Exercise sheet 8
Lecture p-adic Representation Theory

Dr. Daniel Skodlerack

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On this problem sheet F denotes a non-archimedean local field. We further fix an additive character of F which is trivial on $\mathfrak{p}_F$ and non-trivial on o_F. We denote $G := \mathrm{GL}_n(F)$ and $A := \mathrm{M}_n(F)$ and we recall the definition of $\psi_A(a)$ as $\psi_F \circ \mathrm{tr}_{A|F}(x)$.

Aufgabe 1. (5+5+5) We define the dual of a (full) o_F-lattice M of $\mathrm{M}_n(F)$ via

$$M^* := \{a \in \mathrm{M}_n(F) | \forall_{x \in M} : \psi_A(ax) = 1\}$$

1. Show that M^* is again a full o_F-lattice.
2. Show that two o_F-lattices M_1 and M_2 of $\mathrm{M}_n(F)$ satisfy $(M_1 + M_2)^* = M_1^* \cap M_2^*$ and $(M_1^*)^* = M_1$.
3. Let $\mathfrak{a}$ be a hereditary order of G. Prove the equality:

$$(\mathfrak{p}_\mathfrak{a}^m)^* = \mathfrak{p}_\mathfrak{a}^{1-m}.$$

Aufgabe 2. (5+5+5) Suppose $n = 2$ and let $\mathfrak{m}$ be the standard maximal order of G. Consider the strata

$$\Delta := (\mathfrak{m}, 1, \alpha) \ \ \Delta' := (\mathfrak{m}, 1, \alpha'),\ \alpha := \begin{pmatrix} \pi_F^{-1} & 0 \\ 0 & \pi_F^{-1} \end{pmatrix},\ \alpha' := \begin{pmatrix} \pi_F^{-1} & \pi_F^{-1} \\ 0 & \pi_F^{-1} \end{pmatrix}.$$

1. Show that Δ and Δ' intertwine by some element of G.
2. Show that Δ and Δ' are not conjugate by any element of G, i.e. by definition that there is no element g of G such that $g.\Delta := \left(g\mathfrak{m}g^{-1}, 1, g.\alpha g^{-1}\right)$ is equivalent to Δ'.
3. Prove that there is no simple stratum of A which is equivalent to Δ'.

Aufgabe 3. (5+10) We consider the case $n = 2$. Let π be an irreducible smooth representations of G and χ be a smooth character of $F^\times$.

1. Prove $l(\chi\pi) \leq \max\{l(\chi), l(\pi)\}$.
2. Prove $l(\chi\pi) = \max\{l(\chi), l(\pi)\}$, if χ and π have different levels.