Exercise sheet 5
Lecture p-adic Representation Theory

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Let F be non-archimedean local field, G be $\mathrm{GL}_2(F)$, N be the subgroup of upper unipotent matrices and $\bar{N}$ be the subgroup of lower unipotent matrices of G. Further let T be the standard torus of G and $B = TN$. All mentioned representations are complex.

Aufgabe 1. (5+5 points)(N-fixed vector, ([BH, 9.2 Exercise 1])) In the case of $\mathrm{GL}_2(\mathbb{F})$ over a finite field $\mathbb{F}$ we have seen that one can distinguish principal series representations and cuspidal representations just by distinguishing if the representation in question contains the trivial character of the upper unipotent subgroup or not. In the locally compact case this does not work in general as the following problem shows.

1. Show that if a subgroup H of G contains N and $\bar{N}$ then H contains $\mathrm{SL}_2(F)$.
2. Let (π, V) be an irreducible smooth representation of G which contains the trivial character of N. Then π is a character. Hint: You can use that π is admissible. And consider an Iwahori decomposition.

Aufgabe 2. (5+5 points)(Iwahori decomposition and Haar measure, [BH, Exercise 7.6])

1. Let C be the set $\bar{N}TN$. Show that C is open and dense in G and that $\bar{N}\times T\times N \to C$, $(\bar{n}, t, n) \mapsto \bar{n}tn$, is a homeomorphism.
2. Let dg be a Haar measure on G. Show that there are Haar measures $dn, d\bar{n}$ and dt on $N, \bar{N}$ and T, respectively, such that for all $f \in C_c^\infty(G)$

$$\int_G f dg = \int_{\bar{N}} \int_T \int_N \delta_B(t)^{-1} f(\bar{n}tn) dn dt d\bar{n}.$$

Aufgabe 3. (5 points)([BH, 9.2 Exercise 2]) Let (π, V) be an irreducible smooth representation of G, and suppose that V is finite dimensional over $\mathbb{C}$. Show that π is a character. Hint: Show that V has a non-zero N-fixed vector.

Aufgabe 4. (5 points) Let H be a locally profinite group and let (π, V) be a smooth irreducibel representation of H which has a matrix coefficient which is compactly supported mod centre. Show that π is $Z(H)$-compact, i.e. that all matrix coefficients of π are compactly supported mod centre.