# Collection of solutions
# Lecture $p$-adic Representation Theory

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**Aufgabe. 5.2**(Iwahori decomposition and Haar measure, [BH, Exercise 7.6] )

1. Let $C$ be the set $\bar{N}TN$. Show that $C$ is open and dense in $G$ and that $\bar{N}\times T\times N\to C$, $(\bar{n},t,n)\mapsto \bar{n}tn$, is a homeomorphism.

2. Let $dg$ be a Haar measure on $G$. Show that there are Haar measures $dn, d\bar{n}$ and $dt$ on $N, \bar{N}$ and $T$, respectively, such that for all $f\in C_c^\infty(G)$

$$\int_G f dg = \int_{\bar{N}}\int_T\int_N \delta_B^{-1}(t)f(\bar{n}tn)dndtd\bar{n}. \tag{1}$$

   Note that we take as a definition of the modular character of $B$: $\delta_B(b)=(b^{-1}Jb:J)$ for any compact open subgroup $J$ of $N$ and any $b\in B$. In particular this gives $\int_B f(bb')db=\delta_B(b')^{-1}\int_B f(b)db$. This is the way Bushnell and Henniart choose the modular character.

**Solution:**

1. At first we obtain that $C$ is open, because $C=G\setminus wB$ where $w$ is the anti-diagonal matrix with entries 1 in the anti-diagonal. For the homeomorphism property it is enough to show that all products $\bar{N}_iT_itN_i$ are open in $G$ for every $t\in T$. Here $N_i$ is the intersection of $N$ with $K_i := 1+\mathrm{M}_2(\mathfrak{p}_F)$, $i\geq 1$, similar for $T_i$ and $\bar{N}_i$. In fact the set $\bar{N}_iT_itN_i$ is a union of sets of the form $\bar{n}_j(K_i\cap tK_it^{-1})n_jt$ and thus it is open in $\bar{N}TN$. For the density: It is enough to show that $w$ is in the closure of $C$. But this is the case, because

$$\begin{pmatrix}1 & 0\\ \pi_F^{-i} & 1\end{pmatrix}\begin{pmatrix}\pi_F^i & 1\\ 0 & -\pi_F^{-i}+\pi_F^i\end{pmatrix}=\begin{pmatrix}\pi_F^i & 1\\ 1 & \pi_F^i\end{pmatrix}.$$

2. We write $\mu$ for the Haar measure $dg$ on $G$. Recall that $\mu$ is a non-trivial $G$-left invariant Radon measure on $G$. Let $\tilde{\mu}$ be the Radon measure on $G\setminus C$ which is the push forward of the product of measure $d\bar{n}\times\delta_B^{-1}(t)dt\times dn$, where $d\bar{n}, dt, dn$ are chosen Haar measures. Then by Fubini (Note that the measures are locally finite, so we can apply Fubini) we have

$$\int_{G\setminus C} f d\tilde{\mu} = \int_{\bar{N}}\int_T\int_N \delta_B^{-1}(t)f(\bar{n}tn)dndtd\bar{n}.$$

   for every $\tilde{\mu}$-integrable function of $G\setminus C$. We have to show:

   (a) We can choose $d\bar{n}, dt, dn$ such that $\mu$ and $\tilde{\mu}$ agree on the Borel $\sigma$-algebra of $G\setminus C$.

   (b) $\mu(B)=0$, $B=TN$.

   At first we choose $d\bar{n}, dt, dn$ such that $\bar{N}_1, T_1, N_1$ have measure 1. And we can assume without loss of generality that $K_1$ has measure 1. The measures $\mu$ and $\tilde{\mu}$ are Radon measures and $G\setminus C$ has a countable base. So the measures are outer regular (see Elstrodt, Maß- und Intergrationstheorie, Chapter 14, Satz 1.9 and Definition 1.7) and it is enough to show that the measures agree in enough copen (compact open) sets. Both measures agree on $K_1$ and by the bijection

$$K_1/K_i\to\bar{N}_1/\bar{N}_i\times T/T_i\times N_1/N_i$$

which maps $\bar{n}tnK_i$ to $(\bar{n}\bar{N}_i, tT_i, nN_i)$ (Here you need the normality of $K_i$ in $K_1$ and the Iwahori decomposition.) we obtain that both measures also agree on $K_i$ for all positive integers $i$. Take $t \in T$. Then $K_i$ is a disjoint union of sets of the form $\bar{n}(K_i \cap tK_it^{-1})n$- Further $\mu$ and $\tilde{\mu}$ are $\bar{N}$-left and $N$-right invariant ($G$ is unimodular.). So $\mu$ and $\tilde{\mu}$ agree on $K_i \cap tK_it^{-1}$. Suppose we have shown that $\tilde{\mu}$ is $T$-right invariant, then for $\bar{n}, t, n$ and $i$ we have

$$\tilde{\mu}(\bar{n}\bar{N}_iT_itN_in) = \tilde{\mu}(\bar{N}_iT_itN_it^{-1}).$$

which is the disjoint union of sets of the form $\bar{n}_j(K_i \cap tK_it^{-1})n_j$, and we know already that both measures agree on the latter. So they agree on $\bar{n}\bar{N}_iT_itN_in$ and a combinatoric argument shows that both measures agree on a collection of copen sets which have the property that every open subset of $G \setminus C$ is an increasing countable union of a family of them. So the measures agree on the Borel $\sigma$-algebra of $G \setminus C$ by outer regularity. The $T$-right invariance of $\tilde{\mu}$ is obtained, by Fubini, noting that $\delta_B$, $\delta_B(x) := (x^{-1}N_1x : N_1)$, $x \in B$, is the modular character of any left Haar measure of $B$. (Note $dtdn$ defines is a non-trivial $B$-left-invariant Radon measure on $B$, i.e. a left Haar measure, and thus $dtdn\delta_B^{-1}$ is a right Haar measure on $B$.)

Now we only need to show that $B$ and therefore $wB$ is a zero set:

$$\mu(B) = \int_{\{1_{\bar{N}}\}} \int_T \int_N \delta_B(t)^{-1} dndtd\bar{n} = 0$$

because $\{1_{\bar{N}}\}$ is a zero set of $\bar{N}$.

**Aufgabe. 6.2**(Mackey's irreducibility criteria) Let $G$ be a second countable locally profinite group and $H$ be an open subgroup of $G$, such that for all $g \in G$ the space $(H \cap gHg^{-1})\backslash H$ is finite. (For example the condition on $H$ is satisfied if $H$ is open contains the center $Z$ of $G$ and is compact mod $Z$.)

Then the following assertions are equivalent for a smooth irreducible representation $(\sigma, W)$ of $H$.

1. c-$\mathrm{Ind}_H^G\sigma$ is irreducible.
2. The intertwining of $\sigma$ in $G$ is equal to $H$.
3. $\mathrm{End}_G(\text{c-}\mathrm{Ind}_H^G\sigma) \cong \mathbb{C}$.

**Solution:**

- 1. $\Rightarrow$ 3.: If c-$\mathrm{Ind}_H^G\sigma$ is irreducible then $\mathrm{End}_G(\text{c-}\mathrm{Ind}_H^G\sigma) \cong \mathbb{C}$ by Schur.
- 3. $\Leftrightarrow$ 2.: We use the following Mackey-decomposition (Proposition 48):

$$\mathrm{Res}_H^G(\text{c-}\mathrm{Ind}_H^G\sigma) \cong \bigoplus_{H\backslash G/H} \text{c-}\mathrm{Ind}_{H\cap {}^gH}^H \mathrm{Res}_{H\cap {}^gH}^{{}^gH}\, {}^g\sigma,$$

and the second Frobenius reciprocity, as follows:

$$\begin{aligned}
\mathrm{End}_G(\text{c-}\mathrm{Ind}_H^G\sigma) &\cong \mathrm{Hom}_H(\sigma, \mathrm{Res}_H^G \text{c-}\mathrm{Ind}_H^G\sigma) \\
&\cong \mathrm{Hom}_H(\sigma, \bigoplus_{H\backslash G/H} \text{c-}\mathrm{Ind}_{H\cap {}^gH}^H \mathrm{Res}_{H\cap {}^gH}^{{}^gH}\, {}^g\sigma) \\
&\cong \bigoplus_{H\backslash G/H} \mathrm{Hom}_H(\sigma, \text{c-}\mathrm{Ind}_{H\cap {}^gH}^H \mathrm{Res}_{H\cap {}^gH}^{{}^gH}\, {}^g\sigma), \\
&\cong \bigoplus_{H\backslash G/H} \mathrm{Hom}_H(\sigma, \mathrm{Ind}_{H\cap {}^gH}^H \mathrm{Res}_{H\cap {}^gH}^{{}^gH}\, {}^g\sigma), \\
&\cong \bigoplus_{H\backslash G/H} \mathrm{Hom}_{H\cap {}^gH}(\mathrm{Res}_{H\cap {}^gH}^H \sigma, \mathrm{Res}_{H\cap {}^gH}^{{}^gH}\, {}^g\sigma),
\end{aligned}$$

where the 3rd isomorphism exists, since $\sigma$ is of finite type, because it is irreducible, and the replacement from compact induction to induction in the 4th line was possible, since $(H \cap {}^gH)\backslash H$ is finite. Thus $\mathrm{End}_G(\text{c-}\mathrm{Ind}_H^G\sigma) \cong \mathbb{C}$ if and only if there is only one non-zero summend on the right side. The summand for the double coset $H$ is one dimensional by Schur's Lemma, and the summand for the double coset $HgH$ is non-zero if and only if $g$ intertwines $\sigma$.

- 3. $\Rightarrow$ 1.: Suppose that $\text{c-Ind}_H^G\sigma$ is not irreducible, say $\text{c-Ind}_H^G\sigma$ has a proper subrepresentation $\tau$. Let $f_0$ be a non-zero element of $\tau$. The $H$-representation $\delta := \mathcal{H}(H)f_0$ is the subrepresentation of $\text{Res}_H^G\tau$ generated by $f_0$. The finiteness condition implies that there are finitely many right cosets of $H$, say $Hg_i$, $1 \leq i \leq l$, such that every element of $\delta$ has support in $\bigcup_i Hg_i$. ($H\backslash HgH$ is finite because $(H \cap g^{-1}Hg)\backslash H$ is finite.) We can suppose without generality that $H$ is in the support of $f_0$, otherwise replace $f_0$ by a $G$-conjugate of $f_0$.

  Claim: The action of $H$ on $\delta$ is semisimple. By Lemma 1 below it is enough to show that $\text{Res}_{H'}^H\delta$ is semisimple for $H' := \bigcap_i g_i^{-1}Hg_i$, because the index of $H'$ in $H$ is finite. Consider $g = g_i$ for some $i$, and let $\text{Ind}_{Hg}\sigma$ be the set of elements of $\text{c-Ind}_H^G\sigma$ with support in $Hg$. We consider $\text{Ind}_{Hg}\sigma$ as a subrepresentation of $\text{Res}_{H'}^G\text{c-Ind}_H^G\sigma$. The $H'$-representation $\text{Ind}_{Hg}\sigma$ is isomorphic to $\text{Res}_{H'}^{g^{-1}Hg}(\sigma^g)$ by the map which sends $f$ to $f(g)$. The representation $\sigma^g$ is irreducible and thus semisimple. Thus $\text{Res}_{H'}^{g^{-1}Hg}\sigma^g$ is semisimple by Lemma 1 and thus $\text{Ind}_{Hg}\sigma$ is semisimple.

  We let $g$ vary and we see that $\text{Res}_{H'}^H\delta$ is semisimple as a subrepresentation of $\bigoplus_i \text{Ind}_{Hg_i}\sigma$. Thus by Lemma 1 the representation $\delta$ is semisimple.

  Claim: $\sigma$ is a subpresentation of $\delta$. This follows easily because the image of the $H$-morphism

  $$f \in \delta \mapsto f(1)$$

  is non-zero, because $H$ is a subset of the support of $f_0$. Thus this morphism is surjective by the irreducibility of $\sigma$, and the semisimplicity of $\delta$ ensures that this morphism has a right-inverse.

  Now we can conclude the contradiction, because by the last claim and the second Frobenius reciprocity it follows that there is a non-zero morphism of $G$-representations from $\text{c-Ind}_H^G\sigma$ to $\tau$, which contradicts $\text{End}(\text{c-Ind}_H^G\sigma) = \mathbb{C}$.

**Lemma 1** (Lemma 2.7, GL(2), Bushnell Henniart)**.** Let $G$ be a locally profinite group, $H$ be an open subgroup of finite index in $G$ and $\sigma$ be a smooth representation of $G$. Then $\sigma$ is semisimple if and only if $\text{Res}_H^G\sigma$ is semisimple.

This Lemma is a nice exercise.

**Aufgabe. 6.4** Find the cuspidal support of the Steinberg representation of $\text{GL}_2(F)$, i.e. of the infinite dimensional irreducible quotient of $\text{Ind}_B^{\text{GL}_2(F)}\mathbb{1}$, where $B$ is the standard (upper) Borel subgroup.

**Solution:** The representation $\pi := \text{Res}_B^G\text{Ind}_B^G\mathbb{1}$ has the quotient $\delta_B^{-1}$, via $f \mapsto \int_N f(wn)dn$. Thus we have a non-zero morphism $F$ from $\pi$ to $\text{Ind}_B^G\delta_B^{-1}$. The representation $\pi$ has no 1-dimensional quotient by Theorem 64. Thus the Steinberg representation is isomorphic to a subrepresentation of $\text{Ind}_B^G\delta_B^{-1}$ and therefore the cuspidal support of $\text{St}_G$ is the $G$-conjugacy class of $\text{Res}_T^B\delta_B^{-\frac{1}{2}}$, because we have to consider normalized parabolic induction to detect the cuspidal support.