field extensions $E_{i,D}|F$ of degree $\gcd(\sqrt{[D:F]},f(E_i|F))$, $i\in I$.

Theorem 4.16 ([23] 4.48)**.** Suppose there is an element $\tilde{g}\in\prod_i\mathrm{Hom}_D(V^i,V^{\zeta(i)})$ such that $\tilde{g}\Lambda=\Lambda'$ and such that the conjugation with $\tilde{g}$ induces $\bar{\zeta}|_{\kappa_{E_D}}$. Then $\prod_i\mathrm{Hom}_D(V^i,V^{\zeta(i)})$ contains an element u such that $u\Lambda=\Lambda'$ and $u\Delta$ is equivalent to Δ'. We can choose u to further satisfy $u\beta u^{-1}=\beta'$ if β and β' have the same characteristic polynomial over F.

The second part of this Theorem follows directly from the proof of [23, 4.48]. We continue now with its self-dual analogue.

Theorem 4.17. Suppose both strata are self-dual and intertwine by an element of G. Let g be an element of $G\cap\prod_i\mathrm{Hom}_D(V^i,V^{\zeta(i)})$ which satisfies $g\Lambda=\Lambda'$ and such that the conjugation with g induces $\bar{\zeta}|_{\kappa_{E_D}}$. Then there is an element u of $G\cap\prod_i\mathrm{Hom}_D(V^i,V^{\zeta(i)})$ such that $u\Lambda=\Lambda'$ and $u\Delta$ is equivalent to Δ'. We can choose u to further satisfy $u\beta u^{-1}=\beta'$ if β and β' have the same characteristic polynomial over F.

Proof. We apply g and can assume $\Lambda^i=\Lambda'^{\zeta(i)}$, for all i, and $\bar{\zeta}|_{\kappa_{E_D}}=\mathrm{id}|_{\kappa_{E_D}}$ without loss of generality. We can therefore restrict to the case of a single σ-orbit, i.e. I is either a singleton or consists of two non-fixed points of σ. We apply in the latter case Theorem 4.16 for one index i, to get an appropriate element u_i, and $u:=\mathrm{diag}(u_i,\sigma_h(u_i)^{-1})$ satisfies the assertions. So we are left with the singleton case $I=\{i_0\}$, i.e. we assume that both strata are simple. By diagonalization, see 4.3, we can assume that β and β' have the same minimal polynomial over F. The element β is conjugate to β' by some element g_0 of G by Theorem 4.13. The conjugation by g_0 induces $\bar{\zeta}|_{\kappa_{E_D}}=\mathrm{id}_{\kappa_{E_D}}$ because g_0 intertwines Δ with Δ'. Thus, $g_0xg_0^{-1}+\mathfrak{a}_1$ is equal to $x+\mathfrak{a}_1$ in $\mathfrak{a}_0/\mathfrak{a}_1$ for all $x\in o_E$. By [23, 4.39] there is an element v of $P(\Lambda)$ which conjugates β to β'. We consider the $C_{\tilde{G}}(E_D)$-equivariant affine Broussous–Lemaire isomorphism

$$j_{E_D}:\ B_{red}(\tilde{G})^{E_D^\times}\xrightarrow{\sim}B_{red}(C_{\tilde{G}}(E_D))$$

between the reduced Bruhat–Tits buildings, see [4]. The translation classes $[\Lambda]$ and $[g_0^{-1}\Lambda]$ are elements of $B_{red}(\tilde{G})^{E_D^\times}$. They satisfy

$$v^{-1}g_0 j_{E_D}([g_0^{-1}\Lambda]) = j_{E_D}(v^{-1}g_0[g_0^{-1}\Lambda]) = j_{E_D}([\Lambda]).$$

The element $v^{-1}g_0$ acts type-preserving on $B_{red}(C_{\tilde{G}}(E_D))$ because $\nu_{E_D}(\mathrm{Nrd}_{E_D}(v^{-1}g_0))$ vanishes, i.e. the points $j_{E_D}([g_0^{-1}\Lambda])$ and $j_{E_D}([\Lambda])$ have the same simplicial type in $B_{red}(C_{\tilde{G}}(E_D))$. Note that both are points of the building $B(C_G(E_D))$ of the centralizer of E_D in G. By [22, 5.2] there is an element $g_1 \in C_G(E_D)$ such that

$$g_1 j_{E_D}([g_0^{-1}\Lambda]) = j_{E_D}([\Lambda]).$$

The $C_G(E_D)$-equivariance and the injectivity of j_{E_D} imply $[g_1 g_0^{-1}\Lambda] = [\Lambda]$. Thus $g_1 g_0^{-1}\Lambda$ is a translate of Λ, i.e. $g_1 g_0^{-1}$ lies in the normalizer of Λ and hence in $\mathfrak{a}_0^\times$, because it is an element of G. The element $u := g_1 g_0^{-1}$ is an element of $P(\Lambda) \cap G$ which conjugates β' to β. This finishes the proof. □

5. Endo-classes of self-dual strata

Now we compare self-dual semisimple strata which are allowed to correspond to different hermitian spaces where we also allow orthogonal and symplectic spaces over F. We further want to allow the strata to have different parameter r and different E_i-periods, so we repeat the notion of group level and degree. The *degree* of a semi-pure stratum Δ is the F-dimension of E and the *group level* of a semisimple stratum Δ is $\lfloor \frac{r}{e(\Lambda|E)} \rfloor$ where $e(\Lambda|E)$ is the greatest common divisor of $(e(\Lambda^i|E_i))_{i\in I}$ except for the null case where we set the group level to be infinity. Two semisimple strata Δ and Δ' (possibly given on different vector spaces and different skew-fields over F) of the same group level and the same degree are called *endo-equivalent* if there is a bijection $\zeta : I \to I'$ such that $\mathrm{Res}_F(\Delta_i) \oplus \mathrm{Res}_F(\Delta_{\zeta(i)})$ is equivalent to a simple stratum for all indexes $i \in I$, see [23, 6.6,6.7]. We call $\zeta_{\Delta,\Delta'}$ the matching from Δ to Δ'. This is an equivalence relation by [23, 6.7], and note that the used direct sum is a slight generalization, see [23, after 6.3]. The equivalence classes are called *endo-classes* and the endo-class of a semisimple stratum Δ is denoted by $\mathfrak{E}(\Delta)$.

Proposition 5.1 ([23] 6.7)**.** Two endo-equivalent strata Δ and Δ' with matching ζ satisfy

$$e(\Lambda^i|E_i) = e(\Lambda'^{\zeta(i)}|E'_{\zeta(i)}),\ f(\Lambda^i|E_i) = f(\Lambda'^{\zeta(i)}|E'_{\zeta(i)}),$$

for all $i \in I$.

For endo-equivalence to make sense it is important that two intertwining semisimple strata of same degree and group level are endo-equivalent. This is true because the matching results remain true:

Theorem 5.2. Theorem 4.9 and Corollary 4.12 remain true if we replace the conditions: $n = n'$, $r = r'$ and $e(\Lambda|F) = e(\Lambda'|F)$ by the assumption that Δ and Δ' have the same degree and the same group level.

We come to the proof after some preparation.

Remark 5.3. There are two steps to reduce from same group level and same degree to equal parameters.

Step 1: The method of *repeating*. Given a stratum Δ and a positive integer k we can scale the F-period of Λ to $ke(\Lambda|F)$ in the following way. One constructs a new lattice sequence $k\Lambda$ by repeating k-times the lattices which occur in the image of Λ:

$$k\Lambda_j := \Lambda_{\lceil \frac{j}{k} \rceil},\ j \in \mathbb{Z}.$$

The strata $[k\Lambda, \max(nk, kr + j), kr + j, \beta]$, $j = 0, \dots, k-1$ define the same coset as Δ and they are semisimple if and only if Δ is. The case $j = 0$ is used for the definition of the direct sum and $j = k-1$ is useful if one does not want to leave the class of strata which satisfy $n = r+1$. One important consequence: Given two strata Δ and Δ' we can assume without loss of generality that both have the same F-period, and if both are non-null, semisimple and intertwine then we obtain $n = n'$ by [24, 6.9]. This process does not change the group level and the degree.

Step 2: Raising or lowering the parameter r. Suppose Δ and Δ' are two intertwining semisimple strata of the same degree and the same group level, which share the F-period and the parameter n. Suppose further $r \geq r'$. Then the stratum $\Delta'((r-r')+)$, i.e. the stratum obtained from Δ' in raising r' to r, is still semisimple, by the proof of [23, 5.48], and we have $e(\Lambda|E) = e(\Lambda'|E')$

by [23, 4.2]. Thus Δ and $\Delta'((r-r')+)$ share the group level and the degree. Instead of raising r' we can lower r in Δ and the next lemma states that $\Delta((r-r')-)$ and Δ' still intertwine.

An endo-class is invariant under repeating, raising and lowering.

Lemma 5.4. Suppose Δ and Δ' are two intertwining semisimple strata sharing the F-period, the parameter n, the group level and the degree. Suppose $r \geq r'$. Then $\Delta((r-r')-)$ and Δ' intertwine. If further both strata are self-dual and intertwine by some element of G, then Δ' and $\Delta((r-r')-)$ intertwine by some element of G.

Proof. If the common group level is infinite then the identity is a possible intertwiner. So let us suppose that both strata have finite group level, i.e. are non-null. Then all strata

$$\Delta((r-r')-), \Delta, \Delta'((r-r')+), \Delta'$$

have the same group level, because $e(\Lambda|E) = e(\Lambda'|E')$, see 5.3 Step 2 above, and two successive strata are endo-equivalent, the middle two by Theorem 4.9. Thus, by transitivity, the strata $\Delta((r-r')-)$ and Δ' are endo-equivalent, and therefore intertwine by [23, 4.32].

In the self-dual case we apply diagonalization 4.3 on the endo-equivalent strata $\Delta((r-r')-)$ and Δ' to reduce to the case where β and β' have the same characteristic polynomial. So, if Δ and $\Delta'((r-r')+)$ intertwine by some element of G then β and β' are conjugate by some element of G, by Theorem 4.11(i) and Corollary 4.14 and therefore $\Delta((r-r')-)$ and Δ' intertwine by some element of G. □

Proof of Theorem 5.2. This Theorem follows now from Remark 5.3 and Lemma 5.4 and Proposition 5.1. We leave this to the reader as an exercise. □

In the following, if we say "possibly different signed hermitian spaces" it includes possibly different vector spaces and possibly different skew-fields over F. As an example we allow one hermitian form over D and another over F.

Proposition 5.5. Let Δ and Δ' be two endo-equivalent self-dual strata on possibly different signed hermitian spaces h and h' of the first kind and of the same type. Then the matching is equivariant with respect to the adjoint involutions.

Proof. We need to consider the restrictions of Δ and Δ' to F. So, for this proof we just assume that both strata are strata over F. By repeating and raising we can assume that both strata share the F-period, and the parameters n and r. By diagonalization, see Proposition 4.3 for $i \in I_0$ and Proposition 4.2 for $i \in I_+$, we can assume for all $i \in I$ that β_i and $\beta'_{\zeta(i)}$ have the same minimal polynomial, and therefore $-\sigma_h(\beta_i)$ and $-\sigma_{h'}(\beta_{\zeta(i)})$, i.e. $\beta_{\sigma(i)}$ and $\beta'_{\sigma'(\zeta(i))}$, have the same minimal polynomial. Thus $\zeta \circ \sigma = \sigma' \circ \zeta$. □

We denote the class of self-dual semisimple strata which are endo-equivalent to a given self-dual semisimple stratum Δ by $\mathfrak{E}_-(\Delta)$.

6. Self-dual semisimple characters

In this section we study the first building block for explicit constructions of cuspidal irreducible representations of G. These building blocks are the self-dual semisimple characters. We are going to see that they correspond to the semisimple characters of $\tilde{G}$ which are fixed by the adjoint involution. After defining them we turn directly to transfers, followed by the matching theory, results on intertwining, diagonalization theory and conjugacy.

6.1. **First definitions.** Here we define self-dual semisimple characters. There is a big introduction about semisimple characters in section 5 in [23], and we use the notation and definitions from there. We fix an additive character ψ_F from F to $\mathbb{C}^\times$ of level one. One attaches to a semisimple stratum Δ a set $\mathrm{C}(\Delta)$ of complex valued characters on a compact open subgroup $H(\Delta)$ of $\tilde{G}$. Given a character χ on a subgroup of G we write $\sigma.\chi$ for $\chi \circ \sigma$. Recall that p is the odd residue characteristic of F.

Definition 6.1. Let Δ be a self-dual semisimple stratum. Then $H(\Delta)$ and $\mathrm{C}(\Delta)$ are invariant under the action of σ. We define $\mathrm{C}_-(\Delta)$ to be the set of all restrictions of the elements of $\mathrm{C}(\Delta)^\sigma$, i.e. the σ-fixed elements of $\mathrm{C}(\Delta)$, to $H_-(\Delta) := H(\Delta) \cap G$. The restriction map is a bijection by Glaubermann-correspondence using that $H(\Delta)$ is a pro-p-group with odd p. We call an element $\theta \in \mathrm{C}(\Delta)^\sigma$ the *lift* of $\theta|_{H_-(\Delta)}$ to $H(\Delta)$. The elements of $\mathrm{C}_-(\Delta)$ are called *self-dual semisimple characters*. For a stratum Δ' equivalent to Δ we define $\mathrm{C}(\Delta') := \mathrm{C}(\Delta)$ and $\mathrm{C}_-(\Delta') := \mathrm{C}_-(\Delta)$.

Proposition 6.2. Let Δ be a self-dual semisimple stratum and $\theta \in \mathrm{C}(\Delta)$. Then $\theta|_{H_-(\Delta)}$ is an element of $\mathrm{C}_-(\Delta)$.

Proof. If θ_1, θ_2 and θ_3 are elements of $\mathrm{C}(\Delta)$ then $\theta_1\theta_2^{-1}\theta_3$ is an element of $\mathrm{C}(\Delta)$ by the definition of $\mathrm{C}(\Delta)$, see [23, 5.6]. Thus the number of elements of $\mathrm{C}(\Delta)$ which have the same restriction as θ on $H_-(\Delta)$ is independent θ. The number of elements of $\mathrm{C}(\Delta)$ is a power of p and thus there is a non-negative integer s such that for every element $\theta \in \mathrm{C}(\Delta)$ there are exactly p^s elements in $\mathrm{C}(\Delta)$ with the same restriction as θ. Thus the action of σ on the set of those characters must have a fixed point. This finishes the proof. □

The group $\tilde{G}$ acts by conjugation on the set of all complex characters χ on subgroups of $\tilde{G}$ which we denote by $(g, \chi) \mapsto g.\chi$. An element $g \in \tilde{G}$ is said to intertwine a character $\chi : K \to \mathbb{C}$ with a second character $\chi' : K' \to \mathbb{C}$ if the restrictions of χ' and $g.\chi$ coincide on $gKg^{-1} \cap K'$. We denote the set of all elements of $\tilde{G}$ which intertwine χ with χ' by $I(\chi, \chi')$. We adapt the notation $I_H(\chi, \chi')$ for the intersection of $I(\chi, \chi')$ with a subgroup H of $\tilde{G}$. We say that χ and χ' intertwine by some element of H if $I_H(\chi, \chi')$ is non-empty.

6.2. **Transfers.** We now recall the notion of transfer. For more detail consult section [23, section 6.2]. Transfers are defined between strata over different vectors spaces over different skew-fields central and of finite degree over F. Let Δ and Δ' be two endo-equivalent semisimple strata. Let us at first recall the transfer over the same skew-field. The canonically defined restriction maps

$$res_{\Delta\oplus\Delta',\Delta} : \mathrm{C}(\Delta \oplus \Delta') \to \mathrm{C}(\Delta)$$

define a bijection

$$\tau_{\Delta,\Delta'} := res_{\Delta\oplus\Delta',\Delta'} \circ res^{-1}_{\Delta\oplus\Delta',\Delta} : \mathrm{C}(\Delta) \to \mathrm{C}(\Delta')$$

called the *transfer map* from Δ to Δ'.

In the case of different skew-fields the transfer map is defined as follows:

$$\tau_{\Delta,\Delta'}(\theta) := \tau_{\Delta\otimes L,\Delta'\otimes L}(\theta_L)|_{H(\Delta')},$$

where θ_L is any extension of θ to $H(\Delta \otimes L)$. Here we use that L splits the skew-fields. We call $\tau_{\Delta,\Delta'}(\theta)$ the transfer of θ from Δ to Δ'. Let us recall that if V and V' are the same vector spaces and Δ and Δ' intertwine then $\tau_{\Delta,\Delta'}(\theta) = \theta'$ if and only if $I(\Delta, \Delta') \subseteq I(\theta, \theta')$, see [23, 6.10, 5.9].

Proposition 6.3. Transfer commutes with the adjoint involutions, i.e. given two signed hermitian spaces (h, V) and (h', V') of the first kind and the same type with adjoint involutions σ and σ', respectively, and given two endo-equivalent semisimple strata Δ and Δ' in V and V', respectively, then $\sigma' \circ \tau_{\Delta,\Delta'} = \tau_{\#\Delta,\#'\Delta'} \circ \sigma$.

Proof. Without loss of generality we can assume that we work over the same skew-field, because the transfer is defined by reducing to that case, i.e. from different skew-fields to L. Now the result follows, because the restriction maps commute with the adjoint involutions, i.e. $res_{\#\Delta\oplus\#'\Delta',\#'\Delta'} \circ (\sigma \oplus \sigma') = \sigma' \circ res_{\Delta\oplus\Delta',\Delta'}$. □

The transfer map induces for self-dual semisimple strata Δ and Δ' via the Glauberman correspondence a bijection from $\mathrm{C}_-(\Delta)$ to $\mathrm{C}_-(\Delta')$ which maps θ_- to $\tau_{\Delta,\Delta'}(\theta)|_{H_-(\Delta')}$. We denote this map still by $\tau_{\Delta,\Delta'}$.

Let us recall:

Definition 6.4 ([23] 6.3)**.** Two semisimple characters $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$ on possibly different vector spaces over possibly different skew-fields over strata with the same degree and the same group level are called *endo-equivalent* if there are transfers which intertwine, i.e. if there are strata $\tilde{\Delta} \in \mathfrak{C}(\Delta)$ and $\tilde{\Delta}' \in \mathrm{C}(\Delta')$ such that $\tau_{\Delta,\tilde{\Delta}}(\theta)$ and $\tau_{\Delta',\tilde{\Delta}'}(\theta')$ intertwine.

6.3. **Matching, diagonalization and intertwining.** Here we repeat the result on matching and slightly generalize the diagonalization theorem from [23] and we further prove a diagonalization theorem for self-dual semisimple characters. We fix in this section two semisimple strata Δ and Δ' of the same group level and the same degree and two semisimple characters $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$. The restriction of θ to $\mathrm{Aut}_D(V^i) \cap H(\Delta)$ is denoted by θ_i, and it is a simple character for Δ_i.

6.3.1. ***For*** GL. We recall the analogue of Theorem 4.9 for characters.

Theorem 6.5 ([23] 5.48, 6.18,[5] 1.11)**.** Suppose $I(\theta,\theta') \neq \emptyset$. Then there is a unique bijection $\zeta: I \to I'$ such that the set $I(\theta,\theta') \cap \prod_{i\in I} \mathrm{Hom}_D(V^i, V^{\zeta(i)})$ is not empty. The latter non-emptiness condition is equivalent in saying that for all $i \in I$ the characters θ_i and $\theta'_{\zeta(i)}$ are endo-equivalent and the D-dimensions of V^i and $V^{\zeta(i)}$ agree.

We call ζ the *matching* from $\theta \in \mathrm{C}(\Delta)$ to $\theta' \in \mathrm{C}(\Delta')$ and we write $\zeta_{\theta,\Delta,\theta',\Delta'}$, but if there is no reason for confusion then we just skip Δ and Δ'. Further we get a map $\bar{\zeta}_{\theta,\theta'}$ between the residue algebras defined in the similar way as it was done for strata, see [23, 5.52]. $(\zeta,\bar{\zeta})$ is again called the matching pair of $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$.

Note that in the case of intertwining strata it is possible that $\zeta_{\theta,\theta'}$ and $\zeta_{\Delta,\Delta'}$ do not coincide. For an example see [15, 9.7]. But, if θ' is a transfer of θ from Δ to Δ', then both matching pairs coincide.

Diagonalization theorems are important to reduce a statement about semisimple characters to the case of transfers. Here is the GL-version.

Theorem 6.6. Suppose θ and θ' are endo-equivalent and suppose that the strata share the F-period and the parameters r and n. Then there is a semisimple stratum Δ'' on V' with the same associate splitting as Δ', $\Lambda'' = \Lambda'$, $n'' = n$ and $r'' = r$ such that $\Delta \oplus \Delta''$ is semisimple and $\theta \otimes \theta' \in \mathrm{C}(\Delta \oplus \Delta'')$.

The assertions of the Theorem imply that β and β'' have the same minimal polynomial, because $\Delta \oplus \Delta''$ is semisimple, and θ and θ' are endo-equivalent.

Proof. See the proof of [23, 5.42] and use Theorem [23, 6.18] instead of [23, 5.35], to get $\Delta'' = [\Lambda', n, r, \beta'']$ such that β'' has the same minimal polynomial as β and the same associate splitting as β' and such that θ' is the transfer of θ from Δ to Δ''. Then $\theta \otimes \theta' \in \mathrm{C}(\Delta \oplus \Delta'')$. □

We are now able to state the result on intertwining for semisimple characters.

Theorem 6.7. Suppose Δ and Δ' are semisimple strata which share the F-period and the parameters n and r and suppose that $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$ intertwine with matching ζ. Then the following holds.

(i) $I(\theta,\theta') = S(\Delta')(I(\theta,\theta') \cap \prod_{i\in I} \mathrm{Hom}_D(V^i, V^{\zeta(i)}))S(\Delta)$.

(ii) Suppose $I(\Delta,\Delta') \neq \emptyset$ and $\theta' = \tau_{\Delta,\Delta'}(\theta)$, then we have
$$I(\theta,\theta') = S(\Delta')I(\Delta,\Delta')S(\Delta).$$
If further there is an element $\tilde{g}$ of $\tilde{G}$ such that $\tilde{g}\beta\tilde{g}^{-1} = \beta'$, then we have
$$I(\theta,\theta') = S(\Delta')\tilde{g}C_A(\beta)^{\times}S(\Delta). \tag{6.8}$$

Proof. The proof of Theorem 6.7 is the same as for Theorem 4.10 using [23, 5.15] and Proposition 6.6 instead of [23, 4.36] and Proposition 4.2. We use that the set $S(\Delta)$ only depends on θ, Λ and r because it coincides with the intersection of $P_{\min(-(k_0+r),\lfloor\frac{k_0+1}{2}\rfloor)}(\Lambda)$ with $I(\theta)$. □

The next proposition is needed to establish the diagonalization theorem for self-dual characters later.

Proposition 6.9. Suppose that Δ is a stratum such that $\mathrm{C}(\Delta) = \mathrm{C}(\#\Delta)$, Λ is self-dual and $-\sigma_h(\beta)$ has the same minimal polynomial and associate splitting as β. Suppose further that $\tau_{\Delta,\#\Delta}$ is the identity map. Then, there is an element γ of A with the same minimal polynomial and associate splitting as β such that $[\Lambda, n, r, \gamma]$ is self-dual with the same set of semisimple characters as Δ.

Proof. We write Δ' for $\#\Delta$. We assume $I = I'$ without loss of generality. The character $\theta \in \mathrm{C}(\Delta)$ is intertwined by 1. Thus $\bar{\zeta}_{\theta,\Delta,\theta,\Delta'} = \mathrm{id}_{\kappa_E}$. Further θ is its transfer from Δ to Δ' and thus

$$\mathrm{id}_I = \zeta_{\theta,\Delta,\theta,\Delta'} = \zeta_{\Delta,\Delta'}$$

and therefore β_i and β_i' have the same minimal polynomial, and

$$\mathrm{id}_{\kappa_E} = \bar{\zeta}_{\theta,\Delta,\theta,\Delta'} = \bar{\zeta}_{\theta,\tau_{\Delta,\Delta'}(\theta)} = \bar{\zeta}_{\Delta,\Delta'}.$$

Thus by Theorem 4.16 there is an element u of $P(\Lambda)$ which conjugates β to β' and by (6.8) (for $\theta, \Delta, \Delta, \tilde{g} = 1$) we can assume that u is an element of $S(\Delta)$ and therefore the $S(\Delta)$-conjugacy class of β is invariant under the map $-\sigma_h$. Using the latter action we use the pro-p-group property to show by induction that there is a sequence $(s_k)_{k\in\mathbb{N}}$ in $(\prod_i A^{ii}) \cap S(\Delta)$ such that

$$-\sigma_h(s_k\beta s_k^{-1}) \equiv s_k\beta s_k^{-1} =: \beta^{(k)} \bmod \mathfrak{a}_{1+k}$$

and

$$\beta^{(k)} \equiv \beta^{(k-1)} \bmod \mathfrak{a}_k.$$

But then $(\beta^{(k)})_k$ converges to some element γ and by compactness there is a convergent subsequence of $(s_k)_k$, say with limit s_0. We obtain by continuity: $\gamma = s_0\beta s_0^{-1}$. This finishes the proof, noting that $S(\Delta)$ normalizes every element of $\mathrm{C}(\Delta)$. □

6.3.2. ***For*** G. In this part we assume additionally that both strata Δ and Δ' are self-dual with respect to ϵ-hermitian spaces (V, h) and (V', h') over (D, ρ). Suppose further $\theta \in \mathrm{C}(\Delta)^{\sigma}$ and $\theta' \in \mathrm{C}(\Delta')^{\sigma'}$ where σ' is the adjoint involution of h'. We also have a G-version of the diagonalization theorem. Note that there is no G intertwining required in the assumption:

Theorem 6.10. Under the assumptions of Theorem 6.6 we further assume that $\Lambda \oplus \Lambda'$ is self-dual. Then there exists a stratum Δ'' with $\Lambda'' = \Lambda'$, $n'' = n$ and $r'' = r$ such that $\Delta \oplus \Delta''$ is a self-dual semisimple stratum and $\theta \otimes \theta' \in \mathrm{C}(\Delta \oplus \Delta'')$.

Proof of Theorem 6.10. We can choose a stratum Δ'' which satisfies the assertions of Theorem 6.6. The character $\theta \otimes \theta'$ is fixed under the adjoint involution of $h \oplus h'$, by [23, 5.41], in particular we have $\mathrm{C}(\Delta \oplus \Delta'') = \mathrm{C}(\#(\Delta \oplus \Delta''))$ and therefore $\mathrm{C}(\Delta'') = \mathrm{C}(\#\Delta'')$. The elements β'', β and $-\sigma_{h'}(\beta'')$ have the same minimal polynomial. The character θ' is a restriction of $\theta \otimes \theta'$ to Δ'' and to $\#\Delta''$, and thus the transfer map from Δ'' to $\#\Delta''$ is the identity map. Now Lemma 6.9 finishes the proof. □

Remark 6.11. Starting with two self-dual lattice sequences Λ and Λ' of the same F-period we can always obtain $(2\Lambda)_j^{\#} = 2\Lambda_{1-j}$ for all $j \in \mathbb{Z}$ after a translation of the domain of Λ and similar for Λ'. So one can establish that $2\Lambda \oplus 2\Lambda'$ is self-dual.

The same track to Theorem 4.11 leads to:

Theorem 6.12 (see [15] 9.2,9.3 for the case over F)**.** Suppose Δ and Δ' are self-dual and share the F-period and the parameters n and r and suppose $\theta \in \mathrm{C}(\Delta)^{\sigma}$ and $\theta' \in \mathrm{C}(\Delta')^{\sigma}$ intertwine by an element of G with matching ζ. Then the following holds.

(i) $I_G(\theta, \theta') = (G \cap S(\Delta'))(I_G(\theta, \theta') \cap \prod_{i\in I} \mathrm{Hom}_D(V^i, V^{\zeta(i)}))(G \cap S(\Delta))$.
(ii) Suppose Δ and Δ' intertwine by an element of $\tilde{G}$ and $\theta' = \tau_{\Delta,\Delta'}(\theta)$, then we have

$$I_G(\theta, \theta') = (G \cap S(\Delta'))I_G(\Delta, \Delta')(G \cap S(\Delta)). \quad (6.13)$$

If there is further an element $\tilde{g}$ of $\tilde{G}$ such that $\tilde{g}\beta\tilde{g}^{-1} = \beta'$ then

$$I_G(\theta, \theta') = (G \cap S(\Delta'))(G \cap \tilde{g}C_A(\beta)^{\times})(G \cap S(\Delta)). \quad (6.14)$$

Proof. We want to take σ-fixed points in the equations of Theorem 6.7. Now the proof is a complete analogue of the proof of Theorem 4.11 except for (6.13), which follows from diagonalization, see Proposition 4.3, (6.14) and Theorem 4.11. □

Corollary 6.15. Suppose Δ and Δ' are self-dual and $\theta \in \mathrm{C}(\Delta)^\sigma$ and $\theta' \in \mathrm{C}(\Delta')^\sigma$. Suppose $I_G(\theta, \theta') \neq \emptyset$ with matching ζ. Then $I_G(\theta, \theta') \cap \prod_{i \in I} \mathrm{Hom}_D(V^i, V^{\zeta(i)})$ is non-empty and the ϵ-hermitian spaces h_i and $h_{\zeta(i)}$ are isometric for every $i \in I_0$.

Proof. By repeating and lowering, Remark 6.11 and Theorem 6.10 we can assume that β and β' have the same minimal polynomial and that $\theta' = \tau_{\Delta,\Delta'}(\theta)$. After raising, i.e. we consider the third parameter $\max(r, r')$, formula (6.14) implies that β is conjugate to β' over some element of G. Thus $I_G(\theta, \theta')$ contains an element of $\prod_{i \in I} \mathrm{Hom}_D(V^i, V^{\zeta(i)})$. This finishes the proof. □

Corollary 6.16. Intertwining is an equivalence relation on the set of self-dual semisimple characters of G of the same degree and the same group level.

Proof. Suppose $\theta_- \in \mathrm{C}_-(\Delta)$, $\theta'_- \in \mathrm{C}_-(\Delta')$ and $\theta''_- \in \mathrm{C}(\Delta'')$ are three self-dual semisimple characters such that $I_G(\theta_-, \theta'_-) \neq \emptyset$ and $I_G(\theta'_-, \theta''_-) \neq \emptyset$. Then by Glauberman correspondence $I_G(\theta, \theta')$ and $I_G(\theta', \theta'')$ are not empty. By Remark 6.11 and Theorem 6.10 (using repeating and lowering) we can assume without loss of generality that β, β' and β'' have the same minimal polynomials and that the three characters are transfers of each other. From Theorem 6.12 follows now (after repeating and raising) that β, β' and β'' are conjugate under some elements of G. Thus, as transfers, the characters θ and θ'' intertwine by some element of G. □

6.4. **Intertwining and conjugacy for self-dual semisimple characters.** Another important application of Theorem 6.10 is called intertwining and conjugacy theorem for self-dual semisimple characters.

Theorem 6.17. Suppose Δ and Δ' are self-dual semisimple strata sharing the F-period and the parameter r and $\theta \in \mathrm{C}(\Delta)^\sigma$ and $\theta' \in \mathrm{C}(\Delta')^\sigma$ intertwine by an element of G. Let $(\zeta, \bar{\zeta})$ be their matching pair. Let g be an element of $G \cap \prod_i \mathrm{Hom}_D(V^i, V^{\zeta(i)})$ which satisfies $g\Lambda = \Lambda'$ and such that the conjugation with g induces $\bar{\zeta}|_{\kappa_{E_D}}$. Then there is an element u of $G \cap \prod_i \mathrm{Hom}_D(V^i, V^{\zeta(i)})$ such that $u\Lambda = \Lambda'$ and $u\theta = \theta'$. We can choose u to further satisfy $u\beta u^{-1} = \beta'$ if β and β' have the same minimal polynomial over F and $\tau_{\Delta,\Delta'}(\theta) = \theta'$.

Proof. By diagonalization, see Theorem 6.10, we can assume that β and β' have the same minimal polynomial and $\theta' = \tau_{\Delta,\Delta'}(\theta)$. The set $I_G(\Delta, \Delta')$ is not empty by Theorem 6.12 and from the transfer property follows that (θ, θ') and (Δ, Δ') have the same matching pair. Now the theorem follows from Theorem 4.17. □

At the end of this section we show that σ-fixed semisimple character define with a self-dual lattice sequence are always lifts of self-dual semisimple characters:

Proposition 6.18. Suppose Δ is a semisimple stratum with a self-dual lattice sequence such that $\mathrm{C}(\Delta)$ is invariant under the action of σ. Then there is a self-dual semisimple stratum $\Delta' = [\Lambda, n, r, \beta']$ such that $\mathrm{C}(\Delta) = \mathrm{C}(\Delta')$.

Proof. The proof is done by induction on $n - r$. If $n = r$, then $\beta = 0$ by the definition of semisimple strata, and Δ is therefore self-dual. Suppose $n > r$. By induction hypothesis there is a self-dual semisimple stratum $\tilde{\Delta} := [\Lambda, n, r+1, \gamma]$ such that $\mathrm{C}(\tilde{\Delta}) = \mathrm{C}(\Delta(1+))$. By the translation principle [23, 5.43] we can assume without loss of generality that $\Delta(1+)$ is equivalent to $\tilde{\Delta}$. We take semisimple characters $\theta \in \mathrm{C}(\Delta)^\sigma$ and $\tilde{\theta} \in \mathrm{C}(\tilde{\Delta}(1-))^\sigma$. Then there is an element $a \in \mathfrak{a}_{-r-1}$ such that $\theta = \psi_a \tilde{\theta}$. Thus

$$\tilde{\theta}\psi_a = \theta = \sigma.\theta = \sigma.\tilde{\theta}\psi_{-\sigma_h(a)} = \tilde{\theta}\psi_{-\sigma_h(a)}.$$

Thus ψ_a is equal to ψ_{a_-} for $a_- := \frac{a - \sigma_h(a)}{2}$. We are going to prove that $\Delta'' := [\Lambda, n, r, \gamma + a_-]$ is equivalent to a semisimple stratum Δ', because then we can take Δ' to be self-dual, by 4.7, and we get that $\mathrm{C}(\Delta')$, which is $\mathrm{C}(\tilde{\Delta}(1-))\psi_{a_-}$, intersects $\mathrm{C}(\Delta)$ non-trivially and is therefore equal to $\mathrm{C}(\Delta)$ by [23, Proposition 5.38]. We prove that the derived stratum $\partial_\gamma(\Delta'')$ is equivalent to a semisimple multi-stratum, see [23, 4.14, 4.15] for the definition and the usage of the derived multi-stratum. The stratum $\partial_\gamma(\Delta)$ is a semisimple multi-stratum, see Theorem [23, 4.15], because Δ is semisimple. Let s_γ be the chosen corestriction for γ (for forming the derived strata). Then $s_\gamma(a_- + \gamma - \beta)$ is modulo $\mathfrak{a}_{-r}$ congruent to an element of $F[\gamma]$ because it is intertwined by every element of $C_A^\times(\gamma)$ because

$$\mathrm{C}(\tilde{\Delta}(1-)) \cap \mathrm{C}(\tilde{\Delta}(1-))\psi_{a_-+\gamma-\beta} \neq \emptyset.$$

Thus $\partial_\gamma(\Delta'')$ is equivalent to a semisimple multi-stratum. Thus Δ'' is equivalent to a semisimple stratum by [23, 4.15]. □

7. Self-dual semisimple pss-characters

In this section we answer the question, when two transfers of endo-equivalent self-dual semisimple characters intertwine over an element of the corresponding classical group.We will introduce endo-parameters, as in [15].

7.1. Self-dual pss-characters.

7.1. Self-dual pss-characters. To introduce self-dual pss-characters we will take a slightly different path than in [15] because they should not depend on ϵ. We consider all objects of the category $\mathcal{H}$ of orthogonal and symplectic hermitian forms over (D,ρ) and (F,id). We start with what should be the domain of a self-dual semisimple character: Let (Δ,h) be a pair consisting of $h\in\mathcal{H}$ and a self-dual semisimple stratum with respect to h, we also say *h-self-dual*. We define the *self-dual endo-class of* Δ as the following class:

$$\mathfrak{E}_-(\Delta) := \{(\Delta',h') \in \mathfrak{E}(\Delta)\times\mathcal{H} \mid \Delta' \text{ is } h'\text{-self-dual}\}.$$

And we define now self-dual pss-character. For the definition of a pss-character (potentially semisimple character) we refer to [23, 6.3], which is motivated by [5].

Definition 7.1. Let $\mathfrak{E}_-$ be a self-dual endo-class. A self-dual pss-character is a map

$$\Theta_- :\ \mathfrak{E}_- \to \bigcup_{(\Delta,h)\in\mathcal{E}_-} \mathrm{C}_-(\Delta),$$

such that $\Theta_-(\Delta,h)\in \mathrm{C}_-(\Delta)$ for all $(\Delta,h)\in\mathfrak{E}_-$ and such that the values are related by transfer, i.e.

$$\tau_{\Delta,\Delta'}(\Theta_-(\Delta,h)) = \Theta_-(\Delta',h'),$$

for all $(\Delta,h),(\Delta',h')\in\mathfrak{E}_-$.

We attach to a self-dual pss-character Θ_- with domain $\mathfrak{E}_-$ the pss-character Θ whose domain $\mathfrak{E}$ contains a stratum Δ, such that $(\Delta,h)\in\mathfrak{E}_-$ for some $h\in\mathcal{H}$ and such that $\Theta(\Delta)$ is the lift of $\Theta_-(\Delta,h)$. We call Θ the lift of Θ_-. Two pss-characters Θ and Θ' (self-dual pss-characters Θ_- and Θ'_-) of the same degree and the same group level are called *endo-equivalent* if there are strata $\Delta\in\mathfrak{E}$ and $\Delta'\in\mathfrak{E}'$ ($(\Delta,h)\in\mathfrak{E}_-$ and $(\Delta',h)\in\mathfrak{E}'_-$) such that $\Theta(\Delta)$ and $\Theta'(\Delta')$ intertwine ($\Theta_-(\Delta,h)$ and $\Theta'_-(\Delta',h)$ intertwine by an element of $\mathrm{U}(h)$).

Proposition 7.2 ([15] 10.6,[23] 6.18)**.** Two self-dual pss-character of the same degree and the same group level are endo-equivalent if and only if their lifts are endo-equivalent.

The equivalence classes are called *endo-classes* of pss-characters (self-dual pss-characters). A pss-character is called ps-character (potentially simple character) if its values are simple characters. Similar we define self-dual ps-characters. A self-dual pss-character and its endo-class is called *elementary* if I_Θ is a σ-orbit, i.e. either it is simple or $I_\Theta = I_{\Theta,\pm}$ consisting only of two elements.

7.2. Idempotents and Witt groups. We recall the equivalence of categories of hermitian forms. Let (E,σ_E) be a field extension of F together with an F-linear non-trivial involution on E

Proposition 7.3. Let $\mathcal{H}_{\sigma_E\otimes\rho,\epsilon}$ be the category of ϵ-hermitian $E\otimes_F D$-forms with respect to $\sigma_E\otimes\rho$, and let $\mathcal{H}_{\sigma_E,\epsilon}$ the category of ϵ-hermitian E-forms with respect to σ_E. Then $\mathcal{H}_{\sigma_E\otimes\rho,\epsilon}$ is equivalent to $\mathcal{H}_{\sigma_E,\epsilon}$.

Proof. We define a functor $\mathcal{F}:\mathcal{H}_{\sigma_E\otimes\rho,\epsilon}\to\mathcal{H}_{\sigma_E,\epsilon}$ in the following way. We consider an E-algebra isomorphism $\Phi:\ E\otimes_F D\cong \mathrm{M}_2(E)$. The anti-involution $\sigma_E\otimes\rho$ is pushed forward to a unitary anti-involution which can be interpreted as the adjoint anti-involution of a unitary bilinear form. Such a form has a diagonal σ_E-symmetric Gram-matrix $u=\mathrm{diag}(u_1,u_2)$ because the characteristic of F is not 2. So we can choose Φ such that the push forward of $\sigma_E\otimes\rho$ is $u\sigma_E(\ )^Tu^{-1}$. We identify $(E\otimes_F D,\sigma_E\otimes_F\rho)$ with $(\mathrm{M}_2(E),u\sigma_E(\ )^Tu^{-1})$. We consider

$$1^1:=\begin{pmatrix}1&0\\0&0\end{pmatrix}\ 1^2:=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

We define

$$\mathcal{F}(V,\tilde{h}):=(V1^1,\mathrm{tr}_E\circ\tilde{h}|_{V1^1\times V1^1}).$$

Exercise: Show that $\mathcal{F}(V,\tilde{h})$ is non-degenerate.

The inverse functor $\mathcal{G}$ is given by

$$\mathcal{G}(W,h_E):=(W\oplus W,\tilde{h}),\ \tilde{h}((w_1,w_2),(w_1',w_2'))=\begin{pmatrix}h_E(w_1,w_1')&h_E(w_1,w_2')\\u_2u_1^{-1}h_E(w_2,w_1')&u_2u_1^{-1}h_E(w_2,w_2')\end{pmatrix}.$$

The $E\otimes_F D$-action on $W\oplus W$ is given by

$$(w_1,w_2)E_{ij}:=(\delta_{j1}w_i,\delta_{j2}w_i),$$

for $1\le i,j\le 2$, and by the E-action on W, where E_{ij} is the matrix with zero entries except for the entry at (i,j) which is 1. Here the element δ_{ij} is the Kronecker-symbol. □

This Proposition shows that the objects $\mathcal{H}_{\sigma_E\otimes\rho,\epsilon}$ inherit the decomposition properties from the objects of $\mathcal{H}_{\sigma_E,\epsilon}$. In particular we have a Witt group of $E\otimes_F D$ with respect to $\sigma_E\otimes\rho$ and ϵ, we denote this Witt group by $W_\epsilon(\sigma_E\otimes\rho)$.

Proposition 7.4. The Witt-group $W_\epsilon(\sigma_E\otimes\rho)$ is independent of the choice of the isomorphism from $E\otimes_F D$ to $\mathrm{M}_2(E)$.

Proof. Consider two E-algebra isomorphisms $\Phi_i:\ (E\otimes_F D,\sigma_E\otimes\rho)\xrightarrow{\sim}(\mathrm{M}_2(E),u_i\sigma_E(\ )^Tu_i^{-1})$, such that u_i is diagonal and symmetric with respect to σ_E. Let the above functor be denoted by $\mathcal{F}_i$. Since $\mathcal{F}_2\circ\mathcal{G}_1$ sends isometric spaces to isometric spaces, and respects orthogonal sums, we only need to prove that it sends hyperbolic spaces of Witt index 1 to hyperbolic spaces of Witt index 1. We take $(V,\tilde{h})\in\mathcal{H}_{\sigma_E\otimes\rho,\epsilon}$ such that $\mathcal{F}_1(\tilde{h})$ is a hyperbolic space of Witt index 1. Then $\mathcal{F}_1(\tilde{h})$ has a non-zero isotropic vector and thus $\tilde{h}$ has a non-zero isotropic vector $v\in V$. This vector generates a simple $E\otimes_F D$-module $<v>$ on which $\tilde{h}$ vanishes. Let 1_2^1 and 1_2^2 be the idempotents for u_2. There is a simple $E\otimes_F D$-module which does not vanish under 1_2^1, see for example $E\times E$, so $<v>$ does not vanish under 1_2^1, because all simple $E\otimes_F D$-modules are isomorphic to each other. Thus $\mathcal{F}_2(\tilde{h})$ has a non-zero isotropic vector, which finishes the proof. □

The functor $\mathcal{F}$ in the Proof of 7.3 is completely characterized by the idempotent e of $E\otimes_F D$ which is mapped to 1^1 under Φ, more precisely $\mathcal{F}(\tilde{h})=\mathrm{tr}_E\circ\tilde{h}|_{Ve}$. We denote the functor associated to e by $\mathcal{F}_e$ and we take any inverse $\mathcal{G}_e$ constructed in Proposition 7.3. Let $\mathrm{Idemp}(\sigma_E\otimes\rho)$ be the set of rank 1 idempotents of $E\otimes_F D$ which are fixed by $\sigma_E\otimes\rho$. Two elements of $\mathrm{Idemp}(\sigma_E\otimes\rho)$ are conjugate by a similitude g of $\sigma_E\otimes\rho$.

Proposition 7.5. Let $e,f\in\mathrm{Idemp}(\sigma_E\otimes\rho)$ and let g be an element of $(E\otimes_F D)^\times$ such that $geg^{-1}=f$ and $g(\sigma_E\otimes_F\mathrm{id}_D)(g)=s\in E^\times$. Then $\mathcal{F}_f\circ\mathcal{G}_e$ is equivalent to the functor

$$h_E\mapsto s^{-1}h_E.$$

Proof. We take forms $\tilde{h}\in\mathcal{H}_{\sigma_E\otimes\rho,\epsilon}$ and $h_E\in\mathcal{H}_{\sigma_E,\epsilon}$ such that $\mathcal{F}_e(\tilde{h})=h_E$. For $v_1,v_2\in Vf$ we have:

$$\begin{aligned}\mathrm{tr}_E(\tilde{h}(v_1,v_2)) &= \mathrm{tr}_E(f\tilde{h}(v_1,v_2)f)\\ &= \mathrm{tr}_E(geg^{-1}\tilde{h}(v_1,v_2)geg^{-1})\\ &= \mathrm{tr}_E(ge\tilde{h}(v_1gs^{-1},v_2g)eg^{-1})\\ &= s^{-1}h_E(v_1g,v_2g).\end{aligned}$$

Thus $\mathcal{F}_f(\tilde{h})$ is isometric to $s^{-1}h_E$ via $g:\ Vf\to Ve$. Thus the maps $g:\ \mathcal{F}_f(\tilde{h})\to\mathcal{F}_e(\tilde{h})$ form a natural transformation from $\mathcal{F}_f$ to $s^{-1}\mathcal{F}_e$ which is an equivalence. Thus $\mathcal{F}_f\circ\mathcal{G}_e$ is equivalent to $s^{-1}\mathcal{F}_e\circ\mathcal{G}_e$ and the latter is

$$h_E\mapsto s^{-1}h_E$$

by the definition of $\mathcal{G}_e$ in the proof of Proposition 7.3. □

We are now able to describe the elements of $W_\epsilon(\sigma_E\otimes\rho)$ independent of the choice of an idempotent. We call a field extension $E=F[\beta]|F$ *self-dual* if there is an automorphism σ_E of $E|F$ which maps β to $-\beta$.

Definition 7.6. Suppose $E=F[\beta]$ is a self-dual field extension different from F. A *Witt tower* for σ_E,ρ,ϵ is a function

$$wtower:\ \mathrm{Idemp}(\sigma_E\otimes\rho)\to W_\epsilon(\sigma_E)$$

which satisfies $wtower(f)=s^{-1}wtower(e)$ for all $e,f\in\mathrm{Idemp}(\sigma_E\otimes\rho)$ using the similitude of Proposition 7.5. If there is no confusion with ρ and ϵ then we also say Witt tower for σ_E or for β.

We have a canonical bijection from $W_\epsilon(\sigma_E\otimes\rho)$ to the set of Witt towers for σ_E,ρ,ϵ: An element $\tilde{h}_\equiv$ is mapped to $wtower_{\tilde{h}}$ defined via $wtower_{\tilde{h}}(e):=\mathcal{F}_e(\tilde{h})_\equiv\in W_\epsilon(\sigma_E)$. The map is well-defined by Proposition 7.5. We identify $W_\epsilon(\sigma_E\otimes\rho)$ with the set of Witt towers for σ_E,ρ,ϵ. Given a field extension $(E,\sigma_E)|F$ with non-trivial σ_E and an F-linear σ_E-id_F-equivariant non-zero map $\lambda:E\to F$ there is a natural map

$$\mathrm{Tr}_\lambda:\ W_\epsilon(\sigma_E\otimes\rho)\to W_\epsilon(\rho),$$

defined via $Tr_\lambda(\tilde{h}_\equiv):=((\lambda\otimes_F\mathrm{id}_D)\circ\tilde{h})_\equiv$.

7.3. **Witt towers and intertwining.** Given an ϵ-hermitian form h on V with respect to (D,ρ) and an element β in the Lie algebra of G such that $F[\beta]$ is a field, there are unique forms $h_\beta\in\mathcal{H}_{\sigma_E,\epsilon}$ and $\tilde{h}_\beta\in\mathcal{H}_{\sigma_E\otimes\rho,\epsilon}$ such that $\lambda_\beta\circ h_\beta=\mathrm{trd}_{D|F}\circ h$ and $(\lambda_\beta\otimes_F\mathrm{id}_D)\circ\tilde{h}_\beta=h$, where $\lambda_\beta:\ E\to F$ is the F-linear extension of id_F with kernel $\sum_{i=1}^{[E:F]-1}\beta^iF$. We have

$$h_\beta=\mathrm{tr}_E\circ\tilde{h}_\beta \tag{7.7}$$

because:

Lemma 7.8. $\lambda_\beta\circ\mathrm{tr}_E=\mathrm{trd}_{D|F}\circ(\lambda_\beta\otimes_F\mathrm{id}_D)$.

Proof. It is a direct consequence of $\mathrm{trd}_{D|F}(x)=\mathrm{tr}_E(x)$ for all $x\in D$. □

We call the function $wtower_{\tilde{h}_\beta}$ the Witt tower of (β,h) and we also denote it by $wtower_{\beta,h}$.

We want to be able to compare Witt towers for different β and β'.

Let us recall that β is called minimal over F, if $\nu_{F[\beta]}(\beta)$ is co-prime to $e(F[\beta]|F)$ and the residue class of $\beta^{e(F[\beta]|F)}\pi_F^{-\nu_{F[\beta]}(\beta)}$ generates $\kappa_{F[\beta]}$ over κ_F. Define $e_p(\beta)$ as the fraction $\frac{e_{wild}(E|F)}{\gcd(e_{wild}(E|F),\nu_E(\beta))}$ where $e_{wild}(E|F)$ is the wild ramification index. Among those field extensions $E'|F$ in $E=F[\beta]$ which are generated by an element congruent to $\beta^{e_p(\beta)}$ there is a smallest one. (Reason: In a field extension E' in E generated by an element congruent to $\beta^{e_p(\beta)}$ there is an element π_0 with $\nu_E(\pi_0)=\gcd(e(E|F),\nu_E(\beta^{e_p(\beta)}))$ which satisfies

$$\beta^{e_p(\beta)}\equiv\pi_0^{\frac{\nu_E(\beta^{e_p(\beta)})}{\nu_E(\pi_0)}}\iota,\ \pi_0^{\frac{e(E|F)}{\nu_E(\pi_0)}}=\pi_F\lambda$$

where ι and λ are roots of unity with order prime to p. Then $\gamma_{E'} := \pi_0^{\frac{\nu_E(\beta^{e_p(\beta)})}{\nu_E(\pi_0)}}\iota$ is a minimal element over F and generates a sub-extension of E'. For two different such field extensions E'' and E' the element $\gamma_{E'}\gamma_{E''}^{-1}$ is an element of $1+\mathfrak{p}_E$ and a root of unity with order prime to p. Thus $\gamma_{E'}$ and $\gamma_{E''}$ coincide.)

We write $F[\beta_{min,tr}]$ where $\beta_{min,tr}$ is a minimal element over F and congruent to $\beta^{e_p(\beta)}$. In saying $\beta_{min,tr}$ we mean a choice. This choice is not canonical and can be changed to our purpose if necessary, for example if β is skew-symmetric, we choose $\beta_{min,tr}$ skew-symmetric (w.r.t σ_E).

So let us assume that $E=F[\beta]$ and $E'=F[\beta']$ are self-dual field extensions with non-zero β and β'. We are going to define a map from $W_\epsilon(\sigma_E\otimes\rho)$ to $W_\epsilon(\sigma_{E'}\otimes\rho)$. This is a generalization of [15, 5.2]. Let us recall them:

$$w_{\beta,\beta'}:\ W_{-1}(\sigma_E)\to W_{-1}(\sigma_{E'}),\ w_{\beta^2,\beta'^2}:\ W_1(\sigma_E)\to W_1(\sigma_{E'})$$

are bijections which respect the anisotropic dimension and satisfy

$$w_{\beta,\beta'}(\langle\beta\rangle_\equiv)=\langle\beta'\rangle_\equiv,\ w_{\beta^2,\beta'^2}(\langle\beta^2\rangle_\equiv)=\langle\beta'^2\rangle_\equiv.$$

Let us write $w_{\beta,\beta',1}$ for w_{β^2,β'^2} and $w_{\beta,\beta',-1}$ for $w_{\beta,\beta'}$. We need two further assumptions:

(A) Suppose there is an isomorphism from $F[\beta_{min,tr}]|F$ to $F[\beta'_{min,tr}]|F$ which sends $\beta_{min,tr}$ to an element congruent to $\beta'_{min,tr}$, we write "$\beta_{min.tr}\mapsto\beta'_{min,tr}$" for this unique isomorphism.
(B) Suppose that $F[\beta_{min,tr}]_0\subseteq N_{E|E_0}(E)$ if and only if $F[\beta'_{min,tr}]_0\subseteq N_{E'|E'_0}(E')$. (This is a numerical condition as the next lemma states.)

We define the bijection $w_{\beta,\beta',\epsilon}:\ W_\epsilon(\sigma_E\otimes\rho)\to W_\epsilon(\sigma_{E'}\otimes\rho)$ as follows: Let $e\in\mathrm{Idemp}(\sigma_E|_{F[\beta_{min,tr}]}\otimes_F\rho)$ and $e'\in\mathrm{Idemp}(\sigma_{E'}|_{F[\beta'_{min,tr}]}\otimes_F\rho)$ be idempotents such that $e\mapsto e'$ under

$$F[\beta_{min,tr}]\otimes_F D\overset{\text{“}\beta_{min,tr}\mapsto\beta'_{min,tr}\text{”}}{\longrightarrow}F[\beta'_{min,tr}]\otimes_F D.$$

We define $w_{\beta,\beta',\epsilon}(wtower)$ to be the Witt tower of $\sigma_{E'}\otimes\rho$ which satisfies

$$w_{\beta,\beta',\epsilon}(wtower)(e'):=w_{\beta,\beta',\epsilon}(wtower(e)).$$

This map is well-defined, i.e. does not depend on the choice of the idempotent, by Proposition 7.5 and (B).

Lemma 7.9. Let $=F[\beta]|F$ be a field extension with a non-trivial involution σ_E given by $\beta\mapsto-\beta$. Then $F[\beta_{min,tr}]_0$ is a subset of $\mathrm{N}_{E|E_0}(E)$ if and only if $E|F[\beta_{min,tr}]_0$ has even ramification index and even inertia degree. If we do not have the above containment, then $x\in F[\beta_{min,tr}]_0$ is a norm of $F[\beta_{min,tr}]|F[\beta_{min,tr}]_0$ if and only if it is a norm of $E|E_0$.

Proof. We just write E' for $F[\beta_{min,tr}]$. At first we remark that a norm of $E'|E'_0$ is a norm of $E|E_0$ because both have degree two and σ_E and $\sigma_E|_{E'}$ are the Galois generators. Suppose now that $E|E'_0$ has even ramification index and even inertia degree. Then every element of $o_{E'_0}^\times$ is a square in E and therefore a norm of $E|E_0$. If $E'|E'_0$ is ramified then E'_0 contains a uniformizer which is a norm of $E'|E'_0$, and we have the desired containment. In the case of an unramified $E'|E'_0$, since $e(E|E'_0)$ is even there is a uniformizer of the maximal unramified extension in $E_0|E'_0$ which is a square of an element in E_0 and is therefore a norm of $E|E_0$. All elements of $o_{E_0}^\times$ are norms of $E|E_0$ because $E|E_0$ is also unramified. Thus there is a uniformizer of E'_0 which is a norm of $E|E_0$. Suppose for the converse that all elements of E'_0 are norms of $E|E_0$. Then all elements of $\kappa_{E'_0}$ are squares in κ_E, i.e. $f(E|E'_0)$ is even, and there is a uniformizer of E'_0 which is a norm of $E|E_0$, in particular the ν_E-valuation of this uniformizer must be even and therefore $e(E|E'_0)$ is even. In the case where E'_0 is not contained in $\mathrm{N}_{E|E_0}(E)$, say $x\in E'_0$ is not a norm of $E|E_0$, the set of all non-norms of $E'|E'_0$ which is $x\mathrm{N}_{E'|E'_0}(E'^\times)$ is disjoint to $\mathrm{N}_{E|E_0}(E)$ which finishes the proof. □

Definition 7.10 (see [15] For the non-quaternionic case)**.** Let h and h' be two ϵ-hermitian forms over (D,ρ) and suppose that $\beta\in\mathrm{Lie}(U(h))$ and $\beta'\in\mathrm{Lie}(U(h'))$ generate field extensions E and E' different from F, such that (A) and (B) hold. We say the pairs (β,h) and (β',h') *have the same Witt type* if $h_\equiv=h'_\equiv$ in $W_\epsilon(\rho)$ and $w_{\beta,\beta',\epsilon}((\tilde{h}_\beta)_\equiv)=(\tilde{h}'_{\beta'})_\equiv$. Note that having the same Witt type is an equivalence relation on the pairs (β,h).

Remark 7.11. If $g : h \cong h'$ is an isometry then (β, h) and $(g\beta g^{-1}, h')$ have the same Witt type.

Proof. We write ϕ for the map from E to E' satisfying $\phi(x) := gxg^{-1}$. We consider $\tilde{h} := \tilde{h}_\beta$ and $\tilde{h}' := (\phi \otimes \mathrm{id}_D) \circ \tilde{h} \circ (g^{-1} \times g^{-1})$. We have to show $\tilde{h}' = \tilde{h}'_{\beta'}$. For $v_1, v_2 \in V$ we have:

$$\begin{aligned}
\tilde{h}'(v_1, v_2\phi(x)) &= (\phi \otimes \mathrm{id}_D)(\tilde{h}(g^{-1}(v_1), g^{-1}(v_2\phi(x)))) \\
&= (\phi \otimes \mathrm{id}_D)(\tilde{h}(g^{-1}(v_1), (g^{-1} \circ \phi(x))(v_2))) \\
&= (\phi \otimes \mathrm{id}_D)((\tilde{h}(g^{-1}(v_1), (x \circ g^{-1})(v_2))) \\
&= (\phi \otimes \mathrm{id}_D)(\tilde{h}(g^{-1}(v_1), g^{-1}(v_2)x)) \\
&= (\phi \otimes \mathrm{id}_D)(\tilde{h}(g^{-1}(v_1), g^{-1}(v_2))x) \\
&= \tilde{h}'(v_1, v_2)\phi(x)
\end{aligned}$$

Proceeding further this way we see that $\tilde{h}'$ is an ϵ-hermitian form which satisfies $(\lambda_{\beta'} \otimes \mathrm{id}_D) \circ \tilde{h}' = h'$ which finishes the proof. □

Theorem 7.12. Let Δ and Δ' be two non-null self-dual simple strata on (V, h) and $\theta_- \in \mathrm{C}_-(\Delta)$ and $\theta'_- \in \mathrm{C}_-(\Delta')$ be two endo-equivalent self-dual simple characters. Then θ_- and θ'_- intertwine by an element of G if and only if (β, h) and (β', h) have the same Witt type.

We need for the proof the twist h^γ of a signed hermitian form h by a skew-symmetric or symmetric element $\gamma \in \tilde{G}$, see subsection 2.3. If γ is skew-symmetric and invertible then h is symplectic if and only if h^γ is orthogonal

Proof. We choose lifts θ and θ' for θ_- and θ'_-. The field extensions $F[\beta_{min,tr}]|F$ and $F[\beta'_{min,tr}]|F$ are isomorphic by a σ_h-equivariant map which sends $\beta_{min,tr}$ to an element congruent to $\beta'_{min,tr}$. So we can assume without loss of generality that $\beta'_{min,tr}$ is the image of $\beta_{min,tr}$ under this isomorphism. Suppose at first $I_G(\theta, \theta') \neq \emptyset$. Then $\beta_{min,tr}$ and $\beta'_{min,tr}$ are conjugate by an element of G. Thus we can assume $\beta_{min,tr} = \beta'_{min,tr}$ without loss of generality. (Note that conjugation with an element of G does not change the equivalence class of (β, h) by Remark 7.11.) Now take any idempotent e of $F[\beta_{min,tr}] \otimes_F D$ and choose $(\tilde{\Delta}, \mathrm{trd}_{D|F} \circ h|_{Ve}) \in \mathfrak{E}_-(\Delta)$ and $(\tilde{\Delta}', \mathrm{trd}_{D|F} \circ h|_{Ve}) \in \mathfrak{E}_-(\Delta')$ such that $\tilde{\beta} = e\beta$ and $\tilde{\beta}' = e\beta'$. The transfers of θ and θ' to $\tilde{\Delta}$ and $\tilde{\Delta}'$, respectively, intertwine by an element of $\mathrm{Aut}_F(Ve)$ by Theorem [23, 6.1]. Proposition [15, 5.3] implies for the orthogonal case that $(\tilde{\beta}, \mathrm{trd}_{D|F} \circ h|_{Ve})$ and $(\tilde{\beta}', \mathrm{trd}_{D|F} \circ h|_{Ve})$ have the same Witt type and therefore (β, h) and (β', h) have the same Witt type too by (7.7). In the symplectic case we consider the twist $h^{\beta_{min,tr}}$. The latter is orthogonal and the argument of part one shows that $(\beta, h^{\beta_{min,tr}})$ and $(\beta', h^{\beta_{min,tr}})$ have the same Witt type. Thus $wtower_{\beta, h^{\beta_{min,tr}}}(e)$ and $wtower_{\beta', h^{\beta_{min,tr}}}(e)$ are mapped to each other under $w_{\beta,\beta',1}$. The twist with $\beta_{min,tr}$ induces the map

$$W_{-1}(\sigma_E) \to W_1(\sigma_E),\ \langle\beta\rangle_\equiv \mapsto \langle\beta^{e_p(\beta)+1}\rangle_\equiv = \langle(-1)^{\frac{e_p(\beta)-1}{2}}\beta^2\rangle_\equiv,$$

and we have the analogue formula for β'. Now -1 is a norm of $E|E_0$ if and only if it is a norm of $E'|E'_0$ because both $E|F$ and $E'|F$ have the same inertia degree. Thus the pull back of the map $w_{\beta,\beta',1}$ under the twist is the map $w_{\beta,\beta',-1}$. We obtain that (β, h) and (β', h) have the same Witt type.

We now consider the converse. So suppose that (β, h) and (β', h) have the same Witt type. Without loss of generality we can assume that β and β' have the same minimal polynomial by Theorem 6.10 and that the characters are transfers of each other. Note that this did not leave the equivalence class of (β, h) by the first direction of this proof. We have $e \in \mathrm{Idemp}(\sigma_E \otimes \rho)$ and $e' \in \mathrm{Idemp}(\sigma_{E'} \otimes_F \rho)$ which are mapped under "$\beta_{min,tr} \mapsto \beta'_{min,tr}$" to each other such that $\mathcal{F}_e(\tilde{h}_\beta)_\equiv$ is mapped to $\mathcal{F}_{e'}(\tilde{h}_{\beta'})_\equiv$ under $w_{\beta,\beta',1}$. We consider the pullback $\phi^*\tilde{h}_{\beta'}$ of $\tilde{h}_{\beta'}$ along the field-isomorphism $\phi : E \to E'$ with $\phi(\beta) = \beta'$. Then having the same Witt type implies that $\mathcal{F}_e(\tilde{h}_\beta)$ and $\mathcal{F}_e(\phi^*\tilde{h}_{\beta'})$ are isometric. Thus $\tilde{h}_\beta$ and $\phi^*\tilde{h}_{\beta'}$ are isometric and thus there is an element g of G which conjugates β to β'. And this intertwines θ with θ' because the two characters are transfers of each other. □

The last proof shows a nicer version for an equivalent criteria for G-intertwining.

Corollary 7.13. Let Δ and Δ' be two self-dual non-null simple strata for (V,h) and $\theta \in \mathrm{C}(\Delta)^\sigma$ and $\theta' \in \mathrm{C}(\Delta')^\sigma$ be two endo-equivalent simple characters. Then the following conditions are equivalent:

(i) $I_G(\theta,\theta') \neq \emptyset$
(ii) $\Delta((n-r-1)+)$ and $\Delta'((n'-r'-1)+)$ intertwine under some element of G.
(iii) $(\beta_{min,tr},h)$ and $(\beta'_{min,tr},h)$ have the same Witt type.
(iv) (β,h) and (β',h) have the same Witt type

Proof. The first and the last assertion are equivalent by Theorem 7.12 and [15, 7.1]. Further (i) implies (ii), and the assertion (ii) implies (iii) by the implication (i)⇒(iv) for strata (equivalently: characters which are transfers). So let us assume (iii). We can assume without loss of generality that $\beta'_{min,tr}$ is the image of $\beta_{min,tr}$ under $F[\beta_{min,tr}]|F \cong F[\beta'_{min,tr}]|F$. Then (iii) implies that $\beta_{min,tr}$ and $\beta'_{min,tr}$ are conjugate by an element of G, so we can assume without loss of generality that $\beta_{min,tr}$ and $\beta'_{min,tr}$ coincide. Now we proceed as in the proof of Theorem 7.12 to conclude from the endo-equivalence of θ with θ' that (β,h) and (β',h) have the same Witt type. □

We now generalize the formalism of [15] which leads to endo-parameters for quaternionic forms of classical groups.

For that we need to include the case $\beta = 0$. We call $h_{\equiv} \in W_\epsilon(\rho)$ the Witt tower of $(0,h)$. We say that $(0,h)$ and $(0,h')$ have the same Witt type if $h_{\equiv} = h'_{\equiv}$. Let us recall we can identify the index sets of strata of the domain of a pss-character to one set I_Θ, and endo-equivalent pss-characters Θ and Θ' determine a matching $\zeta: I_\Theta \to I_{\Theta'}$, see Theorem [23, 6.18]. Given lifts Θ and Θ' of self-dual pss-characters the index set decomposes into $I_\Theta = I_{\Theta,0} \cup I_{\Theta,\pm}$, and we sometimes choose a disjoint union $I_{\Theta,\pm} = I_{\Theta,+} \cup I_{\Theta,-}$ such that $\sigma(I_{\Theta,+}) = I_{\Theta,-}$. Let us recall that given a bijection $\zeta: I_\Theta \to I_{\Theta'}$, a ζ-comparison pair is a pair $(\Delta,\Delta') \in \mathfrak{E} \times \mathfrak{E}'$ such that both strata are defined over the same skew-field D and $\dim_D V^i = \dim_D V^{\zeta(i)}$ for all $i \in I_\Theta$, see [23, 6.17].

Theorem 7.14. Let Θ_- on $\mathfrak{E}_-$ and Θ'_- on $\mathfrak{E}'_-$ be two endo-equivalent self-dual pss-characters with lifts Θ and Θ' and matching $\zeta = \zeta_{\Theta,\Theta'}$. Then for given pairs $(\Delta,h) \in \mathfrak{E}_-$ and $(\Delta',h) \in \mathfrak{E}_-$ the following assertions are equivalent:

(i) $\Theta_-(\Delta,h)$ and $\Theta_-(\Delta',h)$ intertwine by an element of $U(h)$.
(ii) (Δ,Δ') is a ζ-comparison pair, and (β_i,h_i) and $(\beta'_{\zeta(i)},h_{\zeta(i)})$ have the same Witt type, for all $i \in I_{\Theta,0}$.

Proof. The direction (i)⇒(ii) follows from the Corollaries 6.15 and 7.13. Backwards: The block restrictions Θ_i and $\Theta'_{\zeta(i)}$ are endo-equivalent for all $i \in I_\Theta$. Indeed, Θ and Θ' are endo-equivalent and thus [23, 6.18] and the intertwining formula, see Theorem 6.7, imply that Θ_i and $\Theta_{\zeta(i)}$ are endo-equivalent. So we can restrict to the case where I consists only of one σ-orbit. Theorem [23, 6.18] and Corollary 7.13 now finish the proof. □

Finally we can introduce endo-parameters for quaternionic inner forms of classical groups.

7.4. **Endo-parameters.** At first we generalize the notion of Witt type from [15, Section 13.2]. We fix ρ and ϵ. We consider pairs (β,t) where $F[\beta]$ is a self-dual field extension and $t \in W_\epsilon(\sigma_E \otimes \rho)$. We call (β,t) equivalent to (β',t') if one of the following holds:

- β and β' are zero and $t = t'$
- β and β' are non-zero, fulfil (A) and (B), $\mathrm{Tr}_{\lambda_\beta}(t) = \mathrm{Tr}_{\lambda_{\beta'}}(t')$, and $w_{\beta,\beta',\epsilon}(t) = t'$.

The equivalence classes of these pairs are called (ρ,ϵ)-*Witt types* and the factor set is denoted by $\mathcal{W}_{\rho,\epsilon}$. We add an element 0 to $\mathcal{W}_{\rho,\epsilon}$. We define two maps on $\mathcal{W}_{\rho,\epsilon}$: the anisotropic dimension map and the ρ-Witt tower map via

$$\mathrm{diman}: \mathcal{W}_{\rho,\epsilon} \to \mathbb{N}_0,\ \mathrm{diman}(w) := \begin{cases} \mathrm{diman}(t) & \text{if } w = [\beta,t] \\ 0 & \text{if } w = 0 \end{cases}$$

and

$$WT_\rho : \mathcal{W}_{\rho,\epsilon} \to W_\epsilon(\rho),\ WT_\rho(w) := \begin{cases} \lambda_\beta(t) & \text{if } w = [\beta, t] \\ hyperbolic_{\equiv} & \text{if } w = 0 \end{cases}$$

The second data needed for endo-parameters are certain endo-classes: A semisimple character $\theta \in \mathrm{C}(\Delta)$ is called full if $r = 0$. A pss-character is called full if its domain contains a stratum with $r = 0$, these are the pss-characters of group level zero or infinity. Similarly for self-dual pss-characters. We denote by $\mathcal{E}$ the set of all full endo-classes of ps-characters and by $\mathcal{E}_-$ the set of all elementary full endo-classes. The *degree* of $c_- \in \mathcal{E}_-$ we define to be the degree of a simple block restriction c_1 of c_- where the degree of $c \in \mathcal{E}$ is the degree of any stratum in the domain of an element of c. We write $\deg(c)$ and $\deg(c_-)$.

We want to extract the Witt types which can occur in a given elementary endo-class. So we define: For a simple $c_- \in \mathcal{E}_-$ we put $\mathcal{W}_{c_-} := \{[\beta, t] \in \mathcal{W}_{\rho,\epsilon} |\ \exists_{\Theta_- \in c_-} \exists_{\Lambda,n,h} : ([\Lambda, n, 0, \beta], h) \in \mathrm{dom}(\Theta_-)\}$, and for a non-simple $c_- \in \mathcal{E}_-$ we put $\mathcal{W}_{c_-} := \{0\}$.

We consider maps $f_- = (f_1, f_2) :\ \mathcal{E}_- \to \mathbb{N}_0 \times \mathcal{W}_{\rho,\epsilon}$ which satisfy $f_2(c_-) \in \mathcal{W}_{c_-}$ for all $c_- \in \mathcal{E}_-$. These are global sections of

$$\coprod_{c_- \in \mathcal{E}_-} (\mathbb{N}_0 \times \mathcal{W}_{c_-}) \to \mathcal{E}_-. \tag{7.15}$$

We attach to f_- a map $f : \mathcal{E} \to \mathbb{N}_0$, also called the lift of f_- in the following way: We define for a simple block restriction c of c_-:

$$f(c) := \begin{cases} f_1(c_-) & \text{, if } c_- \text{ is not simple} \\ 2f_1(c_-) + \mathrm{diman}(f_2(c_-)) \frac{\deg(D)}{\gcd(\deg(c_-), \deg(D))} & \text{, else} \end{cases}$$

We say that c_- is in the support of f_- if $f(c) \neq 0$. We can now define the degree of f_- and f via

$$\deg(f_-) := \deg(f) := \sum_{c \in \mathcal{E}} f(c) \deg(c) \in \mathbb{N}_0 \cup \{\infty\}.$$

Definition 7.16. A (ρ, ϵ)-endo-parameter is a section $f_- = (f_1, f_2)$ of (7.15) of finite support such that $f_1(c_-)$ is divisible by $\frac{\deg(D)}{\gcd(\deg(c_-), \deg(D))}$ for every $c_- \in \mathcal{E}_-$.

Recall that a GL-endo-parameter is just a map $f :\ \mathcal{E} \to \mathbb{N}_0$ of finite support. See Definition [23, 7.1]. In particular, the lift of a (ρ, ϵ)-endo-parameter is a GL-endo-parameter.

Theorem 7.17 (see [15] 13.11, for the F-case)**.** The set of intertwining classes of full semisimple characters for $G = \mathrm{U}(h)$ is in canonical bijection to the set of endo-parameters f_- which satisfy:

(i) $\deg(f_-) = \deg(\mathrm{End}_D(V))$.
(ii) $\sum_{c_- \in \mathcal{E}_-} WT_\rho(f_2(c_-)) = h_{\equiv}$.

The map is constructed as follows: Given an intertwining class of $\theta_- \in \mathrm{C}_-(\Delta)$ we define for $c_- \in \mathcal{E}_-$:

$$f_1(c_-) = \begin{cases} (\text{Witt index of } \tilde{h}_{\beta_{c_-}}) \frac{\deg(D)}{\gcd(\deg(c_-), \deg(D))} & \text{if } c_- \text{ is simple} \\ \deg(\mathrm{End}_{E_c \otimes_F D} V^c) & \text{else, where } c \text{ is a simple block restriction of } c_- \end{cases},$$

and

$$f_2(c_-) = \begin{cases} \text{Witt type of } (\beta_{c_-}, h|_{V^{c_-}}) & \text{if } c_- \text{ is simple} \\ 0 & \text{else.} \end{cases}$$

One can interpret the non-simple c_- as GL-parts of the (ρ, ϵ)-endo-parameter.

Proof. The map is well defined by Theorem 7.14. We show at first the injectivity of the map. We consider two full self-dual semisimple characters $\theta_- \in \mathrm{C}_-(\Delta)$ and $\theta'_- \in \mathrm{C}_-(\Delta')$ for $\mathrm{U}(h)$ with the same (ρ, ϵ)-endo-parameter f_-. Then their lifts θ and θ' intertwine by Theorem [23, 7.2] because they have the same GL-endo-parameter. Now Theorem 7.14 implies that θ_- and θ'_- intertwine. Conversely, we have to show that any endo-parameter f_- of the form given in the theorem is attained by a self-dual semisimple character for $\mathrm{U}(h)$. We only consider c_- in the support of f_-.

- For a simple c_- we take a full self-dual semisimple character $\theta_{c_-} \in C_-(\Delta_c)$ whose self-dual ps-character is an element of c_- such that $\tilde{h}_{\beta_{c_-}}$ has Witt index $f_1(c_-)\frac{\gcd(\deg(c_-),\deg(D))}{\deg(D)}$ and Witt type $f_2(c_-)$.
- For a non-simple c_- we consider a simple block c_1 of c_-. We take a simple character $\theta_{c_1} \in C(\Delta_{c_1})$ with endo-parameter supported in c_1 which maps c_1 to $f_1(c_-)$. We construct a hyperbolic ϵ-hermitian space h_{c_-} with Lagrangian V^{c_1}. There is a full semisimple character $\theta_{c_-} \in C(\Delta_{c_-})$ with block restrictions θ_{c_1} and $\theta_{c_1}^{\sigma}$ by [23, 7.3]. We can take the stratum to be self-dual by Proposition 6.18.

At first we can arrange the lattice sequences Λ^{c_-} to have the same D-period and the same duality by Remark 6.11. Again by [23, 7.3] there is a semisimple stratum $\tilde{\Delta}$ with lattice sequence $\oplus_{c_-}\Lambda^{c_-}$ which also splits under this direct sum and such that $\theta := \otimes_{c_-}\theta_{c_-} \in C_-(\tilde{\Delta})$. Note that θ is fixed under the adjoint involution of $\bigoplus_{c_-} h_{c_-}$ by [23, 5.41]. Now by 6.18 there is a self-dual semisimple stratum Δ such that $\theta \in C(\Delta)$. Now h is isometric to $\bigoplus_{c_-} h_{c_-}$ and we can take an isometry to conjugate θ to a character for h say via an isometry g. By construction and Remark 7.11 the character ${}^{g}\theta|_{U(h)\cap H(g.\Delta)}$ has endo-parameter f_-. □

8. Intertwining classes of self-dual embeddings

In this section we answer the following question. Let us underline that in this section we use that D is not a field. Say $\theta_- \in C_-(\Delta)$ is a full self-dual semisimple character with lift $\theta \in C_-(\Delta)^{\sigma}$. How many G-intertwining classes of σ-fixed semisimple characters are contained in the $\tilde{G}$-intertwining class of θ. Let f_- be the endo-parameter of θ_- and f its lift. By Theorem 7.17 we only need to find all endo-parameters f'_- with lift f and such that

$$\sum_{c_-} WT_{\rho}(f'_2(c_-)) = h_{\equiv} \tag{8.1}$$

These endo-parameters only differ in their values for simple $c_- \in \mathcal{E}_-$. We start the simple case. Let us recall: We call an equivariant-F-algebra homomorphism $\phi:\ (E,\sigma_E) \to (\mathrm{End}_D(V),\sigma_h)$ a *self-dual embedding*.

Proposition 8.2. Let $E = F[\beta]$ be a self-dual field-extension different from F and suppose there is a self-dual embedding into $(\mathrm{End}_D(V),\sigma_h)$. Then there are precisely two G-conjugacy classes of self-dual embeddings of (E,σ_E) into $(\mathrm{End}_D(V),\sigma_h)$.

Proposition 7.3 and 7.5 are true if one replaces σ_E with id_E. Note that $\mathrm{id}_E \otimes_F \rho$ is orthogonal and thus the set $\mathrm{Idemp}(\mathrm{id}_E \otimes_F \rho)$ of $\mathrm{id}_E \otimes_F \rho$-fixed idempotents of rank 1 is non-empty. We can again identify the Witt-group $W_{\epsilon}(\mathrm{id}_E \otimes_F \rho)$ with the set of Witt towers, defined as in Definition 7.6, i.e. as maps

$$\mathrm{Idemp}(\mathrm{id}_E \otimes \rho) \to W_{\epsilon}(\mathrm{id}_E),\ \ wtower_{\tilde{h}}(e) := \mathcal{F}_e(\tilde{h})_{\equiv}.$$

For an extension $(E|\sigma_E)|(E'|\sigma_{E'})$, both of even degree over F and an E'-linear non-zero σ_E-$\sigma_{E'}$-equivariant map $\lambda : E \to E'$ we get a map

$$\mathrm{Tr}_{\lambda}:\ W_{\epsilon}(\sigma_E \otimes \rho) \to W_{\epsilon}(\sigma_{E'} \otimes \rho),$$

which in terms of Witt towers is given by

$$wtower_{\mathrm{Tr}_{\lambda}(t)}(e') = \mathrm{Tr}_{\lambda}(wtower_t(e')),\ e' \in \mathrm{Idemp}(\sigma_{E'} \otimes_F \rho).$$

where $\mathrm{Tr}_{\lambda}:\ W_{\epsilon}(\sigma_E) \to W_{\epsilon}(\sigma_{E'})$ is given by $(h_E)_{\equiv} \mapsto (\lambda \circ h_E)_{\equiv}$. We are now able to prove Proposition 8.2.

Proof. There are at most two conjugacy classes because the parity of the anisotropic $E \otimes D$-rank of $\tilde{h}_{\equiv} \in W_{\epsilon}(\sigma_E \otimes \rho)$ such that $(\lambda_{\beta} \otimes \mathrm{id}_D) \circ \tilde{h}$ is isometric to h is determined by the degree of $E|F$ and $\dim_D V$ and there are exactly two Witt towers in $W_{\epsilon}(\sigma_E \otimes \rho)$ with the same parity for the anisotropic rank. It is enough to show that the maximal anisotropic Witt tower is mapped to the hyperbolic Witt tower under $\mathrm{Tr}_{\lambda_{\beta}}$. $E|F$ contains a σ_E-invariant quadratic extension $(E',\sigma_{E'} = \sigma_E|_{E'})$ of F. The image of the maximal anisotropic Witt tower under Tr_{λ} does not depend on the choice of $\lambda :\ (E,\sigma_E) \to F$, non-zero, equivariant and F-linear, by [15, 2.2]. So, we could replace λ_{β} by $\mathrm{tr}_{E'|F} \circ \lambda'$ for some non-zero, equivariant E'-linear $\lambda' : (E,\sigma_E) \to (E',\sigma_{E'})$. The map $\mathrm{Tr}_{\lambda'}$ sends the maximal anisotropic Witt tower to the maximal anisotropic one by [24, 4.4]. So, we have to show that $\mathrm{Tr}_{E'|F}$ sends the maximal anisotropic Witt tower to the hyperbolic one. The maximal anisotropic Witt tower in $W_\epsilon(\sigma_{E'}\otimes\rho)$ can be written in the form $X=(x\tilde{h})_\equiv+\tilde{h}_\equiv$ for some $\tilde{h}_\equiv\in W_\epsilon(\sigma_{E'}\otimes\rho)$ and $x\in F^\times$. By Proposition 2.3 the forms $(\mathrm{tr}_{E'|F}\otimes\mathrm{id}_D)\circ\tilde{h}$ and $x((\mathrm{tr}_{E'|F}\otimes\mathrm{id}_D)\circ\tilde{h})$ are isometric. The Witt group of (D,ρ) is an elementary 2-group and so $\mathrm{Tr}_{E'|F}(X)$ is the hyperbolic Witt tower. □

Two count the number of (ϵ,ρ)-endo-parameters f'_- with lift f we only have to arrange Witt types such that (8.1) holds. The above proposition shows that for the two choices for $f'_2(c_-)$ we have $WT_\rho(f_2(c_-))=WT_\rho(f'_2(c_-))$. So for every non-null simple $c_-\in\mathcal{E}_-$ we have two choices for the Witt type. The Witt type for the null block, if existent, is determined by (8.1). So:

Theorem 8.3. The number of $\mathrm{U}(h)$-intertwining classes of σ-fixed semisimple characters in the $\tilde{G}$-intertwining class of θ is equal $2^{\#I_0}$ if there is no null block restriction for θ and $2^{\#I_0-1}$ if θ has a null block restriction.

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