Semisimple characters for inner forms I: $\mathrm{GL}_m(D)$

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Abstract

The article is about the representation theory of an inner form G of a general linear group over a non-archimedean local field. We introduce semisimple characters for G whose intertwining classes describe conjecturally via the Local Langlands correspondence the behaviour on wild inertia. These characters also play a potential role to understand the classification of irreducible smooth representations of inner forms of classical groups. We prove the intertwining formula for semisimple characters and an intertwining implies conjugacy like theorem. Further we show that endo-parameters for G, i.e. invariants consisting of simple endo-classes and a numerical part, classify the intertwining classes of semisimple characters for G. They should be the counter part for restrictions of Langlands-parameters to wild inertia under the Local Langlands correspondence. MSC2010 [11E95] [20G05] [22E50]

1 Introduction

Semisimple characters occur extensively on the automorphic side of the Local Langlands correspondence, as in the classification of cuspidal irreducible representations of p-adic classical groups of odd residue characteristic or in the description of the restrictions of Langlands parameters to wild inertia on the Galois side (see [12]). For the classification of all irreducible smooth representations of $\mathrm{GL}_m(F)$ via type theory Bushnell and Kutzko developed the theory of simple characters over a non-archimedean local field F, see [7]. These characters are essential in the construction of types, i.e. certain irreducible representations on open compact subgroups which describe the blocks of the Bernstein decomposition. This strategy has been successfully generalized to $\mathrm{GL}_m(D)$ in a series of papers ([10, 18, 4, 19]) for a central division algebra of finite degree over F, by Sécherre, Stevens, Broussous and Grabitz. In the case of p-adic classical groups the simple characters were not sufficient enough for reasonable classification results as the collection of non-split tori which arise from a field extension of F are not enough to construct all irreducible cuspidal representations. So more general non-split tori were needed, more precisely the ones which arise from a product of fields. This leads to the theory of semisimple characters which has been extensively studied by Stevens and the author in [23] and [21], and it succeeded in the classification of irreducible cuspidal representations for p-adic classical groups with odd residual characteristic, see [24] and [12].

The aim of this paper is to generalize the theory of semisimple characters to $\mathrm{GL}_m(D)$. It will have the potential of four major applications:

(i) At first it provides the foundation to generalize to semisimple characters for inner forms of p-adic classical groups which leads to the classification of their irreducible cuspidal representations in odd residue characteristic.

(ii) The endo-parameters which classify intertwining classes of semisimple characters has conjecturally a concrete connection to the Galois side of the Local Langlands correspondence.

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(iii) They also have the potential to help for the study of inner forms of classical groups in even residue characteristic.

(iv) For the study of the representation theory of $\mathrm{GL}_m(D)$, semisimple characters provide a way to decompose and help to describe the category of smooth representations.

Semisimple characters are complex characters defined on compact open subgroups $H(\Delta)$ which are generated by congruence subgroups depending on arithmetic data Δ (so called semisimple strata). Part of the data Δ is an element β of the Lie algebra of $G := \mathrm{GL}_D(V)$ which generates a mod centre anisotropic torus $F[\beta]^\times$ of a Levi subgroup of G. The construction of a semisimple character is done by an inductive procedure using complex characters which factorize through reduced norms of certain centralizers and a character which factorizes through a "β-twisted" reduced trace of $\mathrm{End}_D(V)$. Thus, the "essential" data for the construction are the characters on the tori and the generator β of the torus of Δ. Semisimple characters which are defined with the same essential data but may be on different central algebras over F are called transfers and all semisimple characters which intertwine up to transfer create a class called an endo-class. These endo-classes can be broken into simple endo-classes (the associated torus is the multiplicative group of a field). And conversely given a collection of finitely many simple endo-classes one can unify them to a semisimple endo-class. An endo-parameter for G is a finite collection of simple endo-classes where to every endo-class is attached a non-negative integer.

Theorem 1.1 (2nd Main Theorem 7.2). The set of intertwining classes of full semisimple characters is in canonical bijection to the set of endo-parameters for G.

The Local Langlands correspondence attaches to an irreducible representations of G (automorphic side) a Weil-Deligne representation ϕ (Langlands parameter) of the Weil group W_F in the absolute Galois group of F. Work on $\mathrm{Sp}_{2n}(F)$ by Blondel, Henniart and Stevens suggests that the endo-parameters should describe the collection of possible restrictions of Langlands parameters to the wild inertia group P_F. For G we can define the following diagram:

$$\begin{array}{ccc} \mathrm{Irr}(G) & \stackrel{LLC}{\longrightarrow} & LP_G(W_F) \\ \downarrow & & \downarrow \\ \mathcal{E}_{par}(F) & \stackrel{B.\text{-}H.}{\longrightarrow} & \mathrm{Rep}(P_F)/W_F \end{array}$$

where the above horizontal line is the Local Langlands correspondence, for a description see [11], and the bottom line is the map resulting from Bushnell-Henniart's first ramification theorem [6, 8.2]. The right map is the restriction map from the set $LP_G(W_F)$ of Langlands parameters for G to the set of smooth representations of P_F, up to W_F-conjugacy, and the left map is given in the following way: Given an irreducible representation π of G it should contain a full semisimple character θ and two such semisimple characters must intertwine and they therefore define the same endo-parameter, where $\mathcal{E}_{par}(F)$ is the set of endo-parameters. The conjecture has the following form:

Conjecture 1.2. Diagram (1) is commutative.

A common strategy for the classification for all irreducible representations of $\mathbb{G}(F)$ for a reductive group $\mathbb{G}$ defined over F starts with the Bernstein decomposition of the category $\mathrm{Rep}(\mathbb{G}(F))$ of all smooth representations of $\mathrm{Rep}(\mathbb{G}(F))$.

$$\mathrm{Rep}(\mathbb{G}(F)) = \prod_{\mathfrak{s}} \mathrm{Rep}(\mathbb{G}(F))_{\mathfrak{s}},$$

where $\mathfrak{s}$ passes through all inertia classes of $\mathbb{G}(F)$. We call $\mathrm{Rep}(\mathbb{G}(F))_{\mathfrak{s}}$ the Bernstein block of $\mathfrak{s}$. Now the classification can be broken into three parts:

(i) Exhaustiveness: Find a list of types to cover all Bernstein blocks.

(ii) Rigidity: Describe the relation between two types which give the same Berstein block.

(iii) Description of the Bernstein block (using spherical Hecke algebras).

The construction of types for G starts with a semisimple character, say defined on a group called $H^1 = H(\Delta)$ and it is extended to an irreducible representation κ of a compact open group $J \supseteq H^1$ and tensored with the inflation τ of a cuspidal irreducible representation of a finite quotient J/J^1. It is a similar construction in the case of p-adic classical groups. Beautiful descriptions of this construction can be found in [7], [16] and [24].

Semisimple characters are important for parts (i) and (ii). Given two types which are contained in a cuspidal irreducible representation of G, it is reasonable to ask if they are conjugate by an element of G. As an example to study the non-cuspidal Bernstein blocks for G, Sécherre und Stevens constructed in [19] (maximal) semisimple types but without using semisimple characters, see also the work of Bushnell and Kutzko [8] for the split case. The advantage of the use of the theory of semisimple characters would be to approach the rigidity question. An example where this has been successful is the case of p-adic classical groups [12]. The base of these rigidities is an “intertwining and conjugacy” result for semisimple characters.

We give a sketch of the ingredients and underline the ones which are new. Denote by $\mathrm{C}(\Delta)$ the set of semisimple characters which are constructed with the arithmetic data Δ. Recall that Δ consists of $\beta \in \mathrm{Lie}(G)$ which generates a commutative semisimple algebra E over F, a rational point Λ of the Bruhat–Tits building $\mathfrak{B}(G)$ of G which is in the image of $\mathfrak{B}(C_\beta(G))$, $C_\beta(G)$ being the centralizer of β, by a map constructed by Broussous and Lemaire in [3]. The data Δ consists further of two non-negative integers $r \le n$, where r indicates the biggest integer m such that the semisimple characters are defined on a subgroup of the 1-unit group $P_{m+1}(\Lambda)$, and where n is the smallest integer m such that the semisimple characters are trivial on $P_{m+1}(\Lambda)$. The primitive idempotents of E split $\theta \in \mathrm{C}(\Delta)$ into simple characters θ_i, $i \in I$, for some general linear group $\mathrm{GL}_D(V^i)$, say $\theta_i \in \mathrm{C}(\Delta_i)$, and if two semisimple characters θ and θ' intertwine then the intertwining matches these simple characters to each other:

Theorem 1.3 (Matching Theorem 5.44). Suppose $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$ are intertwining semisimple characters, and suppose $r = r'$ and $n = n'$. Then there is a unique bijection $\zeta : I \to I'$ between the index sets for E and E' such that

(i) $\dim_D V^i = \dim_D V^{\zeta(i)}$, and

(ii) there is a D-linear isomorphism g_i from V^i to $V^{\zeta(i)}$ such that $g_i.\theta_i$ intertwines with $\theta'_{\zeta(i)}$ by a D-linear automorphism of $V^{\zeta(i)}$, for all indexes $i \in I$.

The statement is a generalization of the split case $D = F$, see [21], but the proof heavily needs a fine study of embedding. For the intertwining to force conjugacy we need an extra new ingredient. Two intertwining semisimple characters $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$ induce a canonical bijection $\bar{\zeta}$ between the residue algebras κ_E of E and $\kappa_{E'}$ of $E' = F[\beta']$. When these two characters are conjugate by an element of G then the conjugation has to induce this bijection $\bar{\zeta}$. This leads to the “intertwining and conjugacy” theorem. To avoid to much notation we make the assumption that Δ and Δ' have the same point of $\mathfrak{B}(G)$.

Theorem 1.4 (1st Main Theorem: Intertwining and Conjugacy 5.53). Suppose that $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$ are two semisimple characters such that $n = n'$, $r = r'$ and $\Lambda = \Lambda'$. If there is an element $t \in G$ which normalizes Λ and maps the splitting of V with respect to β to the one with respect to β' such that the conjugation with t induces the map $\bar{\zeta}$ then there is an element g in the normalizer of Λ such that $g.\theta_i = \theta'_{\zeta(i)}$ and such that $g^{-1}t$ fixes Λ. In particular $g.\theta$ and θ' coincide.

This result has great potential to contribute to rigidity for semisimple types for the following open problem:

Problem 1. *Calculate for all Bernstein blocks the types on* $\mathrm{GL}_m(o_D)$.

The work of Henniart in the appendix of [1] on $\mathrm{GL}_2(F)$, Paskunas [15] for cuspidal irreducible representations of $\mathrm{GL}_n(F)$ and Nadimpalli [14] for $\mathrm{GL}_3(F)$ have shown that this problem has in these cases a quite unique solution.

The article is structured as follows: In Section 3 we collect results and study the points in the Bruhat–Tits building of G and some of its centralizers. It is followed by the technical heart of the article: Section 4 about semisimple strata Δ. The highlights of this section is the "intertwining and conjugacy"-result for semisimple strata (4.48). The section is followed by Section 5, where the intertwining formula for transfers (5.15), the matching theorem (5.44) and the "intertwining and conjugacy" theorem for semisimple characters (5.53) is proven. The final section, Section 6, gives the first application of the study of semisimple characters of $\mathrm{GL}_m(D)$: The theory of endo-classes and endo-parameters.

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2 Notation

Let F be a non-archimedean local field and we denote by G the group $\mathrm{GL}_D(V)$ of invertible D-linear automorphisms on a finite dimensional right D-vector space V, where D is a central division algebra over F of finite degree d. We denote the dimension of V by m. Let $L|F$ be a maximal unramified field extension in D, i.e. of degree d, and π_D be a uniformizer of D which normalizes L. The conjugation by π_D is a generator τ of the Galois group $G(L|F)$, i.e. $\tau(x) = \pi_D x \pi_D^{-1}$ for all elements $x \in L$.

3 Bruhat–Tits buildings and Embeddings

3.1 Bruhat–Tits buildings

An o_D-lattice sequence Λ in V is a function from $\mathbb{Z}$ to the set of full o_D sub-modules of V, called lattices, which is decreasing with respect to the inclusion and which is periodic, i.e. there is an integer $e = e(\Lambda|D)$ such that for all integers i the lattice Λ_{i+e} is equal to $\Lambda_i \pi_D$ for any uniformizer π_D of D. The number $e(\Lambda|D)$ is called the D-period of Λ. We call a lattice sequence *strict* if it is injective. The square lattice sequence $\mathfrak{a}_\Lambda$ on A is an o_F-lattice sequence on A defined as follows: $\mathfrak{a}_{\Lambda,i}$ is the set of elements of A which map Λ_j into Λ_{j+i} for all integers j.

One can generalize the notion of lattice sequence to lattice functions, i.e. where the domain is $\mathbb{R}$, such that they can be identified with a point in the Bruhat–Tits building $\mathfrak{B}(G)$ of G. The square lattice function of a point is then the Moy-Prasad filtration. The lattice sequences are then precisely the points with rational barycentric coordinates in a chamber.

Definition 3.1. An o_D-lattice function Λ in V is a function from $\mathbb{R}$ to the set of lattices in V with the following properties:

(i) Λ is decreasing,

(ii) Λ is periodic, i.e. for all elememts $t \in \mathbb{R}$ the lattice $\Lambda_{t+\frac{1}{d}}$ is equal to $\Lambda(t)\pi_D$. for any uniformizer π_D of D.

(iii) Λ is left continuous, i.e. Λ_r is the intersection of all $\Lambda_{r'}$ where r' passes through all real numbers smaller than r.

The square lattice function $\mathfrak{a}_\Lambda$ on A is an o_F-lattice function on A defined as follows: $\mathfrak{a}_{\Lambda,t}$ is the set of elements of A which map Λ_s into Λ_{t+s} for all $s \in \mathbb{R}$.

Definition 3.2. Two lattice functions Λ and Λ' (resp. lattice sequences) are *equivalent* if they are translates of each other, i.e. there is an element $t \in \mathbb{R}$ (resp. $t \in \mathbb{Z}$) such that Λ' is equal to $(\Lambda - t)$ which is defined to be the map which sends an element s of the domain of Λ to Λ_{s+t}. We denote the equivalence class of Λ by $[\Lambda]$. We write $\mathfrak{n}(\Lambda)$ for the normalizer of Λ in G, i.e. the elements g of G for which $g\Lambda$ is a translate of Λ. We denote $(1+\mathfrak{a}_t)^\times$ by $P_t(\Lambda)$ for non-negative t in the domain of Λ, and we write $P(\Lambda)$ for $P_0(\Lambda)$.

We write $C_S(T)$ for the centralizer of T in S where S is a group or an algebra acting on an ambient set of T from both sides, e.g. the centralizer in a subgroup H of G of an element β of the Lie algebra of G is denoted by $C_H(\beta)$. The Bruhat–Tits building of the groups H in question, denoted by $\mathfrak{B}(H)$, is always the enlarged one. If we refer to the reduced Bruhat–Tits building, we write $\mathfrak{B}_{red}(H)$. We write $\mathfrak{b}$ for square lattice functions in the E-centralizer B of A. The reduced Bruhat–Tits building $\mathfrak{B}_{red}(G)$ can be described using translation classes of lattice functions (or equivalently square lattice functions) and the Bruhat–Tits building $\mathfrak{B}(G)$ can be described using lattice functions. For more details see [3]. So we consider $\mathfrak{B}(G)$ and $\mathfrak{B}_{red}(G)$ as the set of lattice functions and their translation classes, respectively.

Theorem 3.3 ([3, 1.1])**.** Let $E|F$ be a field extension in A, then there is exactly one map j_E from the set of $E^\times$-fixed points of $\mathfrak{B}_{red}(G)$ to $\mathfrak{B}_{red}(C_G(E))$ such that

$$\mathfrak{b}_{j_E([\Lambda]),te(E|F)} = \mathfrak{a}_{[\Lambda],t} \cap C_A(E).$$

Further,

(i) j_E is $C_G(E)$-equivariant,

(ii) j_E is affine,

(iii) j_E is bijective.

The map j_E^{-1} can be characterized to be the unique affine, $C_G(E)$-equivariant map from $\mathfrak{B}_{red}(C_G(E))$ to $\mathfrak{B}_{red}(G)$.

The reader can find in [3] a precise description of the map j_E, which we only recall for a special unramified case.

Theorem 3.4 ([3, 3.1])**.** Let $E|F$ be an unramified extension which has an isomorphic field extension of F in D. The map j_E can be described as follows. V is the direct sum of V^i, i passing from 1 to $[E:F]$, where the idempotent 1^i (projections onto V^i) are obtained by the decomposition of $E\otimes_F L$, where L is a maximal unramified extension of F in D. Further the indexing can be adjusted such that V^i is equal to $V^1\pi_D^{i-1}$ for all indexes i, where π_D is a uniformizer of D whose conjugation induces the Frobenius automorphism of $L|F$. Now, given a point $y\in\mathfrak{B}_{red}(C_G(E))$, then the point $x := j_E^{-1}(y)$ is given by the translation class of the lattice function:

$$\Lambda_t := \oplus_{i=0}^{[E:F]-1} M_{(te(E|F)-\frac{ie(E|F)}{d})}\pi_D^i$$

where the translation class of M corresponds to the point y.

Remark 3.5. There is a canonical projection map $[\ ]$ from $\mathfrak{B}(G)$ to $\mathfrak{B}_{red}(G)$ sending Λ to $[\Lambda]$. There is a $C_G(E)$-equivariant, affine and bijective map from $[\ ]^{-1}(\mathfrak{B}_{red}(G)^{E^\times})$ to $\mathfrak{B}(C_G(E))$ which on the level of reduced buildings induces j_E. This map is unique up to translation. We just choose one and we still call this map j_E. One could take the map given by the formula [3, Lemma 3.1], a generalization of the formula in Theorem 3.4.

We will need later a more general notion of lattice sequence:

Definition 3.6. Let $E|F = \prod_j E_j$ be a commutative reduced F-algebra in A. The idempotents split V into $V = \oplus_j V^j$. An o_D-lattice sequence Λ is called an *o_E-o_D-lattice sequence* if Λ is split by the decomposition of V, i.e. Λ is the direct sum of the intersections $\Lambda^j := \Lambda \cap V^j$, and Λ^j is an o_{E_j}-lattice sequence, for every index j. We define $e(\Lambda|E)$ as the greatest common divisor of the integers $e(\Lambda^j|E_j)$.

3.2 Embedding types

The embeddings of buildings j_E from subsection 3.1 can be used to classify conjugacy classes of pairs (E,Λ) consisting of a finite field extension of F in A and a lattice sequence Λ normalized by $E^\times$. Such a pair is called embedding in analogy to the pairs $(E,\mathfrak{a})$ introduced and studied in [2]. Let $E_D|F$ be the unramified sub-extension of $E|F$ whose degree is the greatest common divisor of d and $f(E|F)$. Then two embeddings (E,Λ) and (E',Λ'), of $\mathrm{End}_D(V)$ and $\mathrm{End}_D(V')$, respectively, are called *equivalent* if the field extensions $E_D|F$ and $E'_D|F$ are conjugate by a D-isomorphism g from V to V' such that $g\Lambda$ is equal to Λ' up to translation of $\mathbb{Z}$. The equivalence classes are called (Broussous-Grabitz) *embedding types.*

Theorem 3.7 ([4, 3.3, 3.5])**.** Let ϕ_i, $i=1,2$, be two F-algebra embeddings of a field extension $E|F$ into A and let Λ be a lattice sequence such that $\phi_i(E)^\times$ normalizes Λ for both indexes i. Suppose that the embeddings $(\phi_i(E),\Lambda)$, $i=1,2$, are equivalent. Then there is an element u of the normalizer of Λ in $A^\times$ such that $u\phi_1(x)u^{-1}$ is equal to $\phi_2(x)$, for all $x\in E$. If ϕ_1 and ϕ_2 are equal on E_D then one can choose u in $P(\Lambda)$.

Theorem 3.8 (see [20, 1.2] for the strict case)**.** Suppose (E,Λ) and (E',Λ') are two embeddings for A. Consider Λ and Λ' as points in $\mathfrak{B}(G)$. Then (E,Λ) and (E',Λ') are equivalent if and only if the barycentric coordinates of $j_{E_D}([\Lambda])$ and of $j_{E'_D}([\Lambda'])$ with respect to chambers are equal up to cyclic permutation.

Proof. At first we prove the if-part. There is an element g_1 of $A^\times$ which conjugates E_D to E'_D. The conjugation induces an affine, simplicial isomorphism from $\mathfrak{B}_{red}(G_{E_D})$ to $\mathfrak{B}_{red}(G_{E'_D})$ which respects barycentric coordinates up to cyclic permutation. Thus $j_{E'_D}([g_1\Lambda])$ and $j_{E'_D}([\Lambda'])$ have the same barycentric coordinates up to cyclic permutation, and therefore there is an element g_2 of $G_{E'_D}$ such that $j_{E'_D}([g_2g_1\Lambda])$ and $j_{E'_D}([\Lambda'])$ have the same barycentric coordinates and thus there is an element of $G_{E'_D}$ of reduced norm 1 which conjugates $j_{E'_D}([g_2g_1\Lambda])$ to $j_{E'_D}([\Lambda'])$. Thus, the embeddings (E,Λ) and (E',Λ') are equivalent. For the only-if-part, assume that both embeddings are equivalent, in particular there is an element g of $A^\times$ conjugating E_D to E'_D such that $j_{E'_D}([g\Lambda])$ is equal to $j_{E'_D}([\Lambda'])$. In particular, $j_{E_D}([\Lambda])$ and $j_{E'_D}([\Lambda'])$ have the same barycentric coordinates up to cyclic permutation. □

The last two theorems imply that the existence of an embedding with a fixed lattice sequence only depends on the ramification index and the inertia degree.

Corollary 3.9. Suppose $E|F$ is a field extension and ϕ an F-algebra monomorphism from E into A such that $(\phi(E),\Lambda)$ is an embedding of A. Suppose that $E'|F$ is a field extension in an algebraic closure $\bar{E}$ of E such that $e(E'|F)$ divides $e(E|F)$ and $f(E'|F)$ divides $f(E|F)$ such that $E_D|F$ is equal to $E'_D|F$. Then, there is an F-algebra monomorphism ϕ' from E' into A such that $\phi'(E')^\times$ normalizes Λ and ϕ is equal to ϕ' on $E\cap E'$. In particular, the embeddings $(\phi(E),\Lambda)$ and $(\phi'(E'),\Lambda)$ are equivalent.

Proof. Without loss of generality we can assume that $E|F$ and $E'|F$ have the same ramification index and the same inertia degree. We just define ϕ' to be ϕ on $E\cap E'$ and we go to the centralizer $C_A(E\cap E')$. Thus, using Theorem 3.3, we can assume that $E\cap E'$ is F. In particular both extensions are totally ramified. We extend $E'|F$ to a purely ramified field extension $\tilde{E}'|F$ of degree $\deg(A)$. It can be embedded into A via an F-algebra monomorphisms $\tilde{\phi}'$. The reduced building of $C_A(\phi'(\tilde{E}'))^\times$ only consists of one point, which is given by a mid point of a chamber in $\mathfrak{B}_{red}(G)$, by Theorem 3.3. Let us call this chamber C. By conjugation we can assume that $[\Lambda]$ is an element of the closure of C. We take uniformizers π_E and $\pi_{E'}$. The F-valuation of the reduced norm of the embeddings is for both uniformizers the same. Thus they rotate the Coxeter diagram of $\mathfrak{B}_{red}(G)$ in the same way and thus they transform barycentric coordinates the same way. In particular: The translation class of Λ which coincides with $[\phi(\pi_E)\Lambda]$ has the same barycentric coordinates as $[\tilde{\phi}'(\pi_{E'})\Lambda]$. The latter is still an element of the closure of C and must therefore coincide with $[\Lambda]$. We define ϕ' to be the restriction of $\tilde{\phi}'$ to E'. □

4 Semisimple Strata

4.1 Definitions for semisimple strata

Here we introduce semisimple strata for proper inner forms of the general linear groups, and we introduce a notation which makes it more convenient to work with strata. A very good introduction for semisimple strata for the case of $D = F$ can be foand in [21, Section 6] and [12, Section 2.4, 8.1]. There is a very detailed study of simple strata in [16]. And the aim of this subsection is to state the straight forward generalizations. A stratum is a quadruple

$$\Delta = [\Lambda, n, r, \beta]$$

such that Λ is an o_D-lattice sequence β is an element of $\mathfrak{a}_{\Lambda,-n}$ and $0 \le r \le n$. Given a stratum Δ and an integer j such that $0 \le r + j \le n$ we define $\Delta(j+)$ to be the stratum obtained from Δ replacing r by $r + j$, and we define $\Delta(j-)$ to be $\Delta((-j)+)$ for $0 \le r - j \le n$. If $n = r$ and $\beta = 0$ we call the stratum a *zero-stratum*. A stratum Δ is called *pure* if it is zero or $E := F[\beta]$ is a field and Λ is an o_E-lattice sequence and $\nu_\Lambda(\beta) = -n$. Given a pure stratum Δ we define the *critical exponent* $k_0(\Lambda, \beta)$ similar to [16, Definition 2.3] in the following way: Let $\mathfrak{n}_k(\beta, \Lambda) = \{x \in \mathfrak{a}_{\Lambda,0} : \beta x - x\beta \in \mathfrak{a}_{\Lambda,k}\}$ and define $k_0(\beta, \Lambda)$ by

$$k_0(\beta, \Lambda) = \max\left\{\nu_\Lambda(\beta), \sup\{k \in \mathbb{Z} : \mathfrak{n}_k(\beta, \Lambda) \nsubseteq \mathfrak{a}_{j_E(\Lambda),0} + \mathfrak{a}_{\Lambda,1}\}\right\},$$

for non-zero β and $k_0(0, \Lambda) = -\infty$. This is slightly different from the definition in [16, Definition 2.3] for non-zero elements β of F. If we consider Λ as an o_F-lattice sequence, we write Λ_F, then the critical exponent satisfies

$$k_0(\beta, \Lambda_F) = e(\Lambda|E) k_0(\beta, \mathfrak{p}_E^{\mathbb{Z}}), \tag{4.1}$$

see [22, 5.6].

Notation 4.2. If we write that we are given a stratum Δ' then we mean that the entries have the superscript $'$, i.e. Δ' is equal to $[\Lambda', n', r', \beta']$. This also applies to all other superscripts. For subscripts we have the following rule: Δ_i is the stratum $[\Lambda^i, n_i, r_i, \beta_i]$. This also defines the notation E', V', E_i, V^i etc. .

A pure stratum Δ is called *simple* if $k_0(\beta, \Lambda) < -r$, in particular if $n = r$ then the stratum has to be zero.

Theorem 4.3 ([17, 2.2])**.** Let Δ be a pure stratum. Then there is a simple stratum Δ' equivalent to Δ such that the unramified sub-extension of $E'|F$ is contained in $E|F$.

To define semisimple strata it is convenient to use the direct sum of strata: We recall that the *period* of a stratum Δ is the o_D-period of Λ. Let Δ' and Δ'' be two strata such that $r' = r''$ and with the same period, then we define

$$\Delta' \oplus \Delta'' := [\Lambda' \oplus \Lambda'', \max(n', n''), r', \beta' \oplus \beta''],$$

and if a stratum Δ decomposes in this way, it is called *split* by the decomposition $V' \oplus V''$ with projections $\Delta|_{V'}$ and $\Delta|_{V''}$. A stratum Δ is called *semi-pure* if Δ is a direct sum of pure strata

$$\Delta = \bigoplus_{i \in I} \Delta_i, \; E_i := F[\beta_i],$$

such that β generates over F the product of the fields E_i. We call the strata Δ_i the blocks of Δ. We write B for the centralizer of E in A.

For the definition of semisimple strata we need the equivalence: Two strata Δ and Δ' are called equivalent if $r = r'$ and for all integers $s \le -r$ the coset $\beta + \mathfrak{a}_{\Lambda,s}$ is equal to $\beta' + \mathfrak{a}_{\Lambda',s}$.

Definition 4.4. A semi-pure stratum Δ is called *semisimple* if the direct sum $\Delta_{i_1} \oplus \Delta_{i_2}$ is not equivalent to a pure stratum (or equivalently simple stratum) for all different indexes $i_1, i_2 \in I$ and such that Δ_i is a simple stratum for all $i \in I$.

Given a non-zero semisimple stratum Δ with $r = 0$ we can define the critical exponent $k_0(\beta, \Lambda)$ analogously as in [12, Definition 8.2] to be

$$k_0(\beta, \Lambda) = -\min\{r \in \mathbb{Z} : [\Lambda, n, r, \beta] \text{ is not semisimple}\}.$$

Remark 4.5. Analogously to [12, Remark 8.3] this can be generalized to all pairs (β,Λ) where β generates over F a product of fields which decomposes Λ into o_{E_i}-lattice sequences.

We also write $k_0(\Delta)$ for $k_0(\beta,\Lambda)$. We have extension and restriction of scalars for strata: Given a stratum Δ and a finite field extension $\tilde{F}|L$ we define extension of scalars of Δ to $\tilde{F}$ by $\Delta\otimes_F\tilde{F} = [\Lambda\otimes_{o_L} o_{\tilde{F}}, n, r, \beta\otimes_{o_{\tilde{F}}} 1]$ seen as a stratum with respect to $\mathrm{End}_D(V)\otimes_F\tilde{F}$, and we define the restriction of scalars of Δ to a sub-skewfield $\tilde{D}$ of D to be the stratum $\mathrm{Res}_{\tilde{D}}(\Delta) = [\Lambda, n, r, \beta]$ seen as a stratum of $\mathrm{End}_{\tilde{D}}(V)$. For example $\Delta\otimes L$ and $\mathrm{Res}_F(\Delta)$ are very important. The extension of scalar to $\tilde{F}$ starting from a semi-pure stratum Δ comes equipped with an action of the group $\mathrm{Aut}(\tilde{F}|F)$ (F-linear field automorphisms of $\tilde{F}$) on the set of blocks of $\Delta\otimes\tilde{F}$ induced form the action on $\mathrm{End}_D(V)\otimes\tilde{F}$ on the second factor. We recall that the intertwining from Δ to Δ' is the set $I(\Delta,\Delta')$ of elements $g\in G$ such that $g(\beta+\mathfrak{a}_{-r})g^{-1}$ intersects $(\beta'+\mathfrak{a}'_{-r'})$. We say that an element $g\in G$ intertwines Δ with Δ' if g is an element of $I(\Delta,\Delta')$.

The following construction attaches to a stratum a strict stratum in a canonical way.

Definition 4.6 ([4])**.** Given a stratum Δ we define $\Delta^\dagger = \oplus_{i=0}^{e(\Lambda|F)-1}[\Lambda - i, n, r, \beta]$. Now $\Delta^\dagger$ is strict.

4.2 Fundamental strata

Let Δ be stratum such that $n = r+1$. We recall the construction of the characteristic and the minimal polynomial of Δ which we denote by χ_Δ and μ_Δ, for more details see [19, Definition 2.5]. Let g be the greatest common divisor of $e(\Lambda|F)$ and n. They defined $\mathfrak{y}(\Delta) := \pi_F^{\frac{n}{g}}\beta^{\frac{e(\Lambda|F)}{g}}$ and $\bar{\mathfrak{y}}(\Delta)$ as its class in $\mathfrak{a}_0/\mathfrak{a}_1$, and we denote by χ_Δ and μ_Δ (elements of $\kappa_F[X]$) the characteristic and the minimal polynomial of $\bar{\mathfrak{y}}(\Delta)$, respectively. These polynomials only depend on the equivalence class of Δ. The stratum Δ is called *fundamental* if μ_Δ is not a power of X. We generalize the following proposition.

Proposition 4.7 ([19, Proposition 2.7])**.** Let Δ be a stratum with $n = r+1$. Then Δ is equivalent to a non-zero simple stratum if and if μ_Δ is irreducible different from X.

To state the analogue criteria for semisimple strata we need a multiplication map: For an element $b\in\mathfrak{a}_{-n}(\Lambda)$ and an integer t, the map

$$m_{t,n,b}:\mathfrak{a}_{-tn}/\mathfrak{a}_{1-tn}\to\mathfrak{a}_{-(t+1)n}/\mathfrak{a}_{1-(t+1)n}$$

is defined via multiplication with b.

Proposition 4.8. Let Δ be a stratum with $n = r+1$. Then Δ is equivalent to a semisimple stratum if and only if the minimal polynomial μ_Δ is square-free and for every integer t the kernel of $m_{t+1,n,\beta}$ and the image of $m_{t,n,\beta}$ intersect trivially.

For the proof we need two lemmas.

Lemma 4.9. Let Δ be a stratum with $n = r+1$. Then Δ is equivalent to a zero-stratum if and only if μ_Δ is equal to X and for every integer t the kernel of $m_{t+1,n,\beta}$ and the image of $m_{t,n,\beta}$ intersect trivially.

Proof. We only have to prove the "if"-part. The condition on μ_Δ implies that $\beta^{e(\Lambda|F)}$ is an element of $\mathfrak{a}_{-e(\Lambda|F)n+1}$. We apply the condition on the multiplication maps successively on

$$m_{e(\Lambda|F)-1,n,\beta}\circ m_{e(\Lambda|F)-2,n,\beta}\circ\ldots\circ m_{1,n,\beta}\circ m_{0,n,\beta}$$

successively to obtain that $m_{0,n,\beta}$ is the zero-map. □

Lemma 4.10. Let Λ be an o_D-lattice sequence, and suppose that Λ_F is split by a decomposition $V = V^1+V^2$ into F vector spaces such that the corresponding idempotents 1^1 and 1^2 satisfy $x1^1x^{-1}-1^1\in\mathfrak{a}_1$, for all $x\in D^\times$. Then there is an element $u\in P_1(\Lambda_F)$ such that uV^1 and uV^2 are D-vector spaces.

Proof. The classes $\bar{1}^1$ and $\bar{1}^2$ in $\mathfrak{a}_0/\mathfrak{a}_1$ split the κ_D-vector spaces $\Lambda_i/\Lambda_{e(\Lambda|D)}$ for all $0 \leq i < e(\Lambda|D)$. Thus by the Lemma of Nakayama we find a decomposition $V = \tilde{V}^1 \oplus \tilde{V}^2$ into D-vector spaces such the idempotents satisfy $\bar{\tilde{1}}^i = \bar{1}^i$ for $i = 1,2$. In particular Λ is split by the latter decomposition, and the map u defined via $u(v) := \tilde{1}^1 1^1(v) + \tilde{1}^2 1^2(v)$ is an element of $P_1(\Lambda_F)$ such that $uV^i = \tilde{V}^i$, $i = 1,2$. □

Proof of Proposition 4.8. We prove the "if"-part. The "only-if"-part is easy and left for the reader. As in the proof of [21, Proposition 6.11] $\mathrm{Res}_F(\Delta)$ is equivalent to a stratum Δ'_F which is a direct sum of strata corresponding to the prime decomposition of $\mu_\Delta = \mu_{\mathrm{Res}_F(\Delta)}$. Those direct summands corresponding to the prime factors which are not a power of X are equivalent to a simple stratum by Proposition 4.7 using that μ_Δ is square-free. So, we can assume that those summands are already simple. Using the square-freeness of μ_Δ again, which forces $\kappa_F[\bar{\mathfrak{y}}(\Delta)]$ to be isomorphic to the product of the residue fields corresponding to the summands, we obtain that for every idempotent e of the decomposition of V there is a polynomial $P \in o_F[X]$ such that $P(\mathfrak{y}(\Delta))$ is congruent to e modulo $\mathfrak{a}_{\Lambda_F,1}$. These idempotents satisfy the condition of Lemma 4.10, because β is D-linear. Thus by conjugation with an element $u \in P_1(\Lambda_F)$ we can assume that the constituents of the decomposition of V are in fact D sub-vector spaces. This decomposition also splits Δ into a sum of strata which are equivalent to simple strata by Proposition 4.7 and Lemma 4.9. □

Proposition 4.8 has the following consequence:

Corollary 4.11. Let Δ be a stratum with $n = r + 1$ and let $\tilde{F}|L$ be an unramified field extension. Then the following assertions are equivalent:

(i) The stratum Δ is equivalent to a semisimple stratum.

(ii) The stratum $\Delta \otimes \tilde{F}$ is equivalent to a semisimple stratum.

(iii) The stratum $\mathrm{Res}_F(\Delta)$ is equivalent to a semisimple stratum.

Proof. We only need to study the equivalence to semipure strata. Assertion (i) concludes the remaining assertions by definition reasons. All three strata have the same minimal polynomial, and we therefore have to consider the intersection condition on the multiplications maps. If this intersection condition is satisfied for $\mathrm{Res}_F(\Delta)$ then it is also satisfied by $\Delta \otimes L$ and if it is satisfied for $\Delta \otimes L$ then it is satisfied for Δ, all just by inclusion. Thus for the case $\tilde{F} = L$ all three assertions are equivalent by Proposition 4.8. In the case $\tilde{F} \neq L$ we have that if $\Delta \otimes \tilde{F}$ is equivalent to a semisimple stratum then again by Proposition 4.8 $\Delta \otimes L$ is equivalent to a semisimple stratum, which finishes the proof. □

Corollary 4.12. Let Δ be a stratum with $n = r + 1$. Then the following assertions are equivalent:

(i) The stratum Δ is equivalent to a simple stratum.

(ii) The stratum $\mathrm{Res}_F(\Delta)$ is equivalent to a simple stratum.

Proof. If Δ is equivalent to a simple stratum then $\mathrm{Res}_F(\Delta)$ is equivalent to a pure stratum and thus by Theorem 4.3 equivalent to a simple stratum. We prove now that (i) follows from (ii). Suppose that $\mathrm{Res}_F(\Delta)$ is equivalent to a simple stratum. Thus the polynomial $\mu(\mathrm{Res}_F(\Delta))$ is irreducible and satisfies the trivial intersection property for the multiplication maps by Proposition 4.7 and Lemma 4.9. Thus the same is true for $\mu(\Delta)$ which coincides with $\mu(\mathrm{Res}_F(\Delta))$, i.e. Δ is equivalent to a simple stratum by 4.7 and 4.9. □

One method to prove a question for all semisimple strata is to use strata induction, see [21, 7.2]. It is an induction over r, where one proves at first the statement for all semisimple strata of type $(n, n-1)$ followed by the induction step, where one reduces the problem for a stratum Δ to a problem for $\Delta(1+)$ and a derived stratum which is equivalent to a stratum of type $(r+1, r)$. For that we need to recall

- the tame corestriction and

- the derived stratum.

We have the following natural isomorphisms:

(i) $A \otimes_F \mathrm{End}_A(V) \cong \mathrm{End}_F(V)$ where $\mathrm{End}_A(V) \cong D$,

(ii) $C_A(E) \otimes_F D \cong C_{\mathrm{End}_F(V)}(E)$, for every field extension $E|F$ in A.

Definition 4.13 ([18, Definition 2.25] for the simple case)**.** Given a commutative reduced F-algebra E in A a map s from A to $C_A(E)$ is called a *tame corestriction* of A if the map $s \otimes_F \mathrm{id}_{\mathrm{End}_A(V)}$ is a tame corestriction for $(\mathrm{End}_F(V), E|F)$ in the sense of [21, 6.17], or equivalently in the sense of [7, 1.3.3] if E is a field.

A tuple of strata is called a *multi-stratum*, and we call a multi-stratum *semisimple* if all components are semisimple. Equivalence of two multi-strata is define component-wise. Given a commutative reduced F-algebra $F[\gamma] = \prod_j F[\gamma_j]$ in A such that Λ is an $o_{F[\gamma]}$-lattice sequence, see Definition 3.6, and a tame corestriction s with respect to γ, we write $s(\Delta)$ for the (multi)-stratum $[j_{F[\gamma]}(\Lambda), n, r, s(\beta)]$.

Definition 4.14. We define the *derived (multi)-stratum* of Δ with respect to s and γ in the following way:

$$\partial_\gamma(\Delta) := [j_{F[\gamma]}(\Lambda), \min(n, -\nu_\Lambda(\beta-\gamma)), r, s(\beta-\gamma)].$$

We see $j_{F[\gamma]}(\Lambda)$ as a lattice sequence where $j_{F[\gamma]}(\Lambda)_s$ is defined as $j_{F[\gamma]}(\Lambda_s)$ via formula [3, 3.1]. This is up to doubling the lattice sequence corresponding to its rational barycentric coordinates.

The main theorem for strata induction is:

Theorem 4.15. Let Δ be a stratum for $n \geq r+1$ such that $\Delta(1+)$ is equivalent to a semisimple stratum with last entry γ. Then the following assertions are equivalent:

(i) The stratum Δ is equivalent to a semisimple stratum.

(ii) The stratum $\partial_\gamma(\Delta)$ is equivalent to a semisimple multi-stratum.

We have to postpone the proof of this Theorem to section 4.8, because we need more theory. Nevertheless, the main theorem for simple strata induction is already proven:

Theorem 4.16 ([19, 2.13, 2.14], see also [7, 2.2.8, 2.4.1])**.** Let Δ be a stratum for $n \geq r+1$ such that $\Delta(1+)$ is equivalent to a simple stratum with entry γ and let s be a tame corestriction with respect to γ. Then Δ is equivalent to a simple stratum if and only if $\partial_\gamma(\Delta)$ is equivalent to a simple stratum.

4.3 Level for strata with $n = r + 1$

Definition 4.17. Let Δ be a stratum with $n = r+1$. The *level* $l(\Delta)$ of Δ is defined as $\frac{n}{e(\Lambda|F)}$.

As an immediate consequence we have:

Remark 4.18. Let Δ and Δ' be two strata such that $n = r+1$ and $n = r'+1$. Then

(i) $l(\mathrm{Res}_F(\Delta)) = l(\Delta) = l(\Delta^\dagger) = l(\Delta \otimes L)$.

(ii) The stratum Δ is fundamental if and only if $\Delta^\dagger$ is fundamental if and only if $\mathrm{Res}_F(\Delta)$ is fundamental.

Proposition 4.19 ([21, 6.9] for the split case)**.** Suppose that Δ and Δ' are strata with $n = r+1$ and $n' = r'+1$ and suppose they intertwine.

(i) If both have the same level and Δ is fundamental then Δ' is fundamental.

(ii) If both strata are fundamental then both strata have the same level.

Proof. By doubling we can assume that Λ and Λ' have the same period over D. Recall that Δ is fundamental if and only if $\mathrm{Res}_F(\Delta^\dagger)$ is fundamental. Thus by Remark 4.18 we can assume without loss of generality that Λ and Λ' are lattice chains of the same period and $D = F$. Thus we can conjugate Λ to Λ' by an element of A and we can therefore assume $\Lambda = \Lambda'$.

(i) Both have the same level and $\Lambda = \Lambda'$. Thus $n = n'$. The strata intertwine so they have the same characteristic polynomial. Thus Δ' is fundamental because the characteristic polynomial of Δ is not a power of X.

(ii) Suppose both strata have not the same level, say $l(\Delta) < l(\Delta')$. Then $n < n'$ because $\Lambda = \Lambda'$. Take an element g of A which intertwines Δ with Δ', say

$$g(\beta + a)g^{-1} = \beta' + a' \ a \in \mathfrak{a}_{1-n}, a' \in \mathfrak{a}'_{1-n'}.$$

Then

$$((\beta' + a')^e \pi_F^{n'})^t = g(\beta + a)^{et} \pi_F^{nt} g^{-1} \pi_F^{(n'-n)t}, \ e = e(\Lambda|F),$$

is an element of $g\mathfrak{a}_t g^{-1}$. If t is big enough then $g\mathfrak{a}_t g^{-1}$ is a subset of $\mathfrak{a}'_1$. Thus, the minimal polynomial of $\mathfrak{y}(\Delta')$ is a power of X, Its characteristic polynomial divides a power of the minimal polynomial and is therefore a power of X. Thus Δ' is non-fundamental. A contradiction.

□

Corollary 4.20. Suppose two semisimple strata Δ and Δ' intertwine. Then $[\Lambda, n, n-1, \beta]$ and $[\Lambda', n', n'-1, \beta']$ have the same level if

(i) both strata are non-zero, or

(ii) the lattice sequences Λ and Λ' have the same F-period and $r = r'$.

So in Case (ii) it is not possible that a zero stratum and a non-zero semisimple stratum intertwine.

Proof. (i) If both strata are non-zero then $n > r$ and $n' > r'$, see the definition of semisimple. Then $[\Lambda, n, n-1, \beta]$ and $[\Lambda', n', n'-1, \beta']$ are intertwining fundamental strata. So they have the same level by Proposition 4.19.

(ii) As in (i) we have that both strata have the same level if they are non-zero. If both strata are zero, then the equality follows from the assumptions, because $n = r$ and $n' = r'$. Thus we have to prove that this are the only cases. So we can assume that Δ is non-zero and Δ' is zero, in particular $n > r = r'$. Thus $[\Lambda, n, n-1, \beta]$ intertwines $[\Lambda', n, n-1, 0]$. Thus the stratum $[\Lambda, n.n-1, \beta]$ is not fundamental by *ibid.*. A contradiction, because Δ is non-zero and semisimple. This finishes the proof.

□

4.4 Simple strata

In [16, 2.21] the author introduces pseudo-simple strata. One goal of this subsection is that these strata are semisimple. Further we prove two diagonalization theorems for simple strata, and we recall results about intertwining. So let $\tilde{F}|F$ be a finite unramified field extension such that the index of D divides $[\tilde{F} : F]$. Without loss of generality let us assume that $L|F$ is a sub-extension of $\tilde{F}|F$. In this subsection let Δ be a semi-pure stratum. Then $\Delta \otimes \tilde{F}$ is again a semi-pure stratum.

Proposition 4.21 ([16, 2.23],[7, 2.4.1])**.** Let Δ be a pure stratum then for every simple stratum Δ' equivalent to Δ we have that $e(E'|F)$ divides $e(E|F)$ and $f(E'|F)$ divides $f(E|F)$. Further, the following assertions are equivalent:

(i) Δ is simple.

(ii) $\mathrm{Res}_F(\Delta)$ is simple.

Corollary 4.22 (see [21, 6.1,6.4] for the split case)**.** Let Δ be a pure stratum with $n > r$. Then Δ is simple if and only if there is no pure stratum Δ' equivalent to Δ such that $[E' : F]$ is smaller than $[E : F]$.

Proof. By Proposition 4.21 we only have to prove the "if"-direction. Assume for deriving a contradiction that Δ is not simple, then again by Proposition 4.21 Δ is equivalent to a simple stratum Δ' and $[E : F]$ is equal to $[E' : F]$ by assumption. Thus by the known split case, we obtain that $\mathrm{Res}_F(\Delta)$ is simple. A contradiction to Proposition 4.21. □

Proposition 4.23. If Δ is simple. Then $\Delta \otimes \tilde{F}$ is semisimple, and both strata have the same critical exponent.

Proof. We write $\tilde{\Delta}$ for $\Delta \otimes \tilde{F}$ which also defines $\tilde{\beta}$ and $\tilde{\Lambda}$. In the non-zero case define $k_0'(\tilde{\Delta})$ to be the maximum of all integers k such that $\mathfrak{n}_k(\tilde{\beta}, \tilde{\Lambda})$ is not a subset of $(\mathfrak{b}_0 \otimes_{o_L} o_{\tilde{F}}) + (\mathfrak{a}_1 \otimes_{o_L} o_{\tilde{F}})$. This is equal to $k_0(\Delta)$ by [16, 2.10]. The stratum $\tilde{\Delta}$ is semi-pure with blocks $\tilde{\Delta}_j$, $j \in \tilde{I}$. The element $\tilde{\beta}_i$ is a Galois conjugate of β over F but not over $\tilde{F}$. The definition of $k_0(\tilde{\Delta}_j)$ and $k_0'(\tilde{\Delta})$ imply $k_0(\tilde{\Delta}_j) \le k_0'(\tilde{\Delta})$ for every index j. Thus, by the equality of $k_0'(\tilde{\Delta})$ with $k_0(\Delta)$ we obtain that the blocks $\tilde{\Delta}_i$ are simple. We have to prove that $\tilde{\Delta}_{j_1} \oplus \tilde{\Delta}_{j_2}$ is not equivalent to a simple stratum for $j_1 \neq j_2$ if Δ is simple. If there would be two blocks of $\tilde{\Delta}$ whose sum is equivalent to a simple stratum then we could view these blocks as two simple strata $\tilde{\Delta}_{j_1}$ and $\tilde{\Delta}_{j_2}$ in one vector space over $\tilde{F}$ and they are conjugate up to equivalence, by [21, 8.1], because of conjugate lattice sequences and because they intertwine, by [21, 6.16]. Thus, we can assume without loss of generality that $\tilde{\Delta}_{j_1}$ and $\tilde{\Delta}_{j_2}$ are equivalent. Further, we have that the elements β_{j_1} and β_{j_2} have the same minimal polynomial over F. Thus by Lemma 4.24 β_{j_1} and β_{j_2} have the same minimal polynomial over $\tilde{F}$. A contradiction.

By the proven first assertion we get $k_0(\tilde{\Delta}) \le k_0(\Delta) = k_0'(\tilde{\Delta})$, and by [23, 3.7] we obtain $k_0'(\tilde{\Delta}) \le k_0(\tilde{\Delta})$. Thus the critical exponents of Δ and $\Delta \otimes \tilde{F}$ coincide. □

Lemma 4.24. [For $D = F$] Let $F'|F$ be an unramified field extension in $\mathrm{End}_F(V)$ and assume that Δ_1' and Δ_2' are two equivalent pure strata of $\mathrm{End}_{F'}(V)$ such that β_1' and β_2' have the same minimal polynomial over F. Then β_1' and β_2' have the same minimal polynomial over F'.

Proof. Step 1: We prove at first that we can conjugate β_1' to β_2' by an element g of the normalizer of F' in $\mathrm{End}_F(V)$. We write E_1' for $F[\beta_1']$. By Skolem-Noether there is an element g' of $\mathrm{Aut}_F(V)$ which conjugates β_1' to β_2'. But then $E_1'F'$ and $E_1'g'^{-1}F'g'$ are fields which are isomorphic over E_1'. We apply Skolem-Noether again, to obtain the existence of an E_1'-automorphism g'' of V which conjugates $E_1'g'^{-1}F'g'$ to $E_1'F'$. The uniqueness of unramified sub-extensions implies that F' is normalized by $g''g'^{-1}$ whose inverse is the desired element g. This proves the assertion of Step 1.

Step 2: We take the element g of Step 1. Then there is an $E_1'F'$ automorphism of V which conjugates $g^{-1}\Lambda'$ to Λ' and we can assume that $g\Lambda'$ is equal to Λ'. Thus g normalizes the stratum Δ_1' up to equivalence, and is therefore an element of $P(\Lambda'_{E_1'F'})(1 + \mathfrak{a}_{\Lambda',1})$. In particular, the conjugation by g induces the identity on the residue field of $E_1'F'$. Thus, g centralizes F' which finishes the proof. □

The key theorem to start a study of semisimple strata is the diagonalization theorem. Let us recall that we define for a semisimple stratum Δ the set $\mathfrak{m}(\Delta) = \mathfrak{m}_{-(r+k_0)}(\beta, \Lambda)$ as the intersection of $\mathfrak{n}_{-r}(\beta, \Lambda)$ with $\mathfrak{a}_{-(r+k_0)}$.

Theorem 4.25. Let Δ be a stratum which splits under a decomposition $V = \oplus_j V^j$ into a direct sum of pure strata and suppose that Δ is equivalent to a simple stratum Δ'. Then, there is an element u of $1 + \mathfrak{m}(\Delta')$ such that $u.\Delta'$ is split by the above decomposition.

For the proof we need the intertwining of a simple stratum.

Theorem 4.26. The intertwining of a simple stratum Δ is given by the formula

$$(1+\mathfrak{m}(\Delta))C_{A^\times}(\beta)(1+\mathfrak{m}(\Delta)) \tag{4.27}$$

Proof. We have this formula for the split case $\Delta\otimes L$ by [21, 6.19]. The critical exponents of Δ and $\Delta\otimes L$ coincide by Proposition 4.23. Thus we only need to take the $\mathrm{Gal}(L|F)$-fixed points. Now apply [16, 2.35] using [16, 2.4.3,3.5] to obtain that taking $\mathrm{Gal}(L|F)$-fixed points at (4.27)($\Delta\otimes L$) results in (4.27)(Δ). □

Proof of Theorem 4.25. We write Δ_j for $\Delta|_{V^j}$ as usual. Then $\prod_j I(\Delta_j)$ is a subset of $I(\Delta')$. Thus for the idempotents (1^j) corresponding to the decomposition there are element α_j in the centralizer B' of β' in A such that α_j is congruent to 1^j modulo $\mathfrak{m}(\Delta')$ for all indexes j. Now Lemma [21, 7.14] provides pairwise orthogonal idempotents $1'^j$ in B' which are congruent to α_j modulo $\mathfrak{m}(\Delta')$ and which sum up to 1. The map $u:=\oplus_j 1^j 1'^j$is an element of $1+\mathfrak{m}(\Delta')$ because

$$u-1=\sum_j(1^j-1'^j)1'^j\in\mathfrak{m}(\Delta'),$$

and further $u.\Delta'$ is split by the given splitting and it is equivalent to Δ', ie. to Δ. □

We need a proposition which generalizes Proposition 4.21

Proposition 4.28. Let Δ be a stratum. Then Δ is equivalent to a simple stratum if and only if $\mathrm{Res}_F(\Delta)$ is equivalent to a simple stratum.

Proof of Proposition 4.28. We prove this statement by strata induction. For $n=r$ both strata are zero strata. Suppose now that $n>r$. If Δ is equivalent to a simple stratum then $\mathrm{Res}_F(\Delta)$ is too, by Proposition 4.21. If $\mathrm{Res}_F(\Delta)$ is equivalent to a simple stratum then $\mathrm{Res}_F(\Delta(1+))$ is equivalent to a simple stratum and the induction hypothesis states that $\Delta(1+)$ is equivalent to a simple stratum, say a simple stratum with entry γ. The stratum $\partial_\gamma(\mathrm{Res}_F(\Delta))$ is equivalent to a direct sum of copies of $\partial_\gamma(\Delta)$, by formula [3, 3.1]. Thus $\partial_\gamma(\Delta)$ and $\partial_\gamma(\mathrm{Res}_F(\Delta))$ have the same minimal polynomial which is irreducible by Proposition 4.7 because $\partial_\gamma(\mathrm{Res}_F(\Delta))$ is equivalent to a simple stratum by Lemma 4.16. If the minimal polynomial is different from X then $\partial_\gamma(\Delta)$ is equivalent to a non-zero simple stratum by Proposition 4.7. If the minimal polynomial is X then $\partial_\gamma(\mathrm{Res}_F(\Delta))$ is equivalent to a zero stratum by Proposition 4.7, and in this case $\partial_\gamma(\Delta)$ is equivalent to a zero stratum. Now Lemma 4.16 states that Δ is equivalent to a simple stratum. □

Let us recall the intertwining implies conjugacy result for simple strata:

Theorem 4.29 ([4, 1.9],[2, 4.1.3])**.** Let Δ and Δ' be simple with $n=n'$ and $r=r'$. Suppose (E,Λ) and (E',Λ') have the same embedding type. Then, granted $I(\Delta,\Delta')\neq\emptyset$, there is an element $g\in G$ such that $g\Delta$ is equivalent to Δ'. Moreover, g can be taken, such that it conjugates the maximal unramified extension of F in E to the one in E'.

Proposition 4.28 and the Theorem 4.25 have a finer version of the latter theorem as a consequence:

Theorem 4.30. Let Δ be a stratum which splits under a decomposition $V=\oplus_j V^j$ into a direct sum of pure strata and suppose that $\mathrm{Res}_F(\Delta)$ is equivalent to a simple stratum Δ_F. Then, there is an element u of $1+\mathfrak{m}(\Delta_F)$ such that $u.\Delta_F$ is equivalent to $\mathrm{Res}_F(\Delta)$, split by the above decomposition and such that $u\beta_F u^{-1}$ is D-linear.

Proof. By Theorem 4.25 we can assume that Δ_F is split by $(V^j)_j$ without loss of generality. Thus we can reduce to the case where Δ_F is simple because $\mathfrak{m}(\Delta_{F,j})$ is a subset $1^j\mathfrak{m}(\Delta)1^j$ by $k_0(\Delta_{F,j})\leq k_0(\Delta_F)$. Thus suppose that Δ_F is simple. We can assume that Δ is simple by Theorem 4.3 and thus $\mathrm{Res}_F(\Delta)$ is simple by Proposition 4.21. The strata Δ_F and $\mathrm{Res}_F(\Delta)$ are equivalent and thus Proposition 4.21 implies that $E|F$ and $F[\beta_F]|F$ have the same ramification index and the same inertia degree. Now Corollary 3.9 provides an element $\beta' \in A$ of the same minimal polynomial as β_F such that Λ is normalized by $F[\beta']^\times$ and such that the unramified sub-extensions of $F[\beta']|F$ and $E|F$ coincide. The stratum $\Delta' := [\Lambda, n, r, \beta']$ is simple by Proposition 4.21, because $\mathrm{Res}_F(\Delta')$ is simple by Theorem 4.3, Theorem 4.29 and Corollary 4.22. The direct sum $\Delta' \oplus \Delta$ is equivalent to a simple stratum by Lemma 4.28 because $\mathrm{Res}_F(\Delta \oplus \Delta')$ is equivalent to a simple stratum. Now we can use Theorem 4.25 to obtain a simple stratum Δ'' equivalent to $\Delta \oplus \Delta'$ split by $V \oplus V$. Thus Δ and Δ' intertwine and are conjugate up to equivalence by Theorem 4.29. (Note that both have the same embedding type.) Thus we can assume that Δ and Δ' are equivalent. We have to show that there is an element $u \in 1 + \mathfrak{m}(\Delta_F)$ which conjugates β_F to β'. There is an element $g \in P(\Lambda_F)$ which conjugates β_F to β'. But this element g is an element of $I(\Delta_F) \cap \mathfrak{a}_0^\times = (1 + \mathfrak{m}(\Delta_F))\mathfrak{b}_{F,0}^\times$, i.e. $g = ub$. Thus u conjugates β_F to β'. □

Corollary 4.31. Let Δ, Δ' and Δ'' be three strata which are equivalent to simple strata such that $r = r' = r''$ and $n = n' = n''$ and of the same period.

(i) Suppose $V = V'$ and suppose that $\mathrm{Res}_F(\Delta)$ and $\mathrm{Res}_F(\Delta')$ intertwine. Suppose that (V^j) splits Δ and $(V'^{j'})$ splits Δ'. Then their direct sum is equivalent to a simple stratum which is split by $\bigoplus_j V^j \oplus \bigoplus_j V^{j'}$.

(ii) Suppose that $\Delta \oplus \Delta'$ and $\Delta'' \oplus \Delta'$ are equivalent to a simple strata. Then $\Delta \oplus \Delta''$ and $\Delta \oplus \Delta' \oplus \Delta''$ are equivalent to a simple stratum.

(iii) Suppose that $\Delta \oplus \Delta'$ is a pure stratum. Then Δ, Δ' and their direct sum have the same critical exponent which is equal to $e(\Lambda|E)k_F(\beta)$ where $k_F(\beta)$ is the critical exponent of $(\mathfrak{p}_E^{\mathbb{Z}}, \beta)$.

(iv) Two equivalent simple strata have the same critical exponent, coinciding inertia degrees and coinciding ramification indexes.

Proof. (i) The direct sum of $\mathrm{Res}_F(\Delta)$ and $\mathrm{Res}_F(\Delta')$ is equivalent to simple stratum by Proposition [21, 7.1], and Theorem 4.30 finishes the proof.

(ii) Applying Theorem 4.25 we obtain that $\tilde{\Delta} = \Delta^{\oplus \dim_D V'} \oplus \Delta'^{\oplus \dim_D V''}$ and $\tilde{\tilde{\Delta}} = \Delta'^{\oplus \dim_D V} \oplus \Delta''^{\oplus \dim_D V'}$ intertwine. Thus, by (i), $\tilde{\tilde{\Delta}} \oplus \tilde{\Delta}$ is equivalent to a simple stratum split by the whole splitting. Thus, $\Delta \oplus \Delta''$ and $\Delta \oplus \Delta' \oplus \Delta''$ are equivalent to a pure strata, i.e. to a simple strata by Theorem 4.3.

(iii) We have $k_0(\Delta) \leq k_0(\Delta \oplus \Delta')$ by definition and if Δ would be simple and $\Delta \oplus \Delta'$ not, then the latter is by Corollary 4.22 and Theorem 4.25 equivalent to a simple stratum split by $V \oplus V'$ and of lower degree. But then Δ is equivalent to a simple stratum of lower degree which is not possible by Corollary 4.22. The stratum $\Delta^\dagger = \oplus_{i=0}^{e(\Lambda|F)-1}[\Lambda - i, n, r, \beta]$ is strict with $k_0(\Delta) = k_0(\Delta^\dagger)$ by the first part of this assertion. Thus we only have to prove the formula for the critical exponent in the case where Λ is a lattice chain. By Proposition 4.21 we can restrict to the split case, i.e. $D = F$. This case can be found in [7, 1.4.13(ii)].

(iv) Suppose Δ and Δ' are equivalent simple strata. Then $\mathrm{Res}_F(\Delta^\dagger)$ and $\mathrm{Res}_F(\Delta'^\dagger)$ are equivalent simple strata by Proposition 4.21 and (iii), and they satisfy (iv) by [7, 2.4.1]. Thus Δ and Δ' satisfy (iv) because Δ, $\Delta^\dagger$ and $\mathrm{Res}_F(\Delta^\dagger)$ have the same critical exponent by Proposition 4.21 and (iii).

□

Corollary 4.32. Suppose Δ and Δ' are two simple strata such that $V = V'$, $r = r'$ and $n = n'$. Then the following statements are equivalent:

(i) The stratum $\mathrm{Res}_F(\Delta \oplus \Delta')$ intertwines with a pure stratum (or equivalently with a simple stratum).

(ii) The stratum $\Delta \oplus \Delta'$ is equivalent to a simple stratum which is split under the decomposition $V \oplus V$.

(iii) $I(\Delta, \Delta') \neq \emptyset$.

(iv) $I(\mathrm{Res}_F(\Delta), \mathrm{Res}_F(\Delta')) \neq \emptyset$.

Proof. We leave this as an exercise for the reader. ☐

4.5 Semisimple strata

In this section we apply the section about simple strata to obtain similar results for semisimple strata.

Proposition 4.33. Suppose Δ is a semi-pure stratum. Then strata $\Delta \otimes \tilde{F}$ is semisimple if and only if Δ is semisimple.

Proof. Write $\tilde{\Delta}$ for $\Delta \otimes \tilde{F}$. 1) Suppose Δ is semisimple. The simple case is Proposition 4.23. So let us assume without loss of generality that Δ is semisimple with exactly two simple blocks, i.e. $\Delta = \Delta_1 \oplus \Delta_2$. Write $\tilde{\Delta}_i := \Delta_i \otimes \tilde{F}$. It is semi-pure with blocks $\tilde{\Delta}_{ij}$. From *ibid.* follows that we only have to prove that no two blocks of Δ get intertwining sub-blocks if one extends the scalars to $\tilde{F}$. Let us now assume that there are j_1 and j_2 such that the stratum $\tilde{\Delta}_{1j_1} \oplus \tilde{\Delta}_{2j_2}$ is equivalent to a simple stratum. Then $\mathrm{Res}_F(\tilde{\Delta})$ is equivalent to a simple stratum by Corollary 4.31(ii). But $\mathrm{Res}_F(\tilde{\Delta})$ is a direct sum of copies of $\mathrm{Res}_F(\Delta)$. Thus by Theorem 4.25 the latter stratum is equivalent to a simple stratum, and Δ is equivalent to a simple stratum by Proposition 4.28. This gives a contradiction.

2) Let us now assume that $\tilde{\Delta}$ is semisimple. If Δ is pure then multiply β with a negative power of π_F such that $[\Lambda, n, 0, \pi_F^{-z}\beta]$ is simple. So we can assume without loss of generality that $\Delta(r-)$ is simple. Then the critical exponents of Δ and $\tilde{\Delta}$ coincide by Proposition 4.23. Thus Δ is simple. Suppose Δ is semi-pure with two blocks $\Delta = \Delta_1 \oplus \Delta_2$. Then Δ_1 and Δ_2 are simple by the simple case. If Δ is equivalent to a simple stratum Δ', then it is equivalent to a simple stratum which is split by $V^1 \oplus V^2$. So Δ_1 and Δ_2 have the same inertia degree by Corollary 4.22. Therefore the number of blocks of $\tilde{\Delta}$ is at least twice the number of blocks of $\Delta' \otimes \tilde{F}$. Thus $\tilde{\Delta}$ cannot be semisimple by Proposition [21, Proposition 7.1]. A contradiction. ☐

Lemma 4.34. Suppose $\Delta \oplus \Delta'$ is a semisimple stratum. Then Δ is semisimple.

Proof. The strata Δ and Δ' are semi-pure because $\tilde{\Delta} := \Delta \oplus \Delta'$ is. We consider the associated splitting $(V^i)_{i \in I}$ of Δ. The stratum Δ_i is a direct summand of a simple block of $\tilde{\Delta}$. Thus it is simple by Corollary 4.31(iii). If $\Delta_{i_1} \oplus \Delta_{i_2}$ is equivalent to a simple stratum for different indexes i_1 and i_2, then the sum of the corresponding simple blocks of $\tilde{\Delta}$ is also equivalent to a simple stratum by Corollary 4.31(ii). A contradiction. ☐

Theorem 4.35. Suppose Δ is a semi-pure stratum. The following assertions are equivalent:

(i) The stratum Δ is semisimple.

(ii) The stratum $\Delta \otimes \tilde{F}$ is semisimple.

(iii) The stratum $\mathrm{Res}_F(\Delta \otimes \tilde{F})$ is semisimple.

(iv) The stratum $\mathrm{Res}_F(\Delta)$ is semisimple.

In particular, all mentioned strata have the same critical exponent.

Proof. The first two assertions are equivalent by Proposition 4.33. Assertion (iv) and (i) are equivalent by Proposition 4.21 and Proposition 4.28. The equivalence of the third and the fourth assertion follows from Corollary 4.34, because $\mathrm{Res}_F(\Delta \otimes \tilde{F})$ is a direct summand of $\mathrm{Res}_F(\Delta) \otimes \tilde{F}$ and $\mathrm{Res}_F(\Delta)$ is a direct summand of $\mathrm{Res}_F(\Delta \otimes \tilde{F})$. The last remark of the theorem is now a direct consequence of the definition of the critical exponent. ☐

Theorem 4.35 allows us to transfer many results from [21] about semisimple strata to the case of inner forms. Sécherre's and Stevens' Cohomology argument, see [16] and [22] together using Hilbert 90 for the unramified extension $L|F$ gives the formula for the intertwining for a semisimple stratum over D.

Proposition 4.36. Suppose Δ and Δ' with $\beta = \beta'$ are two semisimple strata. Then the set of D-automorphisms of V which intertwines the first with the second stratum is equal to

$$I(\Delta,\Delta') = (1+\mathfrak{m}'_{-(r'+k_0(\Delta'))})C_{A^\times}(\beta)(1+\mathfrak{m}_{-(r+k_0(\Delta))}).$$

Proof. By [21, 6.22] we have a similar formula for $I(\Delta\otimes L,\Delta'\otimes L)$:

$$(1+\mathfrak{m}'_{-(r'+k_0(\Delta'\otimes L))})C_{(A\otimes L)^\times}(\beta\otimes 1)(1+\mathfrak{m}_{-(r+k_0(\Delta\otimes L))}).$$

From Theorem 4.35 follows that Δ and $\Delta\otimes L$ have the same critical exponent, and similar for Δ'. We intersect the latter formula with $\mathrm{Aut}_D(V)$ and by [16, 2.35] using Hilbert 90 this intersection is the product of the intersections of the three factors with $\mathrm{Aut}_D(V)$. This finishes the proof. □

Corollary 4.37. Let Δ and Δ' be two equivalent semisimple strata. Then there is a bijection $\zeta : I \to I'$ such that 1^i is congruent to $1^{\zeta(i)}$ modulo $\mathfrak{a}_1$. And there is an element $u\in 1+\mathfrak{m}_{-r-k_0}$ such that uV^i is equal to $V'^{\zeta(i)}$ for all indexes $i\in I$.

Proof. Suppose $k_0\geq k'_0$, then $\mathfrak{m}_{-r-k'_0}$ is a subset of $\mathfrak{m}'_{-r-k_0}$ by the equivalence of the strata. By Proposition 4.36 we have the equality

$$(1+\mathfrak{m}_{-r-k_0})C_A(\beta)(1+\mathfrak{m}_{-r-k_0}) = (1+\mathfrak{m}_{-r-k_0})C_A(\beta')(1+\mathfrak{m}_{-r-k_0}),$$

which concludes that for every index $i\in I$ there is an element $e_i\in C_A(\beta')\cap\mathfrak{a}_0$ which is congruent to 1^i modulo $\mathfrak{m}_{-r-k_0}$. Now Lemma [21, 7.13] implies that there is an idempotent $\tilde{e}_i$ in $C_A(\beta')$ congruent to e_i and thus to 1^i modulo $\mathfrak{m}_{-r-k_0}$. Lemma [21, 7.16] (applied to the algebra $C_A(\beta')$) shows that $\tilde{e}_i$ is central in $C_A(\beta')$. Symmetrically $1^{\zeta(i)}$ is congruent to a central idempotent of $C_A(\beta)$. Thus, there is a bijection ζ from I to I' such that 1^i is congruent to $1^{\zeta(i)}$ modulo $\mathfrak{m}_{-r-k_0}$. The map $u := \sum_i 1^{\zeta(i)}1^i$ is an element of $1+\mathfrak{m}_{-r-k_0}$ which satisfies $uV^i = V'^{\zeta(i)}$. Therefore $u.\Delta'_{\zeta(i)}$ is equal to Δ_i. Now u is an element of $I(\Delta')\cap\mathfrak{a}_0^\times$ which is $\mathfrak{b}_0'^\times(1+\mathfrak{m}'_{-r-k'_0})$, i.e. u is of the form $u=b'u'$. Thus u' is an element of $(1+\mathfrak{m}'_{-r-k'_0})$ with the desired properties. □

4.6 Strata and Embedding types

The aim of this subsection is to establish an equivalent description of embedding type, see Theorem 4.40.

Theorem 4.38 ([2, 5.2],[19, 2.11])**.** If two simple strata Δ and Δ' are equivalent then they have the same embedding type, and the embeddings (Λ, E_D) and (Λ, E'_D) are conjugate by an element of $P_1(\Lambda)$.

Proposition 4.39. Let ϕ_i, $i=1,2$, be to F-algebra embeddings of a field extension $E|F$ into A, and let Λ be a lattice sequence such that $\phi_i(E)^\times$ normalizes Λ for both indexes i. Suppose further that there is an element g of $P(\Lambda)$ such that the conjugation by g seen as an endomorphism of $\mathfrak{a}_{\Lambda,0}/\mathfrak{a}_{\Lambda,1}$ restricts on the residue field of $\phi_1(E_D)$ to the map induced by $\phi_2\circ\phi_1^{-1}$. Then there is an element $u\in P(\Lambda)$ such that $u\phi_1(x)u^{-1}$ is equal to $\phi_2(x)$, for all $x\in E$. If $E=E_D$ then we can choose u in $gP_1(\Lambda)$.

Proof. We can restrict to the case where E is equal to E_D, by Theorem 3.7. Further by conjugating with g, we can assume that g is the identity. We take an element β_1 of $\phi_1(E_D)$ whose residue class generates the residue field extension and an element β_2 of $\phi_2(E_D)$ which has the same residue class as β_1 in $\mathfrak{a}_{\Lambda,0}/\mathfrak{a}_{\Lambda,1}$ and the same minimal polynomial over F as β_1. The simple strata $[\Lambda, e(\Lambda|F), e(\Lambda|F)-1, \beta_i\pi_F^{-1}]$ are equivalent and by Theorem 4.38 they have the same embedding type. We therefore find an element t of the normalizer of Λ which conjugates β_1 to β_2, see Theorem 3.7. By Proposition 4.36 the element t can be written as a product of an element of $P_1(\Lambda)$ and of $C_A(\beta_1)$. Thus, we can choose t to be an element of $P_1(\Lambda)$. □

With the above theory we now can prove an equivalent description of two embeddings to be equivalent.

Theorem 4.40. Two embeddings (E,Λ) and (E',Λ') are equivalent if and only if there is a D-automorphism g of V which maps Λ to a translation of Λ' such that the conjugation with g induces a field isomorphism between the residue fields of E_D and E'_D. The element t of $A^\times$ which conjugates (E_D,Λ) to (E'_D,Λ') (up to translation of Λ') can be chosen such that gt^{-1} is an element of $P_1(\Lambda')$.

Proof. We can assume $E = E_D$ and $E' = E'_D$, in particular E and E' are isomorphic. We only have to prove the if-part. Applying g we can assume that both lattice sequences coincide and the residue fields coincide in $\mathfrak{a}_{\Lambda,0}/\mathfrak{a}_{\Lambda,1}$. We take two F-algebra monomorphisms ϕ and ϕ' from E into A such that the image of ϕ is E (e.g. $\phi = \mathrm{id}$) and the image of ϕ' is equal to E', such that the maps coincide on the residue field of E. Then we obtain from Proposition 4.39 that both embeddings are equivalent and the conjugating element can be taken from $P_1(\Lambda)$. □

4.7 Intertwining and Conjugacy for Semisimple strata

In this subsection we prove an intertwining and conjugacy result for semisimple strata. This generalize the result for the split case, see [21, Theorem 8.3]. For that we fix in the whole subsection two semisimple strata Δ and Δ' such that $n = n'$, $r = r'$ and $e(\Lambda|D)$ is equal to $e(\Lambda'|D)$ and we assume that $I(\Delta,\Delta')$ is non-empty.

The first important step is the statement of the existence of a matching for two intertwining semisimple strata over D.

Proposition 4.41 (see [21, Theorem 7.1] for the split case)**.** There is a unique bijection ζ from the index set I for β to the index set I' for β' such that $\Delta_i \oplus \Delta'_{\zeta(i)}$ is equivalent to a simple stratum, for all $i \in I$. Furthermore the dimensions of V^i and $V'^{\zeta(i)}$ coincide.

We call the map ζ a matching.

Proof. If $I(\Delta,\Delta')$ is non-empty then $I(\mathrm{Res}_F(\Delta),\mathrm{Res}_F(\Delta'))$ is non-empty and by [21, Theorem 7.1] there is a unique matching ζ for $\mathrm{Res}_F(\Delta)$ and $\mathrm{Res}_F(\Delta')$. This is also a matching for Δ and Δ' which follows from Theorem 4.35. □

Corollary 4.42. Under the assumptions of this subsection we have

(i) $e(\Lambda|E) = e(\Lambda'|E')$ and $k_0(\Delta) = k_0(\Delta')$.

(ii) $e(\Lambda^i|E_i) = e(\Lambda'^{\zeta(i)}|E'_{\zeta(i)})$ and $e(E_i|F) = e(E'_{\zeta(i)}|F)$ and $f(E_{\zeta(i)}|F) = f(E'_{\zeta(i)}|F)$ and $k_0(\Delta_i) = k_0(\Delta'_{\zeta(i)})$ for all $i \in I$.

Proof. (ii) follows directly from Proposition 4.41, Corollary 4.32 and Corollary 4.31(iv)(iii). We now prove (i). The equality $e(\Lambda|E) = e(\Lambda'|E')$ follows directly from the second assertion. Thus we are left with the equation for the critical exponents. Suppose for a contradiction that $k_0(\Delta) > k_0(\Delta')$ ("Δ' is semisimpler") Take a positive integer j such that $r + j = -k_0(\Delta)$. Then $\Delta(j+)$ is not semisimple but $\Delta'(j+)$ is still semisimple and both strata are equivalent. Moreover the stratum $\Delta_i(j+)$ is still simple, because $k_0(\Delta_i)$ is equal to $k_0(\Delta'_{\zeta(i)})$ by the second assertion. Thus for $\Delta(j+)$ not being semisimple there exists two indexes $i_1, i_2 \in I$ such that $\Delta_{i_1}(j+) \oplus \Delta_{i_2}(j+)$ is equivalent to a simple stratum. But then $\Delta'_{\zeta(i_1)}(j+) \oplus \Delta'_{\zeta(i_2)}(j+)$, which intertwines with $\Delta_{i_1}(j+) \oplus \Delta_{i_2}(j+)$, by Theorem 4.25, must intertwine with a simple stratum. This leads to a contradiction to Proposition 4.41, because $\Delta'_{\zeta(i_1)}(j+) \oplus \Delta'_{\zeta(i_2)}(j+)$ has two blocks and the intertwining simple stratum only one block. □

Definition 4.43. We define the *group level* of a non-zero semi-pure stratum Δ as $\lfloor \frac{r}{e(\Lambda|E)} \rfloor$, and we put the group level of a zero-stratum to be infinity. Caution: Equivalent semi-pure strata can have different group levels. The *degree* of a semi-pure stratum Δ is defined as $\dim_F E$.

Corollary 4.42 implies:

Corollary 4.44. Granted $e(\Lambda|F) = e(\Lambda'|F)$ and $r = r'$, then two intertwining semisimple strata Δ and Δ' have the same group level if they intertwine.

We need to refine the intertwining and conjugacy theorem for simple strata, Theorem 4.29, more precisely we need to control the valuation of the conjugating element to generalize Theorem 4.29 to intertwining semisimple strata. The solution is motivated by Theorem 4.40. For a product of field extensions $E = \prod_i E_i$ of F we define the residue-algebra to be the product of the residue fields

$$\kappa_E := \prod_i \kappa_{E_i}.$$

Proposition 4.45 (see [7, 2.4.12] for the strict simple split case)**.** Suppose that Δ and Δ' are equivalent and that Λ is equal to Λ'. Then the residue algebras of $F[\beta]$ and $F[\beta']$ coincide on $\mathfrak{a}_{\Lambda,0}/\mathfrak{a}_{\Lambda,1}$.

Proof. If Δ and Δ' are equivalent then $\mathrm{Res}_F(\Delta)^\dagger$ and $\mathrm{Res}_F(\Delta')^\dagger$ are equivalent, and we can therefore restrict to the strict split case, i.e. Λ is a lattice chain and $D = F$. By Corollary 4.37 there is an element of $1 + \mathfrak{m}_{-(k_0+m)}$ which conjugates the splitting of β to the one of β', and we are reduced to the simple case which is done in [7, 2.4.12] using Corollary 4.42(ii). □

Lemma 4.46. The conjugation by $g \in I(\Delta,\Delta')$ induces a canonical isomorphism form the residue algebra of E to the one of E', and the isomorphism does not depend on the choice of the intertwining element.

Proof. Let us at first define the map between the residue algebras: Take $g \in I(\Delta,\Delta')$. The residue algebra κ_E is canonically embedded into $\mathfrak{a}_0/\mathfrak{a}_1$ and analogously for $\kappa_{E'}$. Conjugation by g maps $\mathfrak{a}_0/\mathfrak{a}_1$ into $(g\mathfrak{a}_0g^{-1} + \mathfrak{a}'_0)/(g\mathfrak{a}_1g^{-1} + \mathfrak{a}'_1)$. There is a canonical map from $\mathfrak{a}'_0/\mathfrak{a}'_1$ into $(g\mathfrak{a}_0g^{-1} + \mathfrak{a}'_0)/(g\mathfrak{a}_1g^{-1} + \mathfrak{a}'_1)$. We compose the inverse of the embedding of $\kappa_{E'}$ with the g-conjugation on κ_E. We have to prove

(i) for the existence that the g-conjugate of κ_E is the image of $\kappa_{E'}$ in $(g\mathfrak{a}_0g^{-1} + \mathfrak{a}'_0)/(g\mathfrak{a}_1g^{-1} + \mathfrak{a}'_1)$, and

(ii) the independence of the map from the choice of the intertwining element g.

At first we simplify the situation. Proposition 4.41 allows to reduce to the case that the block decomposition of β and β' are the same and where the matching map ζ does not permutes the blocks. We identify the index sets, so that ζ is the identity. By Proposition 4.45 we can replace Δ and Δ' by equivalent strata, and thus we can assume that β and β' have the same minimal polynomial by Theorem 4.25. Thus, after conjugation, we can assume without loss of generality that β is equal to β'.

Let us prove Assertion (i): Consider the set of intertwining elements in Proposition 4.36. The property (i). is true for intertwining elements $t \in C_A(\beta)(1 + \mathfrak{m}_{-(m+k-0)})$, in particular, $t\kappa_E t^{-1}$ is in the image of $\mathfrak{a}'_0/\mathfrak{a}'_1$ in $(t\mathfrak{a}_0t^{-1} + \mathfrak{a}'_0)/(t\mathfrak{a}_1t^{-1} + \mathfrak{a}'_1)$. Conjugation with elements of $1 + \mathfrak{m}'_{-(m+k_0)}$ acts as the identity on $\mathfrak{a}'_0/\mathfrak{a}'_1$, and therefore we have Assertion (i) for all elements of $I(\Delta,\Delta')$.

We now prove the uniqueness: By direct calculation all elements in $C_A(\beta)(1 + \mathfrak{m}_{-(m+k_0)})$ induce the identity on the residue algebra. Further conjugation with elements of $1 + \mathfrak{m}'_{-(m+k_0)}$ fixes all elements of $\mathfrak{a}'_0/\mathfrak{a}'_1$. Thus all intertwining elements induce the identity on the residue algebra. □

Notation 4.47. We denote the map of Lemma 4.46 by $\bar{\zeta}$ and call it the matching of the residue algebras of the intertwining semisimple strata in question. And we write $(\zeta,\bar{\zeta})$ and call it the *matching pair.*

We now have all ingredients to formulate and prove the natural intertwining implies conjugacy theorem for semisimple strata. Let us remark, that we do not need the notion of embedding type any more. We write E_D for the product of the $(E_i)_D$.

Theorem 4.48. Let $(\zeta,\bar{\zeta})$ be the matching pair of Δ and Δ'. Suppose there is a D-automorphism t of V which maps Λ to a translation of Λ' such that the conjugation with t induces $\bar{\zeta}|_{\kappa_{E_D}}$ and tV^i is equal to $tV^{\zeta(i)}$ for all indexes $i \in I$. Then there is an element g of G such that $g\Delta$ is equivalent to Δ', $gV^i = V^{\zeta(i)}$ and gt^{-1} is an element of $P(\Lambda')$.

Proof. Conjugating with t allows us to reduce to the case where $\Lambda = \Lambda'$, $V^i = V^{\zeta(i)}$ and $\bar{\zeta}|_{\kappa_{E_D}}$ is induced by conjugation with 1. We are now reduced to the simple case, and let us therefore assume that both strata are simple. By Corollary 4.32 and Theorem 4.25 and Proposition 4.45 we can assume that β and β' have the same minimal polynomial. Now, take an element of $P_1(\Lambda)$ which conjugates E_D to E'_D, by Theorem 4.40, and we can therefore restrict to the case that E_D is equal to E'_D. We want to apply Proposition 4.39. For that we consider two embeddings of E with $\phi_1(\beta) = \beta$ and $\phi_2(\beta) = \beta'$. There is an element of G which conjugates β to β', i.e. which is on E the map ϕ_2. This conjugation induces on residue fields the identity because $\bar{\zeta}$ is the identity. Thus, $\phi_2 \circ \phi_1^{-1}$ is on E_D the identity, and Proposition 4.39 provides an element of $P(\Lambda)$ which conjugates β to β'. This finishes the proof. □

4.8 Proof of the main theorem of strata induction

For the proof of Theorem 4.15 we need a lot of preparation.

Lemma 4.49. Let Δ be a stratum such that $\Delta(1+)$ is equivalent to a semisimple stratum with entry γ. Let 1^j be the idempotents for the associated decomposition of V for γ. Then there is an element u of $1 + \mathfrak{m}_{-(r+1+k_0(\gamma,\Lambda))}$ such that $u.\Delta$ is equivalent to a stratum which is split by the associated decomposition of V for γ.

For the proof we need the map $a_\gamma : A \to A$ defined via $a_\gamma(x) = \gamma x - x\gamma$.

Proof. We have the decomposition $A = \oplus_{jj'} A^{jj'}$ where $A^{jj'} = 1^j A 1^{j'}$. We use this notation also for elements of A. By [21, 6.21] we have for non-equal indexes j and j' that the restriction of a_γ to $A^{jj'}$ is a bijection and that for all integers $s \geq k_0(\gamma,\Lambda)$ the pre-image of $\mathfrak{a}_s$ under a_γ is equal to $\mathfrak{n}_s(\gamma,\Lambda)$. In particular for $j \neq j'$ there is an element $a^{jj'} \in \mathfrak{n}_s(\gamma,\Lambda)^{jj'}$ such that $a_\gamma(a^{jj'}) = \beta^{jj'}$. Thus for $x = \sum_{j\neq j'} a^{jj'}$ we have

$$\beta = \sum_j \beta^{jj} + a_\gamma(x).$$

The last equation implies that $\sum_j \beta^{jj}$ is congruent to $(1+x)^{-1}\beta(1+x)$ modulo $\mathfrak{a}_{-r}$, see [21, 7.6] for the calculations. This finishes the proof. □

For a non-zero semisimple stratum Δ and a tame corestriction s with respect to β we have the following sequence

$$\mathfrak{n}_{-r} \cap \mathfrak{a}_{-r-k_0} \overset{\alpha_\beta}{\to} \mathfrak{a}_{-r}/\mathfrak{a}_{-r+1} \overset{s}{\to} (\mathfrak{a}_{-r} \cap C_A(\beta))/(\mathfrak{a}_{-r+1} \cap C_A(\beta)) \to 0 \tag{4.50}$$

Lemma 4.51 (see [21, 7.6] together with Lemma 4.49 for the split case)**.** The sequence (4.50) is exact.

Proof. We denote the sequence (4.50) as $\mathrm{Seq}(\Delta)$. The sequence $\mathrm{Seq}(\Delta \otimes L)$ is exact by the split case. $\mathrm{Seq}(\Delta)$ is obtained from $\mathrm{Seq}(\Delta \otimes L)$ by taking the $\mathrm{Gal}(L|F)$-fixed points. Hilbert 90 for the trace states that the cohomology group $H^1(\mathrm{Gal}(L|F), \kappa_L)$ is trivial, which forces the exactness of $\mathrm{Seq}(\Delta)$. □

Let Λ be a lattice sequence and $1 = \sum_j 1^j$ a decomposition into pairwise orthogonal idempotents $1^j \in \mathfrak{a}_0$. Given an integer r, we say that an element $a \in \mathfrak{a}_{-r}$ is *split via* $1 = \sum_j 1^j$ *modulo* $\mathfrak{a}_{1-r}$ if a is congruent to $\sum_j 1^j a 1^j$ modulo $\mathfrak{a}_{1-r}$.

Lemma 4.52. Let Δ be a simple stratum with a corestriction s_β, $1 = \sum_j 1^j$ a decomposition in $\mathfrak{a}_0 \cap C_A(\beta)$ and let a be an element of $\mathfrak{a}_{-r}$ such that $s_\beta(a)$ is split by $1 = \sum_j 1^j$ modulo $C_A(\beta) \cap \mathfrak{a}_{-r+1}$. Then there is an element $u = 1 + x$ of $1 + \mathfrak{m}_{-(r+k_0)}$ such that $u(\beta + a)u^{-1}$ is split by the 1^j modulo $\mathfrak{a}_{-r+1}$.

Proof. This follows directly from the exactness of the sequence (4.50), in the following way: $\tilde{a} = \sum_j 1^j a 1^j$ and a have the same image under s_β, and thus by the exactness of sequence (4.50) there is an element x in $\mathfrak{n}_{-r} \cap \mathfrak{a}_{-(r+k_0)}$ such that $a_\beta(x)$ is congruent to $a - \tilde{a}$ modulo $\mathfrak{a}_{-r+1}$ which is equivalent to the conclusion of the lemma. □

Lemma 4.53. Let Δ be a semisimple stratum and let s be a tame corestriction with respect to β. Given $a, a' \in \mathfrak{a}_{-r}$, the element $s(a)$ is congruent to $s(a')$ modulo $\mathfrak{a}_{-r+1}$ if and only if there is an element $u \in 1 + \mathfrak{m}_{-r-k_0}$ such that $u(\beta + a)u^{-1}$ is congruent to $\beta + a'$ modulo $\mathfrak{a}_{-r+1}$.

Proof. This follows directly of the exactness of the sequence (4.50) and standard calculations. □

Proof of 4.15. **Part 1: 4.15(i) implies 4.15(ii):** Here we can assume that Δ is a semisimple stratum. Then there is a semisimple stratum equivalent to $\Delta(1+)$ which is split by the splitting of β and by Corollary 4.37 and Lemma 4.53 we can assume that γ is split by the splitting of β. By Lemma 4.16 $\partial_\gamma(\Delta)$ equivalent to multi-stratum of semisimple strata.
Part 2: 4.15(ii) implies 4.15(i): The stratum $\Delta(1+)$ is equivalent to a semisimple stratum $\tilde{\Delta}$ with splitting $1 = \sum_j \tilde{1}^j$ and $\tilde{\beta} = \gamma$. By Lemma 4.49 the stratum Δ is conjugate to a stratum split by $(\tilde{1}^j)_j$ modulo $\mathfrak{a}_{-r}$ by an element of $1 + \mathfrak{m}_{-r-1-\tilde{k}_0}$. This conjugation does not change the equivalence class of the derived stratum by Lemma 4.53. So, we can assume without loss of generality that Δ is split by $(\tilde{1}^j)_j$. Thus we can restrict to the case where $F[\gamma]$ is a field. We assume by (ii) that $\partial_\gamma(\Delta)$ is equivalent to a semisimple stratum which has again its own associated splitting say $1 = \sum_i 1^i$, and thus Δ is conjugate to a stratum split by $(1^i)_i$ modulo $\mathfrak{a}_{-r}$ by an element of $1 + \mathfrak{m}_{-r-1-\tilde{k}_0}$ by Lemma 4.52 which allows us to assume that $\partial_\gamma(\Delta)$ is equivalent to a simple stratum. Lemma 4.16 states in this case that Δ is equivalent to a simple stratum. □

As a corollary we obtain the subtle generalization of Theorem 4.35.

Theorem 4.54. Let Δ be a stratum and let $\tilde{F}|L$ be a finite unramified field extension. Then the following assertions are equivalent:

(i) The stratum Δ is equivalent to a semisimple stratum.

(ii) The stratum $\Delta \otimes \tilde{F}$ is equivalent to a semisimple stratum.

(iii) The stratum $\mathrm{Res}_F(\Delta \otimes \tilde{F})$ is equivalent to a semisimple stratum.

(iv) The stratum $\mathrm{Res}_F(\Delta)$ is equivalent to a semisimple stratum.

We need one lemma:

Lemma 4.55. Given two strata Δ and Δ' with $r = r'$ and $e(\Lambda) = e(\Lambda')$ and suppose that $\Delta \oplus \Delta'$ is equivalent to a semisimple stratum. Then Δ is equivalent to a semisimple stratum.

Proof. Without loss of generality we can assume $n \geq n'$. We prove by induction on $n - r$ that Δ and Δ' are equivalent to semisimple strata. The case $n = r$: Δ and Δ' are equivalent to a zero-stratum. The case $n = r+1$ follows directly from Proposition 4.8. The case $n > r+1$: $\Delta \oplus \Delta'$ is equivalent to a semisimple stratum implying that by induction hypothesis $\Delta(1+)$ and $\Delta'(1+)$ are equivalent to semisimple strata $\tilde{\Delta}$ and $\tilde{\Delta}'$ with entries γ and γ', respectively. By Theorem 4.25 we can choose $\tilde{\Delta}$ and $\tilde{\Delta}'$ such that $\tilde{\Delta} \oplus \tilde{\Delta}'$ is semisimple. Thus $\partial_{\gamma \oplus \gamma'}(\Delta \oplus \Delta') = \partial_\gamma(\Delta) \oplus \partial_{\gamma'}(\Delta')$ is equivalent to a semisimple multi-stratum. The induction start implies that $\partial_\gamma(\Delta)$ and $\partial_{\gamma'}(\Delta')$ are equivalent to semisimple multi-strata. Finally Theorem 4.15 finishes the proof. □

Proof of Theorem 4.54. We have (i)⇒(ii)⇒(iii) by Theorem 4.35. The implication (iii)⇒(iv) follows directly from Lemma 4.55, because $\mathrm{Res}_F(\Delta \otimes \tilde{F})$ is equivalent to a direct sum of $[\tilde{F} : L]$ many copies of $\mathrm{Res}_F(\Delta)$. We prove (iv)⇒(i) by induction on $n - r$. Case $n = r$ is trivial and Case $n = r+1$ follows from Corollary 4.11. The stratum $\mathrm{Res}_F(\Delta)$ is equivalent to a semisimple stratum and therefore $\mathrm{Res}_F(\Delta(1+))$ which is $\mathrm{Res}_F(\Delta)(1+)$ is equivalent to a semisimple stratum too. Then $\Delta(1+)$ is equivalent to a semisimple stratum $\tilde{\Delta}$, denote $\gamma := \tilde{\beta}$, and by Lemma 4.49 we can conjugate to the case where Δ is equivalent to a stratum which is split by the splitting of γ, i.e. we can assume $\tilde{\Delta}$ to be simple without loss of generality. The stratum $\partial_\gamma(\mathrm{Res}_F(\Delta))$ is equivalent to a semisimple stratum by Theorem 4.15 and it is equivalent to a direct sum of copies of $\mathrm{Res}_{F[\gamma]}(\partial_\gamma(\Delta))$ by formula [3, Lemma 3.1]. Thus Δ is equivalent to a semisimple stratum by Lemma 4.55, the induction start and Theorem 4.15. □

In the next section we will need the defining sequence of a stratum:

Definition 4.56. A *defining sequence* for Δ is a sequence of semisimple strata $\Delta(j) = [\Lambda, n, r+j, \beta(j)]$, equivalent to $\Delta(j+)$, $j = 0, \ldots, n-r$, such that, $\beta = \beta(0)$ and $\Delta(j+1)$ is split by the associated splitting of $\Delta(j)$. Such a sequence always exists by coarsening and Theorem 4.25. And it is not unique. Thus if we write $\Delta(j)$ there is always a fixed choice behind it. We call a defining sequence k_0*-controlled* if $\beta(j)$ is equal to $\beta(j+1)$ for the indexes j satisfying $k_0(\Delta(j)) = k_0(\Delta(j+1))$, i.e. if one increases j the last entry in the stratum changes only if the stratum reaches its critical exponent. Given a defining sequence of Δ the decreasing finite sequence of integers $(r_l) = (r + j_l)_{l=0,..}$ which begins with r and where we have $k_0(\Delta(j_l - 1)) > k_0(\Delta(j_l))$, for each positive l, is called the *jump sequence* of Δ. Members of the jump sequence are called jumps. The *core approximation* of Δ is a stratum $\Delta(j)$ where $r+j$ consists to the jump sequence such that $r+j$ is the biggest number in the jump sequence which satisfies $r \geq \lfloor \frac{-k_0(\Delta(j-1))}{2} \rfloor$. In the case of $r < \lfloor \frac{-k_0(\Delta)}{2} \rfloor$, we define Δ to be its own core approximation, see [7, 3.5.5].

5 Semisimple characters

In this section we generalize semisimple characters for $\mathrm{End}_F(V)$, see [21], and simple characters for $\mathrm{End}_D(V)$, see [16], to semisimple characters for $\mathrm{End}_D(V)$, and generalize the properties of semisimple strata from the last section to semisimple characters. Initially the concept of these kind of characters is introduced in [7] where they only considered the strict split simple case, but the generalization is straight forward.

5.1 Definitions

Here we define semisimple characters for $\mathrm{End}_D(V)$ as restrictions of semisimple characters from $\mathrm{End}_L(V)$ to $\mathrm{End}_D(V)$.

Notation 5.1. We need to fix an additive character ψ_F of F of level 1, i.e. $\psi_F|_{o_F}$ is non-trivial and factorizes through the residue field. The following constructions depend on the choice of ψ_F, and we denote the character $\psi_F \circ \mathrm{trd}_{A|F}$ by ψ_A. For an element $c \in A$ we define the map $\psi_c : A \to \mathbb{C}$ via $\psi_c(x) := \psi_A(c(x-1))$.

Let Δ be a semisimple stratum.

5.1.1 Split case

Let us recall the constructions of $\mathrm{C}(\Delta)$ for the case $\mathbf{D} = \mathbf{F}$, see [21, Section 9] for details. At first we recall the definition. We write $\lfloor x \rfloor$ for the greatest integer which is not greater than a given real number x.

Definition 5.2 ([21] 9.5)**.** Let Δ be semisimple stratum. The definition of $\mathrm{C}(\Delta)$ goes along an induction on $k_0(\beta, \Lambda)$.

(i) At first we define the relevant groups. If Δ is a zero-stratum then one defines $H(\beta, \Lambda)$ and $J(\beta, \Lambda)$ as $P(\Lambda)$. If Δ is not a zero stratum then we take a semisimple stratum $[\Lambda, n, -k_0(\beta, \Lambda), \gamma]$ which is equivalent to $[\Lambda, n, -k_0(\beta, \Lambda), \beta]$ and one defines $H(\beta, \Lambda)$ as $P(j_E(\Lambda))(H(\gamma, \Lambda) \cap P_{\lfloor \frac{k_0(\beta,\Lambda)}{2} \rfloor + 1}(\Lambda))$ and $J(\beta, \Lambda)$ as $P(j_E(\Lambda))(H(\gamma, \Lambda) \cap P_{\lfloor \frac{k_0(\beta,\Lambda)+1}{2} \rfloor}(\Lambda))$. For non-negative integers i the intersections of $H(\beta, \Lambda)$ with $P_i(\Lambda)$ is denoted by $H^i(\beta, \Lambda)$, and similar is $J^i(\beta, \Lambda)$ defined.

(ii) Here we define the set $\mathrm{C}(\Delta)$, also denoted by $\mathrm{C}(\Lambda, r, \beta)$, of semisimple characters for Δ. They are defined on the group $H^{r+1}(\beta, \Lambda)$ which we denote as $H(\Delta)$. If Δ is a zero-stratum then $\mathrm{C}(\Delta)$ is the singleton consisting of the trivial character on $H(\Delta)$. If Δ is not a zero stratum we take $[\Lambda, n, -k_0(\beta, \Lambda), \gamma]$ as above. Then $\mathrm{C}(\Delta)$ is defined to be the set of those *complex* characters θ which satisfy the following properties:

(a) θ is normalized by $\mathfrak{n}(j_E(\Lambda))$.

(b) The restriction of θ to $P_{r+1}(j_E(\Lambda))$ factorizes through the determinant $\det\colon C_{A^\times}(\beta) \to E^\times$.

(c) Put r_0 to be the maximum of r and $\lfloor \frac{k_0(\beta,\Lambda)}{2} \rfloor$. The restriction of θ to $H^{1+r_0}(\gamma,\Lambda)$ coincides with $\psi_{\beta-\gamma}\theta_0$ for some element θ_0 of $\mathrm{C}(\Lambda, r_0, \gamma)$.

Remark 5.3. Stevens gave a different definition of a semisimple character for the split case, see [23, 3.13], which is equivalent to Definition 5.2 by [21, 9.6, 9.7].

It is not surprising but a subtle statement that $\mathrm{C}(\Delta)$ only depends on the equivalence class of Δ. Following the definition one obtains that for $r \geq \lfloor \frac{n}{2} \rfloor + 1$ the set $\mathrm{C}(\Delta)$ is a singleton consisting of the element ψ_β. We will later refer to this definition to give the analogue for the non-split case. A priori one can formulate Definition 5.2 for the non-split case if we replace the determinants by reduced norms. We are going to define $\mathrm{C}(\Delta)$ in the non-split case differently and show that this coincides with Definition 5.2 if generalized to the non-split case.

5.1.2 General case

We now generalize $\mathrm{C}(\Delta)$ to an **arbitrary** D: We fix an additive character ψ_L of L of level 1 whose restriction to F is equal to ψ_F.

Definition 5.4 ([16] for the simple case)**.** We define $H(\Delta)$ as the intersection of $H(\Delta \otimes L)$ with G. The set $\mathrm{C}(\Delta)$ is defined to be the set of characters of $H(\Delta)$ which can be extended to an element of $\mathrm{C}(\Delta \otimes L)$. We call the elements of $\mathrm{C}(\Delta)$ *semsimple characters for* Δ. Note that a semisimple character always comes equipped with a stratum Δ! A semisimple character is called *simple* if Δ is simple.

Remark 5.5. The set $\mathrm{C}(\Delta)$ only depends on the equivalence class of Δ.

Because of the fact that $\mathrm{C}(\Delta)$ only depends on the equivalence class of Δ, we put $\mathrm{C}(\Delta') := \mathrm{C}(\Delta)$ for every not necessarily semisimple stratum Δ' which is equivalent to Δ. Analogously we use notation $H(\Delta')$, $J(\Delta')$ and so on. For later induction purposes we introduce the notation $\theta(j+)$ to be the restriction of θ into $\mathrm{C}(\Delta(j+))$ for all positive integers j smaller than $n-r$.

Proposition 5.6. Let Δ be a semisimple stratum then the inductive definition, see 5.2, of $\mathrm{C}(\Delta \otimes L)$ restricts canonically to an inductive definition of $\mathrm{C}(\Delta)$. The same is true for the groups $H(\beta,\Lambda)$ and $J(\beta,\Lambda)$. In particular, $\mathrm{C}(\Delta)$ does not depend on the choice of the extension ψ_L of ψ_F.

Proof. We have to prove two assertions:

(i) The assertions for $H(\beta,\Lambda)$ and $J(\beta,\Lambda)$, and

(ii) That every character given by the inductive definition extends to a semisimple character in $\mathrm{C}(\Delta \otimes L)$.

(The remaining assertions for $\mathrm{C}(\Delta)$ follow directly by restriction.) We start with (i). We only prove the assertion for $H(\beta,\Lambda)$. The proof for $J(\beta,\Lambda)$ is similar. We consider the inductive definition of $H(\beta\otimes 1,\Lambda)$ and take the $\mathrm{Gal}(L|F)$-fixed points. The only subtlety is the equality

$$(P(\Lambda_{F[\beta\otimes 1]})H^{\lfloor \frac{k_0}{2} \rfloor+1}(\gamma\otimes 1,\Lambda))^{\mathrm{Gal}(L|F)} = P(\Lambda_{F[\beta\otimes 1]})^{\mathrm{Gal}(L|F)}H^{\lfloor \frac{k_0}{2} \rfloor+1}(\gamma\otimes 1,\Lambda)^{\mathrm{Gal}(L|F)}.$$

This equality follows from the next lemma.

We now prove (ii) by induction on the critical exponent. For $k_0 = -\infty$ the trivial character extends to a trivial character. Suppose for the induction step that $k_0 \geq -n$. Suppose at first that $r \geq \lfloor \frac{-k_0}{2} \rfloor$. In this case we take a semisimple stratum Δ_γ in a defining sequence for Δ with $r_\gamma = -k_0$, and an inductive defined character θ with respect to Δ can be extended into $\mathrm{C}(\Delta \otimes L)$ because we can extend $\psi_{\gamma-\beta}\theta$ by induction hypothesis. The case of $r < \lfloor \frac{-k_0}{2} \rfloor$ is proven by induction on r, using the case $r = \lfloor \frac{-k_0}{2} \rfloor$ as the induction start. Let θ be an inductively defined character with respect to Δ. Then $\theta(1+)$ is extendible to a semisimple character $\theta(1+)_L \in \mathrm{C}(\Delta(1+)\otimes L)$. For the sake of simplicity let us assume that Δ is a simple stratum, i.e. $E\otimes L$ is a product of fields E_j which are over L isomorphic to $E\tilde{L}$ where $\tilde{L}$ is an unramified field extension of F isomorphic to $L|F$ in an algebraic closure of E. The restriction of $\theta(1+)_L$ to $P_{r+2}(\Lambda_{E\otimes L})$ factorizes through the reduced norm with character $\otimes_j\chi_j$ on $\prod_j P_{\lfloor\frac{r+1}{e(\Lambda|E)}\rfloor+1}(o_{E_j})$. Let further χ be the character on $P_{\lfloor\frac{r}{e(\Lambda|E)}\rfloor+1}(o_E)$ which comes from the restriction of θ to $P_{r+1}(j_E(\Lambda))$. Then we can extend $\otimes_j\chi_j$ to $\prod_j P_{\lfloor\frac{r}{e(\Lambda|E)}\rfloor+1}(o_{E_j})$ say to $\otimes_j\chi'_j$ (using extensions of the χ_j) such that for every element x of $P_{\lfloor\frac{r}{e(\Lambda|E)}\rfloor+1}(o_E)$ the product of the $\chi'_j(1^jx)$ is equal to $\chi(x)$, and we use $\otimes_j\chi'_j$ to extend $\theta(1+)_L$ to an element θ_L of $\mathrm{C}(\Delta\otimes L)$. □

Lemma 5.7. Let Q_1 and Q_2 be two subgroups of an ambient group Q such that Q_1 normalizes Q_2. Assume further that a group Γ is acting on Q. Then $(Q_1Q_2)^\Gamma$ is equal to $Q_1^\Gamma Q_2^\Gamma$ if the (non-abelian group) cohomology group $H^1(\Gamma, Q_1\cap Q_2)$ is trivial.

Proof. The cohomology group being trivial is equivalent to the exactness of the sequence

$$1\to(Q_1\cap Q_2)^\Gamma\to Q_2^\Gamma\to(Q_2/(Q_1\cap Q_2))^\Gamma\to 1.$$

which implies the equation $(Q_1Q_2)^\Gamma = Q_1^\Gamma Q_2^\Gamma$. □

Semisimple characters respect certain Iwahori decompositions.

Proposition 5.8 ([24] 5.5)**.** Let $\bigoplus_j V^j = V$ be a splitting of V such that the idempotents commute with β. Let $\tilde{M}$ be the Levi subgroup and $\tilde{U}_+$ and $\tilde{U}_-$ be the unipotent subgroups of G with respect to the splitting (V^j). Then $H(\Delta)$ has an Iwahori decomposition with respect to $\tilde{U}_-\tilde{M}\tilde{U}_+$ and every element θ of $\mathrm{C}(\Delta)$ is trivial on the groups $H(\Delta)\cap\tilde{U}_+$ and $H(\Delta)\cap\tilde{U}_-$.

Further, we have the following restriction maps

$$\mathrm{C}(\Delta)\to\mathrm{C}(\Delta_i),\theta\mapsto\theta_i := \theta|_{H(\beta_i,\Lambda^i)}$$

for all indexes $i\in I$ and we call the θ_i, $i\in I$ the *block restrictions* of θ. Two semisimple characters $\theta,\theta'\in\mathrm{C}(\Delta)$ coincide if block-wise the block restrictions coincide.

5.2 Transfers, and Intertwining

In this subsection we fix two semisimple strata Δ and Δ'. Take $\theta\in\mathrm{C}(\Delta)$ and $\theta'\in\mathrm{C}(\Delta')$. We denote by $I(\theta,\theta')$ the set of elements g of $A^\times$ which intertwine θ with θ', i.e. $g.\theta$ and θ' coincide on $g.H(\Delta)\cap H(\Delta')$. We denote by $\theta(1+)\in\mathrm{C}(\Delta(1+))$ the restriction of θ to $H(\Delta(1+))$. Let us recall that we denote by $\Delta(1)$ a first member of an arbitrary fixed defining sequence for Δ (There is a choice hidden.) The stratum $\Delta(1)$ is equivalent to $\Delta(1+)$ and $\Delta(1)$ can be chosen to be $\Delta(1+)$ if $\Delta(1+)$ is semisimple.

Definition 5.9. Suppose that Δ and Δ' have the same group level and the same degree. We call θ' a *transfer* of θ from Δ to Δ' if $I(\Delta,\Delta')$ is a non-empty subset of $I(\theta,\theta')$.

Notation 5.10. In the following of this subsection we assume that Δ and Δ' satisfy $r = r'$ and $e(\Lambda|F) = e(\Lambda'|F)$. We further assume that Δ and Δ' intertwine. The latter implies $n = n'$, see 4.20(ii), and that both strata have the same group level, see 4.44.

Remark 5.11. The following statements would all be true if we just would assume that Δ and Δ' have the same group level and the same degree, but many of the referred results are not stated in terms of group levels and degree in the literature.

Remark 5.12. Granted $I(\Delta,\Delta')\neq\varnothing$, then, by Theorem 4.25, there are semisimple strata $\tilde{\Delta}$ and $\tilde{\Delta}'$ equivalent to Δ and Δ', respectively, such that $\tilde{\beta}$ and $\tilde{\beta}'$ are conjugate to each other. Say, they are conjugate by a D-automorphism u of V. Then $C_A(\tilde{\beta}')^\times u$ is contained in $I(\Delta,\Delta')$.

Remark 5.13. There is a way to construct a transfer using the transfer which is already understood in the split case [23, 3.26]: Take $\theta \in \mathrm{C}(\Delta)$ and an extension $\theta_L \in \mathrm{C}(\Delta \otimes L)$. By the preceding remark we can suppose that β and β' are conjugate. Let θ'_L be the transfer of θ_L from $\Delta \otimes L$ to $\Delta' \otimes L$. Then $\theta'_L|_{H(\Delta')}$ is a transfer of θ from Δ to Δ'.

Proposition 5.14. Suppose for $\theta \in \mathrm{C}(\Delta)$ and two semisimple characters θ' and θ'' in $\mathrm{C}(\Delta')$ that $I(\Delta, \Delta') \cap I(\theta, \theta')$ and $I(\Delta, \Delta') \cap I(\theta, \theta'')$ are non-empty, then θ' and θ'' coincide and are transfers of θ from Δ to Δ'. In particular there is exactly one transfer of θ from Δ to Δ'.

In the proof we need the process of interior lifting and the process of base change which are introduced on [4].

Proof. We assume $\beta = \beta'$ without loss of generality, by Remark 5.12. The transfer exists by Remark 5.13, and by transitivity we can assume that θ' is this transfer. Then $I(\theta, \theta')$ contains $C_{A^\times}(\beta)$. Further $C_{A^\times}(\beta) \cap I(\theta.\theta'')$ is non-empty, because $1 + \mathfrak{m}(\Delta)$ and $1 + \mathfrak{m}(\Delta')$ normalize θ and θ'', respectively. So we can assume that 1 intertwines θ with θ''. Let us start with the simple case. Using an interior lifting with respect to E_D and the base change with respect to $L|E_D$ moves the setting to the split case, where the statement is already known. (For example apply a †-construction to reduce to the case of block-wise principal lattice chains and apply [7, 3.6.1].) The semisimple case follows from the simple case because the characters respect the Iwahori-decomposition given by β and are trivial on the unipotent parts, see 5.8. □

We denote the transfer map by $\tau_{\Delta,\Delta'} : \mathrm{C}(\Delta) \to \mathrm{C}(\Delta')$, i.e. $\tau_{\Delta,\Delta'}(\theta)$ is the transfer of θ from Δ to Δ'. The map is well defined by the last proposition.

For the next theorem we need the following subset:

$$S(\Delta) := (1 + \mathfrak{m}(\Delta)) J^{\lfloor \frac{-k_0(\Delta)+1}{2} \rfloor}(\Delta).$$

It is the intersection of $S(\Delta \otimes L)$ with G because $1 + \mathfrak{m}(\Delta \otimes L)$ normalizes the second factor $J^{\lfloor \frac{-k_0+1}{2} \rfloor}(\Delta \otimes L)$ and we can apply Lemma 5.7. We have further the equality

$$S(\Delta) = J^{\lfloor \frac{-k_0(\Delta)+1}{2} \rfloor}(\Delta),$$

if $r \le \lfloor \frac{-k_0(\Delta)+1}{2} \rfloor$, because $1 + \mathfrak{m}(\Delta)$ is contained in $J^{\lfloor \frac{-k_0(\Delta)+1}{2} \rfloor}(\Delta)$ for that case of r, by [23, 3.10(i)] and restriction to G. Further $S(\Delta)$ is a subset of the normalizer of θ because it is a subset of the normalizer of every extension $\theta_L \in \mathrm{C}(\Delta \otimes L)$ of θ by [23, 3.16].

Theorem 5.15. Granted that Δ and Δ' intertwine, suppose that $\theta' = \tau_{\Delta,\Delta'}(\theta)$. Then the set of intertwining elements from θ to θ' is the sets

$$S(\Delta') I(\Delta, \Delta') S(\Delta).$$

If $\beta = \beta'$ the formula simplifies to

$$I(\theta, \theta') = S(\Delta') C_{A^\times}(\beta) S(\Delta). \tag{5.16}$$

For the proof of the intertwining formula we need the Lie algebra of $H(\Delta)$: The group $H(\Delta)$ can be written as $1 + \mathfrak{h}(\Delta)$ with a bi-$P(j_E(\Lambda))$-order $\mathfrak{h}(\Delta)$ in A. We also write $\mathfrak{h}^j(\beta, \Lambda)$ for $H^j(\beta, \Lambda) - 1$, for positive j. For subsets S of A we have the duality operation:

$$S^* := \{a \in A | aS \subseteq \ker(\psi_A)\}.$$

As an example consider $S = \mathfrak{h}(\Delta)$. It is a full o_F-lattice in A and the set of $\mathrm{Gal}(L|F)$-fixed points of the $\mathrm{Gal}(L|F)$-invariant o_L-module $\mathfrak{h}(\Delta \otimes L)$. Thus $\mathfrak{h}(\Delta \otimes L)$ is equal to $\mathfrak{h}(\Delta) \otimes_{o_F} o_L$. And therefore $\mathfrak{h}(\Delta)^{*,\psi_F}$ is a subset of $\mathfrak{h}(\Delta \otimes L)^{*,\psi_L}$.

Proof. The last assertion is a consequence of Proposition 4.36. Let $\theta_L \in \mathrm{C}(\Delta \otimes L)$ and $\theta'_L \in \mathrm{C}(\Delta' \otimes L)$ be extensions of θ and θ' and suppose that θ'_L is a transfer of θ_L, see Remark 5.13. Then $A^\times \cap I(\theta_L, \theta'_L)$ is a subset of $I(\theta, \theta')$. Without loss of generality we can assume that $\beta = \beta'$ by Remark 5.12. We have formula (5.16) for the split case by [12, 9.3]. Taking $\mathrm{Gal}(L|F)$-fixed points, a cohomology argument implies that $A^\times \cap I(\theta_L, \theta'_L)$ is equal to $S(\Delta')C_{A^\times}(\beta)S(\Delta)$.

We need to show that $I(\theta, \theta')$ is a subset of $I(\theta_L, \theta'_L)$. This is done by an induction on the critical exponent $k_0(\Delta)$ which is equal to $k_0(\Delta')$ by Corollary 4.42(i), because the strata are semisimple, intertwine, $r = r'$ and $e(\Lambda|F) = e(\Lambda'|F)$. The start of the induction is the statement for zero strata which is evident. There are two cases in the induction step.

Case 1: $r \geq \lfloor \frac{-k_0}{2} \rfloor$. By the end of the proof of Lemma [7, (3.3.5)] we have the following: If x is an element of $I(\theta, \theta')$ then $\psi_{x\beta x^{-1}}$ is equal to ψ_β on $x.H(\Delta) \cap H(\Delta')$. The latter holds if and only if $x\beta x^{-1} - \beta$ is an element of $(x.\mathfrak{h}(\Delta) \cap \mathfrak{h}(\Delta'))^*$ which is $x.\mathfrak{h}(\Delta)^* + \mathfrak{h}(\Delta')^*$. Thus x is an element of $I^+(\Delta, \Delta')$, the set of elements of $A^\times$ which intertwine $\beta + \mathfrak{h}(\Delta)^*$ with $\beta + \mathfrak{h}(\Delta')^*$. In particular, x is an element of $I^+(\Delta \otimes L, \Delta' \otimes L)$ because $\mathfrak{h}(\Delta)^{*,\psi_F}$ is a subset of $\mathfrak{h}(\Delta \otimes L)^{*,\psi_L}$. A †-construction, intertwining implies conjugacy [21, 10.2] and [7, 3.3.8] (Analogous proof of $I(\theta_L) = I^+(\Delta_L)$ for the split semisimple case.) imply that $I^+(\Delta_L, \Delta'_L)$ is a subset of $I(\theta_L, \theta'_L)$, Thus x is an element of $I(\theta_L, \theta'_L)$.

Case 2: $r < \lfloor \frac{-k_0}{2} \rfloor$. In this case $\Delta(1+)$ and $\Delta'(1+)$ are still semisimple. Then the proof follows by induction. We have that $I(\theta, \theta')$ is contained in $I(\theta(1+), \theta'(1+))$ which is contained in $I(\theta_L(1+), \theta'_L(1+))$ which is the same as $I(\theta_L, \theta'_L)$, by the formula (5.16), which is known in the split case, see [12, 9.3], using that $S(\Delta \otimes L)$ and $S((\Delta \otimes L)(1+))$ coincide with $J^{\lfloor \frac{-k_0+1}{2} \rfloor}(\Delta \otimes L)$ for these r. □

The proof of the last Theorem is very similar to the proof of the split simple counterpart in [7, 3.3.2].

Corollary 5.17. Suppose that $C(\Delta)$ and $\mathrm{C}(\Delta')$ intersect non-trivially, then there is an element of $S(\Delta)$ which conjugates the first associated splitting to the second.

Proof. Consider the two descriptions of the intertwining of a semisimple character of $\mathrm{C}(\Delta)$ and take limits in the-p-adic topology. We get

$$S(\Delta)C_A(\beta)S(\Delta) = S(\Delta')C_A(\beta')S(\Delta').$$

As in the proof of [21, 9.9(iv)] we use congruence between the central simple idempotents of $\mathrm{C}_A(\beta)$ and $\mathrm{C}_A(\beta')$ to obtain an element of $S(\Delta)$ which conjugates the associated splitting of Δ to the one of Δ'. □

5.3 The †-construction

Suppose $\theta \in \mathrm{C}(\Delta)$ is a semisimple character, and fix an integer k and integers $s_l,\ l = 1, \dots k$. Then we define $\Delta^{\dagger,(s_k)_k}$ as the direct sum of the strata $[\Lambda - s_k, n, r, \beta]$ where k passes from 1 to k. Let $U_- M U_+$ be the Iwahori decomposition with respect to the direct sum $V^\dagger = \bigoplus_k V$. Then $H(\Delta^{\dagger,(s_k)_k})$ respects this Iwahori decomposition and we define a map $\theta^{\dagger,(s_k)_k}$ on $H(\Delta^{\dagger,(s_k)_k})$ via

$$\theta^{\dagger,(s_k)_k}(u_- x u_+) := \prod_k \theta(x_k).$$

Proposition 5.18. The map $\theta^{\dagger,(s_k)_k}$ is an element of $\mathrm{C}(\Delta^{\dagger,(s_k)_k})$.

Proof. The †-construction and this proposition is already known in the split case. See for example [13, 3.3]. Thus, for an extension $\theta_L \in \mathrm{C}(\Delta \otimes L)$ the character $\theta_L^{\dagger,(s_k)_k}$ is an element of $\mathrm{C}(\Delta^{\dagger,(s_k)_k} \otimes L)$, and its restriction to $H(\Delta^{\dagger,(s_k)_k})$ is $\theta^{\dagger^{(s_k)_k}}$. □

One important application is the construction of sound strata. A stratum Δ is called *sound* if $[\Lambda]$ and $j_E([\Lambda])$ are mid points of the facets of principal lattice chains, i.e. Λ is a lattice chain such that $\mathfrak{b}_0$ is the order of a principal lattice chain and such that $\mathfrak{n}(\mathfrak{b}_0)$ coincides with $\mathfrak{n}(\mathfrak{a}_0) \cap B^\times$.

Definition 5.19 ([4] 2.16,2.17)**.** Suppose we are given a simple stratum Δ. Let e be the F-period of Λ. This coincides with the F-period of $j_E(\Lambda)$. Take the sequence $s_k = k$ for $k = 0,\dots,e-1$. Then $\Delta^{\ddagger} := \Delta^{\dagger,(s_k)_k}$ is a sound stratum. Analogously we define $\theta^{\ddagger}$.

Sound simple characters have been studied in [10].

5.4 Derived characters

Let Δ and Δ' be semisimple strata of the same period and with $r = r'$.

Definition 5.20. We need an analogue to the strata induction. For that we need the derived character: Let $\theta_0 \in \mathrm{C}(\Delta(1)(1-))$ be an extension of $\theta(1+)$. Then there is a $c \in \mathfrak{a}_{-r-1}$ such that θ is equal to $\psi_{\beta-\beta(1)+c}\theta_0$. Let s be a tame corestriction with respect to $\beta(1)$. We write $\partial_{\beta(1),\theta_0}$ for the character $\psi_{s(\beta-\gamma+c)}$ defined on $P_{r+1}(j_{E(1)}(\Lambda))$.

Proposition 5.21. Suppose θ and θ' intertwine and that we can choose $\Delta(1)$ and $\Delta'(1)$ such that $\beta(1) = \beta'(1) =: \gamma$. Suppose further that $\theta'(1+)$ is a transfer of $\theta(1+)$ from $\Delta(1)$ to $\Delta'(1)$. Let $\theta_0 \in \mathrm{C}(\Delta(1)(1-))$ be an extension of $\theta(1+)$ and $\theta'_0 := \tau_{\Delta,\Delta'}(\theta_0)$. Then we have the following assertions:

(i) If an element $g = uxv \in I(\theta(1+),\theta'(1+))$ (with decomposition from equation (5.16)) intertwines θ with θ' then x intertwines $\partial_{\gamma,\theta_0}\theta$ with $\partial_{\gamma,\theta'_0}\theta'$.

(ii) For every $x \in I(\partial_{\gamma,\theta_0}\theta,\partial_{\gamma,\theta'_0}\theta')$, there are elements $u \in S(\Delta'(1))$ and $v \in S(\Delta(1))$ such that uxv intertwines θ with θ'.

For the proof we need the following generalization of Sequence (4.50).

Lemma 5.22. Let Δ and Δ' be semisimple strata of the same period with $\beta = \beta'$ and $r = r'$ and let s be a tame corestriction with respect to β. Then the sequence

$$\mathfrak{m}(\Delta) + \mathfrak{m}(\Delta') \xrightarrow{\alpha_\beta} (\mathfrak{a}_{-r} + \mathfrak{a}'_{-r})/(\mathfrak{a}_{-r+1} + \mathfrak{a}'_{-r+1}) \xrightarrow{s} (\mathfrak{b}_{-r} + \mathfrak{b}'_{-r})/(\mathfrak{b}_{-r+1} + \mathfrak{b}'_{-r+1}) \to 0$$

is exact.

Proof. The proof is analogous to the proof of Lemma 4.51. Because the exactness of this sequence is known in the split case by [21, 6.21,7.6]. □

Proof of Lemma 5.21. (i) The element x intertwines $v.\theta$ with $u^{-1}.\theta'$. We have $\theta = \theta_0\psi_{\beta-\gamma+c}$ and $\theta = \theta_0\psi_{\beta'-\gamma+c'}$ Thus x intertwines the restriction of $v.\theta$ and $u^{-1}.\theta'$ to $P_{r+1}(j_{F[\gamma]})$. So we have to calculate these restrictions.

We write θ and θ' as in Definition 5.20 using the corestriction s and elements $c \in \mathfrak{a}_{-1-r}$ and $c' \in \mathfrak{a}_{-1-r}$. Claim: $v.\theta$ coincides with $\psi_{s(\beta-\gamma+c)}\theta_0$ on $P_{r+1}(j_{F[\gamma]})$. The claim and the analogous statement for θ' would imply that x intertwines $\psi_{s(\beta-\gamma+c)}$ with $\psi_{s(\beta'-\gamma+c')}$, because x intertwines θ_0 with θ'_0. Now let us prove the claim:

$$\begin{aligned} v.\theta &= v.\psi_{\beta-\gamma+c}v.\theta_0 && (5.23)\\ &= \psi_{\beta-\gamma+c}v.\theta_0 && (5.24)\\ &= \psi_{\beta-\gamma+c}\theta_0\psi_{v\gamma v^{-1}-\gamma}\theta_0 && (5.25)\end{aligned}$$

In general for an element a of $\mathfrak{a}_{-r-1}$ the restriction of ψ_a to $P_{r+1}(j_{F[\gamma]})$ is equal to $\psi_{s(a)}$. So we have to show that $s(v\gamma v^{-1})$ congruent to $s(\gamma)$ modulo $\mathfrak{a}_{-r}$. We have

$$\begin{aligned} v\gamma v^{-1} &= (\gamma v + v\gamma - \gamma v)v^{-1} && (5.26)\\ &\equiv \gamma + (v\gamma - \gamma v) \bmod \mathfrak{a}_{-r}, && (5.27)\\ & && (5.28)\end{aligned}$$

because $v\gamma-\gamma v$ is an element of $\mathfrak{a}_{-1-r}$ and v is a 1-unit of $\mathfrak{a}_0$. We now apply s to obtain that $s(v\gamma v^{-1})$ is congruent to $s(\gamma)$ modulo $\mathfrak{a}_{-r}$, because $v\gamma - \gamma v$ is in the kernel of s.

(ii) It is the reverse of the first part. Suppose $x \in B_\gamma^\times$ intertwines $\psi_{s(\beta-\gamma+c)}$ with $\psi_{s(\beta'-\gamma+c')}$. Then then $s(x(\beta-\gamma+c)x^{-1})$ is congruent to $s(\beta'-\gamma+c')$ modulo $x\mathfrak{a}_{-r}x^{-1}+\mathfrak{a}'_{-r}$. By Lemma 5.22 there are elements $y_v \in \mathfrak{m}(\Delta)$ and $y_u \in \mathfrak{m}(\Delta')$ such that

$$x(\beta-\gamma+c)x^{-1}-(\beta'-\gamma+c') \equiv \alpha_\gamma(xy_vx^{-1})+\alpha_\gamma(y_u)$$

modulo $x\mathfrak{a}_{-r}x^{-1}+\mathfrak{a}'_{-r}$. Define u by $1-y_u$ and v by $1-y_v$. Then

$$xv(\beta+c)v^{-1}x^{-1} \equiv u^{-1}(\beta'+c')u.$$

Thus $v.\psi_{\beta+c}$ and $u^{-1}.\psi_{\beta'+c'}$ intertwine by x. And

$$v.\psi_{\beta+c} = v.\psi_{\beta+c-\gamma}\psi_{v\gamma v^{-1}-\gamma}\psi_\gamma.$$

The element x intertwines ψ_γ (on G) and thus $v.\psi_{\beta+c-\gamma}\psi_{v\gamma v^{-1}-\gamma}$ with $u^{-1}.\psi_{\beta'+c'-\gamma}\psi_{u^{-1}\gamma u-\gamma}$. Further θ_0 and θ'_0 are intertwined by x. Thus x intertwines $v.\theta$ with $u^{-1}.\theta'$ and thus $uxv \in I(\theta,\theta')$.

□

5.5 Simple characters

Let us recall that a semisimple characters $\theta \in \mathrm{C}(\Delta)$ is called simple if Δ is a simple stratum. In this section we fix two simple strata Δ and Δ' and simple characters $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$.

Here we collect some facts about simple characters.

Proposition 5.29 (see [10, 9.1, 9.9], for sound strata)**.** Suppose $\Lambda = \Lambda'$, $r = r'$ and $\mathrm{C}(\Delta)\cap\mathrm{C}(\Delta') \neq \emptyset$. Then both strata have coinciding intertia degrees, coinciding ramification indexes and coinciding critical exponents.

Proof. By Proposition 4.19 either both strata are zero or both are non-zero, because of intertwining, $r = r'$ and $\Lambda = \Lambda'$. In the first case there is nothing to prove. So we can assume that both simple strata are non-zero. The intertwining of an element of $\theta \in \mathrm{C}(\Delta)\cap\mathrm{C}(\Delta')$ can be described with Δ and Δ':

$$I(\theta) = S(\Delta)B^\times S(\Delta) = S(\Delta')B'^\times S(\Delta').$$

We intersect both sides with $\mathfrak{a}^\times$ and we factorize by $1+\mathfrak{a}_1$ to obtain:

$$\prod_i \mathrm{GL}_{s_i}(\tilde\kappa) = \prod_{i'} \mathrm{GL}_{s'_i}(\tilde\kappa').$$

Wedderburn implies $\sum_i s_i = \sum_{i'} s_{i'}$ and we denote this sum as $\tilde m$. We have $B = M_{\tilde m}(D_\beta)$ where D_β is the skew-field which is Brauer equivalent to B and $E\otimes_F D$. In particular $\deg D_\beta = \frac{d}{\gcd(d,[E:F])}$ The double centralizer theorem states

$$\deg A = \deg B[E:F] = \tilde m\frac{d}{\gcd(d,[E:F])}[E:F].$$

Further $[\tilde\kappa:\kappa_F] = f(E|F)\frac{d}{\gcd(d,[E:F])}$. We have the same equations for Δ' and we get by the quotient of the first with the second equation that both strata have the same ramification index. For the intertia degrees we claim that the minimum of $\mathrm{im}(\nu_F\circ\mathrm{Nrd}_{A|F}(I(\theta)))$ in $\mathbb{R}^{>0}$ is equal to $f(E|F)$ and $f(E'|F)$. This minimum is realized if one takes an element x of minimal positive reduced norm $\mathrm{Nrd}_{B|E}$ in B. Such an element x satisfies $\mathrm{Nrd}_{B|E}(x) \in o_E^\times\pi_E$. Thus $\mathrm{Nrd}_{A|F}(x) \in \pi_F^{f(E|F)}o_F^\times$. Which finishes the proof of the equality of the inertia degrees. The equality for the critical exponents follows now directly from Corollary 4.22, because of if for example $\Delta(1+)$ is still simple, in particular $n > r+1$, but $\Delta'(1+)$ not, then take $\tilde\Delta$ is equivalent to $\Delta'(1+)$ and simple. Then $[E:F] = [\tilde E:F] < [E':F] = [E:F]$ where the equalities are from Part 1 or the proof. A contradiction. □

From that Proposition follows now:

Theorem 5.30. Suppose Λ and Λ' have the same F-period, $r = r'$ and θ and θ' intertwine then $e(E|F) = e(E'|F)$, $f(E|F) = f(E'|F)$ and both strata have the same critical exponent.

Proof. We consider at first the case where $\Lambda = \Lambda'$. By [4, 4.5] we can assume without loss of generality that Δ and Δ' are sound with the same Fröhlich invariant. So we want to apply Theorem [9, 10.3]. But this Theorem uses [9, 9.1] with a gap in the proof, which is filled by Proposition 5.29. Now [9, 10.3], see also [4, 1.16], implies $e(E|F) = e(E'|F)$ and $f(E|F) = f(E'|F)$, and θ_1 is conjugate to θ_2 by an element of the normalizer of Λ and Proposition 5.29 finishes the proof for the case $\Lambda = \Lambda'$. Suppose now that Λ is different from Λ'. Then a †-construction, see Proposition 5.18, such that Λ and Λ' are conjugate principal lattice chains reduces to the case where both lattice sequences are coinciding lattice chains. See Corollary 4.31(iii) for invariance of the critical exponent under this construction. □

Recall that $\left\lfloor \frac{r}{e(\Lambda|E)} \right\rfloor$ is the group level of a semisimple stratum.

Corollary 5.31. Suppose two Δ and Δ' have the same degree and the same group level. And suppose that θ and θ' intertwine. Then $f(E|F) = f(E'|F)$, $e(E|F) = e(E'|F)$ and $k_0(\Delta) = k_0(\Delta')$.

Proof. By doubling we can assume that both lattice sequences have the same period. If $r = r'$ then we can apply Theorem 5.30. Suppose now that $r < r'$. Then there is a simple stratum $\Delta(r'-r)$ equivalent to $\Delta((r'-r)+)$ and $\theta((r'-r)+)$ still intertwines θ'. Thus $E(r'-r)$ and E' have the same degrees and the same inertia degrees by Theorem 5.30. Thus $E|F$ and $E(r'-r)|F$ have the same ramification index by Proposition 4.21,i.e. $E|F$ and $E'|F$ have the same ramification index. Thus the stratum $\Delta((r'-r)+)$ is still simple because

$$-k_0(\Delta) \geq e(\Lambda|E)(1 + \lfloor \frac{r}{e(\Lambda|E)} \rfloor) = e(\Lambda|E')(1 + \lfloor \frac{r'}{e(\Lambda|E')} \rfloor) > r'.$$

We can therefore apply Theorem 5.30 to $\theta((r'-r)+) \in \mathrm{C}(\Delta((r'-r)+))$ and $\theta' \in \mathrm{C}(\Delta')$. □

Proposition 5.32. Suppose Δ is a semisimple stratum and j a positive integer such that

$$\lfloor \frac{r}{e(\Lambda|E)} \rfloor = \lfloor \frac{r+j}{e(\Lambda|E)} \rfloor,$$

i.e. both strata Δ and $\Delta(j+)$ have the same group level. Then the restriction map $res_{\Delta,\Delta(j+)}$ from $\mathrm{C}(\Delta)$ to $\mathrm{C}(\Delta(j+))$ is bijective.

Proof. It is enough to show the statement for $j = 1$. For zero-strata there is nothing to prove. For the case $-k_0(\beta) \geq n$ we move to a core approximation to reduce to the case where $r < \lfloor \frac{-k_0(\Delta)}{2} \rfloor$. Then $H(\Delta) = P_{r+1}(j_E(\Lambda))H(\Delta(1+))$. Take an element $\theta \in \mathrm{C}(\Delta(1+))$. Then we only have to prove that there is a unique extension of $\theta|_{P_{r+2}(j_E(\Lambda))}$ to $P_{r+1}(j_E(\Lambda))$. This follows immediately form having the same group level because the image of Nrd_{E_i} on $P_r(j_{E_i}(\Lambda^i))$ is equal to $P_{\lfloor \frac{r}{e(\Lambda^i|E^i)} \rfloor}(o_{E_i})$, so it is the same for r and for $r+1$ by both strata having the same group level. Thus there is exactly one extension of θ into $\mathrm{C}(\Delta)$. □

The last two statements allow us to describe the transfer from Δ to Δ' if both strata have the same degree and the same group level and both strata intertwine. Say $r \leq r'$ and that both strata have the same F-period. The restriction map from $\mathrm{C}(\Delta)$ to $\mathrm{C}(\Delta((r'-r)+))$ is a bijection. The transfer from Δ to Δ' is given by the map

$$\tau_{\Delta,\Delta'}(\theta) := \tau_{\Delta((r'-r)+),\Delta'}(\theta((r'-r)+)), \theta \in \mathrm{C}(\Delta).$$

This map is a bijection, being composed by two bijections. Given $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$ such that θ' is a transfer of θ, i.e. $I(\Delta,\Delta') \subseteq I(\theta,\theta')$ then $I(\Delta((r'-r)+),\Delta') \subseteq I(\theta((r'-r)+),\theta')$ and thus $\tau_{\Delta((r'-r)+),\Delta'}(\theta((r'-r)+)) = \theta'$. Thus for $\theta \in \mathrm{C}(\Delta)$ exists exactly one transfer in $\mathrm{C}(\Delta')$ and for $\theta' \in \mathrm{C}(\Delta')$ exists exactly one transfer in $\mathrm{C}(\Delta)$.

Theorem 5.33 ([4] 4.16)**.** Suppose $\Lambda = \Lambda'$ and $r = r'$. Then the sets $C(\Delta)$ and $C(\Delta')$ coincide if they have a non-empty.

Proposition 5.34 ([4] 1.12)**.** Suppose Δ and Δ' are simple with the same embedding type, $r = r'$ and $\Lambda = \Lambda'$. Let E_{ur} (resp. E'_{ur}) be the maximal unramified field extension in $E|F$ (resp. $E'|F$). Let $\theta \in C(\Delta)$ and $\theta' \in C(\Delta')$ be two simple characters which intertwine. Then θ is conjugate to θ' by an element of the normalizer of Λ which conjugates E_{ur} to E'_{ur}.

Proposition 5.35 ([4] 1.11,8.3)**.** Intertwining is an equivalence relation on the set of all simple characters for G with same group level and the same degree.

Proof. [4, 1.11] only implies this for simple characters for the same lattice sequence. Corollary 5.31 provides that the simple characters in question have the same degree and the same ramification index. So we can transfer to the case where all characters are attached to the same lattice sequence and all have the same embedding type by Proposition 3.9, if we have proven the following statement:

Suppose Δ, Δ' and Δ'' are three simple strata of the same degree and the same group level. Suppose that $\beta' = \beta''$, $\Lambda = \Lambda'$ and that Δ and Δ' have the same embedding type. Suppose further that $\theta \in C(\Delta), \theta' \in C(\Delta')$ and $\theta'' \in C(\Delta'')$ are simple characters such that θ' and θ'' are transfers of each other. Then θ intertwines θ' if and only if θ intertwines θ''.

Proof of this statement: Suppose at first that θ and θ' intertwine, say $r \geq r'$. Then θ and $\theta'((r-r')+)$ intertwine and thus they are conjugate by Proposition 5.34. Thus θ and θ'' intertwine because $\theta'((r-r')+)$ is the transfer of θ''. If $r < r'$ then we take the transfer $\tilde{\theta}'$ of θ'' from Δ'' to $\Delta'((r-r')+)$. Then $\tilde{\theta}'$ is the unique extension of θ'. Now θ and $\tilde{\theta}'$ intertwine by [4, 1.11] and we proceed as in the case $r \geq r'$. We prove now the other direction. Suppose now that θ and θ'' intertwine. Using doubling and a †-construction to construct conjugate principal lattice chains we see that $\theta^\dagger$ and $\theta''^\dagger$ intertwine. Then by [4, 1.11] θ and θ' intertwine. □

5.6 Equality of sets of semisimple characters

Here we are given two semisimple strata Δ and Δ' with $\Lambda = \Lambda'$, $r = r'$ and $n = n'$.

Lemma 5.36. The strata Δ and Δ' have the same jump sequence if $C(\Delta) \cap C(\Delta')$ is non-empty. In particular Δ and Δ' have the same critical exponent.

Proof. The second assertion follows from the first one by the definition of the jump sequence, see 4.56. Thus we only have to prove the first assertion. At first we remark that we can assume that both strata have the same associated splitting by corollary 5.17, and we take the same indexing for both strata, i.e. $I = I'$. If there is a jump at $r + j_s$ in the defining sequence of Δ then the number of blocks decreases or for one stratum Δ_i there is a jump at $r + j_s$. Now, $\Delta'(j)$ has the same number of blocks as $\Delta(j)$, because this number is determined by the intertwining of any element of $C(\Delta'(j)) \cap C(\Delta(j))$. Thus in the first case $\Delta'(j_s)$ has less blocks than $\Delta'(j_s - 1)$. In the second case there is also a jump for the stratum Δ'_i at $r + j_s$, by Lemma 5.29. □

For the next Lemma we need the core approximation, see Definition 4.56. The idea of a core approximation $\tilde{\Delta}$ of Δ is that we have

$$H(\Delta) = H^{r+1}(\tilde{\beta}, \Lambda) = P_{r+1}(j_{\tilde{E}}(\Lambda))H^{r+2}(\tilde{\beta}, \Lambda).$$

Lemma 5.37 ([7, 3.5.7])**.** Let θ and θ' elements $C(\Delta)$. Suppose further that $\tilde{\Delta}$ is a core approximation of Δ. Then θ/θ' is intertwined by $S(\tilde{\Delta})C_{A^\times}(\tilde{\beta})S(\tilde{\Delta})$.

Proposition 5.38 (see [7, 3.5.8] and [4, 4.16] for the simple case)**.** $C(\Delta)$ and $C(\Delta')$ coincide if they intersect non-trivially.

The strategy of the proof is from [7, 3.5.8].

Proof. The proof is done by induction on $n-r$. Let us suppose that not both sets $\mathrm{C}(\Delta)$ and $\mathrm{C}(\Delta')$ are singletons, because otherwise the statement is trivial, in particular we are in the case of non-zero strata and $r < \lfloor \frac{n}{2} \rfloor$. By induction hypothesis we can assume that $\mathrm{C}(\Delta(1+))$ is equal to $\mathrm{C}(\Delta'(1+))$. Let $\tilde{\theta}$ be an element of $\mathrm{C}(\Delta) \cap \mathrm{C}(\Delta')$, and take core approximations $\tilde{\Delta}$ and $\tilde{\Delta}'$ of Δ and Δ', respectively. We have to show that $\mathrm{C}(\Delta)$ is contained in $\mathrm{C}(\Delta')$. So let θ be an element of $\mathrm{C}(\Delta)$. Then there are elements $b \in \mathfrak{a}_{-r-1}$ and $\theta' \in \mathrm{C}(\Delta')$ such that θ is equal to $\theta'\psi_b$. This is possible, because the restriction map from $\mathrm{C}(\Delta')$ to $\mathrm{C}(\Delta'(1+))$ is surjective. Then the qoutients $\theta/\tilde{\theta}$ and $\theta'/\tilde{\theta}$ are intertwined by the intertwining of every element of the non-empty intersection $C(\tilde{\Delta}) \cap \mathrm{C}(\tilde{\Delta}')$ by Lemma 5.37. Thus the character

$$\psi_b = \theta/\tilde{\theta}(\theta'/\tilde{\theta})^{-1}$$

is intertwined by $C_{A^\times}(\tilde{\beta}')$. And by [19, 2.10] we obtain that $s_{\tilde{\beta}'}(b)$ is congruent to an element of $\tilde{E}'$ modulo $\mathfrak{a}_{-r}$ and thus the restriction of ψ_b to $\mathrm{P}(j_{\tilde{E}'}(\Lambda))$ factorizes through the reduced norm. Thus the restriction of $\phi := \theta\psi_{\tilde{\beta}'-\beta'}$ to $\mathrm{P}(j_{\tilde{E}'}(\Lambda))$ factorizes through the reduced norm. Secondly, $\phi|_{H(\Delta(1+))}$ coincides with $\theta'(1+)\psi_{\tilde{\beta}'-\beta'}$ and thus an element of $\mathrm{C}(\tilde{\Delta}'((\tilde{r}'-r-1)-))$ and thus ϕ is an element of $\mathrm{C}(\tilde{\Delta}'((\tilde{r}'-r)-))$ because $r < \lfloor \frac{-k_0(\tilde{\Delta}')}{2} \rfloor$. Thus θ is an element of

$$\psi_{\beta'-\tilde{\beta}'}\mathrm{C}(\tilde{\Delta}'((\tilde{r}'-r)-)) = \mathrm{C}(\Delta').$$

□

Lemma 5.39 (see [4] 4.12 for the case of simple Strata)**.** Suppose that $\mathrm{C}(\Delta)$ and $\mathrm{C}(\Delta')$ coincide and $r \geq 1$. Then $H(\Delta(1-))$ is equal to $H(\Delta'(1-))$.

The statement of this lemma is proven in [4, 4.12] for the case where Δ and Δ' are simple strata. See also the proof in [7, 3.5.9] for the case of split simple strata.

Proof. The proof is done by induction on r. At first we can assume that Δ and Δ' have the same associated splitting, by Corollary 5.17, and we take for both strata the same indexing. We also write k_0 for both critical exponents, which agree by Lemma 5.29. At first let us assume that r is not greater than $\lfloor \frac{-k_0}{2} \rfloor$. Then we have the following equalities

$$H(\Delta(1-)) = H(\Delta)(\prod_i H(\Delta_i(1-))) \text{ and } H(\Delta'(1-)) = H(\Delta')(\prod_i H(\Delta_i'(1-))).$$

From the equality of $\mathrm{C}(\Delta)$ with $\mathrm{C}(\Delta')$ we obtain a block wise equality, i.e. $\mathrm{C}(\Delta_i)$ and $\mathrm{C}(\Delta_i')$ coincide. The already proven simple case, see [4, 4.12], implies that $H(\Delta_i(1-))$ is equal to $H(\Delta_i'(1-))$ for all indexes $i \in I$ and thus we obtain the desired equality of $H(\Delta(1-))$ with $H(\Delta'(1-))$. Secondly, we need to consider the case $r > \lfloor \frac{-k_0}{2} \rfloor$. The proof is the same as in [7, 3.5.9] using the jump sequences, which equal by Lemma 5.36, and core approximations. Let us repeat the argument. We take core approximations $\tilde{\Delta}$ and $\tilde{\Delta}'$ for $\Delta(1-)$ and $\Delta'(1-)$, respectively. We have that the group $H(\Delta(1-))$ and $H^r(\tilde{\beta},\Lambda)$ coincide, by [21, 9.1(ii)] and restriction to G. Then $\tilde{r} > r$ because Δ is semisimple. Thus, the sets of characters $\mathrm{C}(\tilde{\Delta})$ and $\mathrm{C}(\tilde{\Delta}')$ coincide. The closure on both sides of the formulas for the intertwining of a semisimple character followed by an intersection with $\mathfrak{a}_r(\Lambda)$ and an additive closure implies

$$\mathfrak{a}_r(j_{\tilde{E}}(\Lambda)) + S(\tilde{\Delta}) - 1 = \mathfrak{a}_r(j_{\tilde{E}'}(\Lambda)) + S(\tilde{\Delta}') - 1.$$

We add on both sides $\mathfrak{h}(\Delta)$, which is $\mathfrak{h}(\Delta')$, and we obtain

$$\mathfrak{h}(\Delta(1-)) = \mathfrak{h}^r(\tilde{\beta},\Lambda) = \mathfrak{a}_r(j_{\tilde{E}}(\Lambda)) + \mathfrak{h}(\Delta) = \mathfrak{a}_r(j_{\tilde{E}'}(\Lambda)) + \mathfrak{h}(\Delta') = \mathfrak{h}(\Delta'(1-)).$$

This finishes the proof. □

Lemma 5.40. Suppose that $V = \bigoplus_k V^k$ is a splitting which refines the associated splittings of Δ and Δ', suppose that $\mathrm{C}(\Delta(1+))$ is equal to $\mathrm{C}(\Delta'(1+))$ and that $\mathrm{C}(\Delta_k)$ is equal to $\mathrm{C}(\Delta_k')$ for all indexes k. Let $a \in \mathfrak{a}_{-r-1} \cap \prod_k \mathrm{End}_D(V^k)$, $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$ be given such that θ_k coincides with $\theta_k'\psi_{a_k}$ for all indexes k. Then $\mathrm{C}(\Delta) = \mathrm{C}(\Delta')\psi_a$.

Proof. The group $H(\Delta)$ is the same as $H(\Delta')$ by Proposition 5.39. We show that $\Delta'' = [\Lambda, n, r, \beta' + a]$ is equivalent to a semisimple stratum. Then $\mathrm{C}(\Delta'')$ is equal to $\mathrm{C}(\Delta')\psi_a$ and it has a non-trivial intersection with $\mathrm{C}(\Delta)$ and Proposition 5.38 would finish the proof. Now let s' be a tame corestictions with respect to $\beta'(1)$. Then the multi-stratum $[j_{E'(1)}(\Lambda), r+1, r, s'(\beta' + a - \beta'(1))]$ is equivalent to a semisimple multi-stratum, because $[j_{E'(1)}(\Lambda), r+1, r, s'(\beta' - \beta'(1))]$ is and $s'(a)$ is congruent to an element of $\prod_k 1^k E'(1)$ because $\psi_{\beta'(1)_k - \beta'_k}\theta'_k\psi_{-a_k} \in \mathrm{C}(\Delta'(1)(1-)_k)$. Thus, Δ'' is equivalent to a semisimple stratum by Theorem 4.15. □

Corollary 5.41. Suppose that $V = \bigoplus_k V^k$ is a splitting which refines the associated splittings of Δ and Δ', and suppose that there are semisimple characters $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$ such that for every index k the characters θ_k and θ'_k coincide. Then $\mathrm{C}(\Delta)$ is equal to $\mathrm{C}(\Delta')$.

Proof. This follows inductively from Lemma 5.40 with $a = 0$. □

5.7 Matching theorem

In this section we are given two semisimple strata Δ and Δ' with $r = r'$ and $n = n'$ and the same period.

Proposition 5.42. Suppose that Δ and Δ' have the same associated splitting and suppose $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$ intertwine by an element of $(\prod_i A^{ii})^\times$. Then there is a stratum Δ'' with the same splitting as Δ such that $\Lambda'' = \Lambda$, $n'' = n$, $r'' = r$ and $\mathrm{C}(\Delta) = \mathrm{C}(\Delta'')$ and β''_i has the same minimal polynomial as β'_i, and $\theta = \tau_{\Delta',\Delta''}(\theta')$.

Proof. We underline at first that Δ_i and Δ'_i have the same interia degree and the same ramification index by Proposition 5.30. By Corollary 3.9 there are simple strata $\tilde{\Delta}_i$ with $\Lambda^i = \tilde{\Lambda}^i$ and $\tilde{r}_i = r$ such that $\tilde{\beta}_i$ is a conjugate of β'_i and which has the same embedding type as Δ_i. Now, by Proposition 5.34, there is an element g_i of the normalizer of Λ which conjugates $\tau_{\Delta'_i,\tilde{\Delta}_i}(\theta'_i)$ to θ_i. Define Δ'' as the direct sum of the strata $[\Lambda^i, n, r, g_i\tilde{\beta}_i g_i^{-1}]$. Then, $\tau_{\Delta',\Delta''}(\theta')$ and θ have the same block restrictions. And thus, by Corollary 5.41, the set of semisimple characters $\mathrm{C}(\Delta)$ and $\mathrm{C}(\Delta'')$ coincide and θ is the transfer of θ' from Δ' to Δ''. □

Theorem 5.43. (Translation principle, see [19, 3.3] for simple strata) Suppose that $\tilde{\Delta}$ is a semisimple stratum which has the same associated splitting and the same lattice sequence as $\Delta(1)$, such that $\tilde{r} = r - 1$ and $\mathrm{C}(\Delta(1))$ and $\mathrm{C}(\tilde{\Delta})$ coincide. Then there is a semisimple stratum Δ' and an element $u \in (1 + \mathfrak{m}(\tilde{\Delta})) \cap \prod_{\tilde{j}} A^{\tilde{j},\tilde{j}}$ satisfying $\Delta'(1) = u.\tilde{\Delta}$, such that $\mathrm{C}(\Delta')$ is equal to $\mathrm{C}(\Delta)$.

The Theorem 5.43 has been proven for the case when Δ and $\Delta(1)$ are simple in [19, 3.3] and in the split case [21, 9.16].

Proof. This proof follows the concept of [21, 9.16]. For the start let us firstly treat the case where $\Delta(1)$ is simple. By Lemma 4.37 we can always replace $\tilde{\Delta}$ by an equivalent stratum. By the simple version of the translation principle, Theorem [19, 3.3], we can assume that $\Delta(1)(1-)$ and $\tilde{\Delta}(1-)$ give the same set of simple characters. Let us take tame corestrictions s and $\tilde{s}$ with respect to $\beta(1)$ and $\tilde{\beta}$ as in Lemma [19, 3.5]. The derived stratum $\partial_{\beta(1)}(\Delta)$ is equivalent to a semisimple stratum, by Theorem 4.15. Write $c := \beta - \beta(1)$. Thus, by Proposition 4.8 and Lemma [19, 3.5(iii)(iv)], the stratum $[j_{\tilde{E}}(\Lambda), r, r+1, \tilde{s}(c)]$ is equivalent to a semisimple stratum, say with splitting, $(1'^{\prime i})_i$, and therefore, by Theorem 4.15, the stratum $[\Lambda, n, r, \tilde{\beta} + \sum_i 1''^i c 1''^i]$ is equivalent to a semisimple stratum Δ'' with the associated splitting $(1''^i)_i$ and Lemma 4.52 provides an element u of $1 + \mathfrak{m}(\tilde{\Delta})$ which conjugates β'' to $\tilde{\beta} + c$ modulo $\mathfrak{a}_{-r}$. We define Δ' as $u.\Delta''$ and we have

$$\mathrm{C}(\Delta) = \mathrm{C}(\Delta(1)(1-))\psi_c = \mathrm{C}(\tilde{\Delta}(1-))\psi_c = \mathrm{C}(\Delta').$$

Suppose now that $\tilde{\Delta}$ is not simple. We apply the simple case block-wise to obtain a semisimple stratum Δ' and a desired element u such that there is a defining sequence for Δ' with $u.\tilde{\Delta}$ as $\Delta'(1)$, and such that $C(\Delta'_{\tilde{j}})$ and $C(\Delta_{\tilde{j}})$ coincide. We apply Corollary 5.17 block-wise for the simple blocks of $\tilde{\Delta}$ to conjugate the associated splitting of β to the one of β' by an element of $S(\Delta)\cap\prod_{\tilde{j}} A^{\tilde{j},\tilde{j}}$. For a semisimple character θ for Δ and an extension $\theta'\in \mathrm{C}(\Delta')$ of $\theta(1+)$ there is an element $a\in\mathfrak{a}_{-r-1}\cap\prod_{i'}A^{i',i'}$, such that $\theta\psi_a$ coincides with θ'. Now Lemma 5.40 finishes the proof. □

We now state the matching theorem.

Theorem 5.44. Suppose $\theta\in\mathrm{C}(\Delta)$ and $\theta'\in\mathrm{C}(\Delta')$ are intertwining semisimple characters. Then there is a unique bijection $\zeta: I\to I'$ such that

(i) $\dim_D V^i = \dim_D V^{\zeta(i)}$, and

(ii) there is a D-linear isomorphism g_i from V^i to $V^{\zeta(i)}$ such that $g_i.\theta_i$ intertwines with $\theta'_{\zeta(i)}$ by a D-linear automorphism of $V^{\zeta(i)}$,

for all indexes $i\in I$.

The map ζ is called a *matching* for θ and θ' from Δ to Δ'.

Proof. We prove the existence by an induction along $n-r$. If $n=r$ there is nothing to prove, and the case $n=r+1$ follows from the theory of strata, see Proposition 4.41. Suppose now that n is greater than $r+1$. By induction hypothesis we have a matching for $\theta(1+)$ and $\theta'(1+)$ from $\Delta(1)$ to $\Delta'(1)$ and by Proposition 5.42 there is a semisimple stratum $\tilde{\Delta}$ with the same associated splitting as $\Delta'(1)$ and the same set of semisimple characters such that $\tilde{\beta}$ is conjugate to $\beta(1)$ by an element of G which is diagonal with respect to the associated splitting $(\prod_j A^{j,j})^\times$ of $\tilde{\Delta}$, such that $\theta'(1+)$ is the transfer of $\theta(1+)$ from $\Delta(1)$ to $\tilde{\Delta}$. We apply the translation theorem 5.43 to $(\Delta',\Delta'(1))$ and $\tilde{\Delta}$ to reduce to the case where $\beta(1)$ and $\beta'(1)$ are conjugates by an element of $\prod_j A^{j,j}$. Thus we can assume without loss of generality that $\beta(1)$ and $\beta'(1)$ coincide. We apply Lemma 5.21(i) then Theorem 4.41 followed by Lemma 5.21(ii) to obtain a matching between θ and θ'.

We now prove the uniqueness: Because of the existence of a matching ζ we can assume by conjugation that ζ is the identity and $V^i=V'^i$ for all indexes $i\in I$. If there is a second matching then, we obtain by Proposition 5.35 two indexes $i_1,i_2\in I$ such that θ_{i_1} and θ_{i_2} intertwine by an isomorphism from V^{i_1} to V^{i_2}. And thus the simple character $\theta_{i_1}\otimes\theta_{i_2}$ intertwines with $\theta_{i_1}\otimes\theta_{i_1}$. This is a contradiction because two intertwining semisimple characters have the same number of blocks by the existence part. □

Definition 5.45. We call the bijection $\zeta_{\theta,\theta'}: I\to I'$ the matching from $\theta\in\mathrm{C}(\Delta)$ to $\theta'\in\mathrm{C}(\Delta')$.

Corollary 5.46. Suppose that $\theta\in\mathrm{C}(\Delta)$ and $\theta'\in\mathrm{C}(\Delta')$ are transfers then $\bar{\zeta}_{\Delta,\Delta'}=\bar{\zeta}_{\theta,\theta'}$.

Proof. By Proposition 4.41 there is an element $g\in\prod_i \mathrm{End}_D(V^i, V'^{\zeta_{\Delta,\Delta'}(i)})$ which intertwines Δ with Δ' and thus θ with θ', because they are transfers. □

Corollary 5.47. Granted $e(\Lambda|F)=e(\Lambda'|F)$ and $r=r'$, two intertwining characters $\theta\in\mathrm{C}(\Delta)$ and $\theta'\in\mathrm{C}(\Delta')$ have the same group level, the same degree and the same critical exponent. Moreover $e(E_i|F)=e(E_{\zeta(i)}|F)$ and $f(E_i|F)=f(E_{\zeta(i)}|F)$ and $k_0(\Delta_i)=k_0(\Delta'_{\zeta(i)})$ for all indexes i.

Proof. This follows directly from Theorem 5.44 and Theorem 5.30. □

5.8 Semisimple characters of same degree and same group level

At first we generalize the matching to semisimple characters of the same group level and the same degree.

Proposition 5.48. Suppose that $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$ are two intertwining semisimple characters of the same group level and the same degree. Then there is a unique bijection $\zeta : I \to I'$ and a D-linear isomorphism of V such that $gV^i = V'^{\zeta(i)}$ and such that $g.\theta_i$ intertwines with $\theta'_{\zeta(i)}$ for all indexes $i \in I$. Further $e(E_i|F) = e(E_{\zeta(i)}|F)$ and $f(E_i|F) = f(E_{\zeta(i)}|F)$. If Λ and Λ' have the same period then $k_0(\Delta) = k_0(\Delta')$ and $k_0(\Delta_i) = k_0(\Delta'_{\zeta(i)})$ for all $i \in I$.

Proof. By doubling we can assume that Λ and Λ' have the same period. If $r \geq r'$, then θ and $\theta'((r-r')+)$ intertwine. Thus there is a matching for θ and $\theta'((r-r')+) \in \mathrm{C}(\Delta'(r-r'))$ by Theorem 5.44. Thus $E(r-r')$ and E' have the same F-dimension, by Corollary 5.47, and therefore $\Delta'(r-r')$ and Δ' have the same degree. Thus $\Delta'(r-r')$ has the same number of blocks as Δ' and $E'_{i'}|F$ has the same inertia degree and the same ramification index as $E'(r-r')_{i'}|F$, by Proposition 4.21. Thus $e(\Lambda|E) = e(\Lambda'|E')$ and thus $\Delta'((r-r')+)$ is still semisimple because both strata have the same group level. The matching theorem 5.44 provides an element $g \in \prod_i \mathrm{End}_D(V^i, V^{\zeta(i)})$ such that $g.\theta_i$ and $\theta'_{\zeta(i)}((r-r')+)$ intertwine. By Proposition 5.35 the characters $g.\theta_i$ and $\theta'_{\zeta(i)}$ intertwine. The uniqueness comes from the uniqueness of the matching for θ and $\theta'((r-r')+)$. The remaining assertions follow from Corollary 5.47 applied on θ and $\theta'((r-r')+)$. □

Corollary 5.49. Intertwining is an equivalence relation among all semisimple characters of the same group level and the same degree.

Proof. This follows directly from Proposition 5.35 and Proposition 5.48. □

5.9 Intertwining and Conjugacy

In this subsection we fix two semisimple characters $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$ which intertwine. We further assume that $n = n'$ and $r = r'$. The Theorem 5.44 provides a matching $\zeta_{\theta,\theta'} : I \to I'$ from θ to θ'.

Lemma 5.50 (see [19, 3.5] for the simple case)**.** Suppose that $\mathrm{C}(\Delta) = \mathrm{C}(\Delta')$ with $\Lambda = \Lambda'$. Then the residue algebras κ_E and $\kappa_{E'}$ coincide.

Proof. By Corollary 5.17 we can conjugate the associated splittings to each other by an element of $S(\Delta)$ and thus we can assume that both associated splittings are the same. By Theorem [19, 3.5] the residue fields of E_i and E'_i coincide which implies the assertion. □

Lemma 5.51. The conjugation by $g \in I(\theta, \theta')$ induces a canonical isomorphism $\bar{\zeta}$ form the residue algebra of E to the one of E', and the isomorphism does not depend on the choice of the intertwining element.

Proof. By Theorem 5.44 and Proposition 5.42 and Lemma 5.50 we can assume that θ and θ' are transfers. The conjugation with $S(\Delta)$ does fixes κ_E and similar fixes the conjugation with $S(\Delta')$ the algebra $\kappa_{E'}$. Thus we only need to show that the conjugation with an element of $C_{A^\times}(\beta)$ defines a map from κ_E to $\kappa_{E'}$ which does not depend on the chioce of an element of $C_{A^\times}(\beta)$. This is done in Lemma 4.46. □

Notation 5.52. We denote the map of Lemma 4.46 by $\bar{\zeta}$ or $\bar{\zeta}_{\theta,\theta'}$ and call it the matching of the residue algebras from θ to θ'. And we write $(\zeta, \bar{\zeta})$ and call it the matching pair. By definition this depends on $\theta, \theta', n = n', r = r', \Lambda$ and Λ', but not on β or β'. In fact one can show that it only depends on $\theta, \theta', \Lambda, \Lambda'$ and the group level and the degree.

We are now able to formulate and prove the first main theorem of this article about intertwining and conjugacy of semisimple characters.

Theorem 5.53 (1st Main Theorem). Let $(\zeta, \bar{\zeta})$ be the matching pair from θ to θ'. If there is a element of $t \in G$ and an integer s such that $t\Lambda^i_j = \Lambda^{\zeta(i)}_{j+s}$ for all $i \in I$ and all integers j such that the conjugation with t induces the map $\bar{\zeta}|_{\kappa_{E_D}}$ then there is an element g of G such that $g.\theta_i = \theta'_{\zeta(i)}$ and $g^{-1}t \in P(\Lambda)$. In particular $g.\theta$ and θ' coincide.

Proof. We can conjugate θ and Δ with t and obtain the case where $\bar{\zeta}_{E_D}$ is induced by 1. Thus we can assume and $t = 1$. By Corollary 5.41 it is enough to prove the theorem for the simple case. In this case we have to show that θ is conjugate to θ' by an element of $P(\Lambda)$. By Proposition 5.42 we can assume (by changing β') that θ' is the transfer of θ from Δ to Δ'. Then $\bar{\zeta}_{\Delta,\Delta'} = \bar{\zeta} = \mathrm{id}$ by Corollary 5.46. Thus there is an element of $P(\Lambda)$ which conjugates Δ to Δ' by Theorem 4.48 and this element conjugates θ to the transfer, i.e. to θ'. □

6 Endo-classes for GL

In this section we generalize the theory of endo-classes to semisimple characters of general linear groups. The theory of endo-classes for simple characters can be found in [4] and for the split case semisimple endo-classes are fully studied in [12]. The section is structured into four parts. At first we study restriction maps between sets of semisimple characters, secondly we define transfer between for strata on different vector spaces and then between different division algebras, then we define and study semisimple endo-classes and at the end we classify intertwining classes of semisimple characters using so-called endo-parameters which are finite sums of weighted simple endo-classes.

6.1 Restrictions of semisimple characters

Theorem 6.1. Let Δ be a semisimple stratum with associated splitting $\oplus_{i\in I} V^i$ (as usual) and suppose further that Δ is split by $V^0 \oplus V^1$ with a non-zero vector space V^0, say $\Delta = \Delta_0 \oplus \Delta_1$. Then

(i) the restriction map $\mathrm{C}(\Delta) \overset{res_{\Delta,\Delta_0}}{\longrightarrow} \mathrm{C}(\Delta_0)$ is surjective.

(ii) if for all indexes $i \in I$ the intersection of V^i with V^0 is non-zero then res_{Δ,Δ_0} is bijective.

Lemma 6.2 ([4] 2.17(ii),2.10)**.** Under the assumptions of Theorem 6.1 if Δ is simple then res_{Δ,Δ_0} is bijective.

Lemma 6.3 ([12] 13.4)**.** If Δ is a semisimple stratum split by $V^0 \oplus V^1$ and let $\tilde{\theta} \in \mathrm{C}(\Delta(1+))$ and $\theta_1 \in \mathrm{C}(\Delta_0)$ be semisimple characters such that their restrictions to $H(\Delta_0(1+))$ coincide. Then there is a semisimple character $\theta \in \mathrm{C}(\Delta)$ such that $\theta(1+) = \tilde{\theta}$ and with restriction θ_0 on $H(\Delta_0)$.

Proof. The proof goes along an induction on the critical exponent and is completely the same as in [12, 13.4], except with a small change in the case $m < \lfloor \frac{-k_0(\Delta)}{2} \rfloor$ where we use Lemma 6.2 instead of transfers. In fact this is no real change, because Lemma 6.2 was proven by using transfers for simple characters on different vector spaces. □

Proof of Theorem 6.1. (i) The proof uses an induction along r. If $r = n$ then $\mathrm{C}(\Delta)$ and $\mathrm{C}(\Delta_0)$ are singletons. Thus the restriction map is surjective. In the case $r < n$ take a semisimple character $\theta_0 \in \mathrm{C}(\Delta_0)$ and use the induction hypothesis to obtain an extension $\tilde{\theta} \in \mathrm{C}(\Delta(1+))$ of $\theta_0(1+)$. By Lemma 6.3 there exists a common extension of $\tilde{\theta}$ and θ_0 into $\mathrm{C}(\Delta)$. This finishes the proof of the first part.

(ii) Statement (i) provides the surjectivity. For the injectivity consider the following diagram

$$\begin{array}{ccc} \mathrm{C}(\Delta) & \to & \mathrm{C}(\Delta_0) \\ \downarrow & \circlearrowleft & \downarrow \\ \prod_{i\in I} \mathrm{C}(\Delta_i) & \overset{\sim}{\longrightarrow} & \prod_{i\in I} \mathrm{C}(\Delta_{0,i}) \end{array},$$

where the bottom map is a bijection by Lemma 6.2. This proves the injectivity.

□

We need a generalization of the direct sum of strata. For two strata Δ and Δ', we define their direct sum to be the following stratum (write e for $\frac{e(\Lambda|F)}{\gcd(e(\Lambda|F),e(\Lambda'|F))}$ and e' for $\frac{e(\Lambda'|F)}{\gcd(e(\Lambda|F),e(\Lambda'|F))}$):

$$[e'\Lambda \oplus e\Lambda', \max(ne', n'e), \max(re', r'e), \beta \oplus \beta'].$$

Taking this definition into account we have to be a little bit careful with restrictions. Suppose Δ and Δ' are two semisimple strata of the same period and suppose $r = r' + 1$. The restriction of an element $\theta \in \mathrm{C}(\Delta \oplus \Delta')$ to $\mathrm{End}_D(V')$ is an element of $\mathrm{C}(\Delta'(1+))$. Thus we need to extend $\theta|_{H(\Delta'(1+))}$ to $H(\Delta')$. Proposition 5.32 states that this extension is unique if Δ' and $\Delta'(1+)$ have the same group level.

Definition 6.4. Given two semisimple strata Δ and Δ' such that $(\Delta \oplus \Delta')|_V$ has the same group level as Δ, we define the restriction from $\Delta \oplus \Delta'$ to Δ as follows. For simplicity let us assume $e(\Lambda|F) = e(\Lambda'|F)$.

$$res_{\Delta\oplus\Delta',\Delta} := res^{-1}_{\Delta,\Delta(k-r)} \circ res_{\Delta\oplus\Delta',\Delta(k-r)},\ k = \max(r, r')$$

This is the usual restriction followed by extension. We call these maps *external restrictions*, and we call the usual restrictions as *internal restrictions*. Thus every internal restriction can be interpreted as an external one, but not every external restrictions comes from an internal one.

The †-construction has the following influence on external restrictions.

Proposition 6.5. Let Δ_i, $i \in I$ be semisimple strata such that there are two indexes i_1, i_2 such that Δ_{i_1} is equal to Δ_{i_2}. Suppose further that $(\oplus_I \Delta_i)|_{V^{i_1}}$ has the same group level as Δ_{i_1}. Then the maps $res_{\oplus_I\Delta_i,\Delta_{i_1}}$ and $res_{\oplus_I\Delta_i,\Delta_{i_2}}$ coincide.

Proof. At first the restriction $res_{\Delta_{i_1}\oplus\Delta_{i_2},\Delta_{i_1}}$ and $res_{\Delta_{i_1}\oplus\Delta_{i_2},\Delta_{i_2}}$ coincide because they have the same inverse which is the †-map $()^{\dagger,(0,0)}$. The statement follows now from

$$res_{\oplus_I\Delta_i,\Delta_{i_j}} = res_{\Delta_{i_1}\oplus\Delta_{i_2},\Delta_{i_j}} \circ res_{\oplus_I\Delta_i,\Delta_{i_1}\oplus\Delta_{i_2}}$$

for $j = 1, 2$. □

6.2 Generalization of transfer

The idea of transfer is the following. Let us illustrate this roughly for the simple case: Say Δ is a simple stratum with defining sequence $(\Delta(j))_j$. A character $\theta \in \mathrm{C}(\Delta)$ is constructed inductively using the characters $\psi_{\beta(j)}$, determinants and characters χ_j on $E(j)^\times$. Now, given another semisimple stratum Δ' of the same group level as Δ such that β' has the same minimal polynomial as β one just takes the same data χ_j to construct a character θ'. This θ' is called the transfer of θ from Δ to Δ'. This idea does defines a bijection between $\mathrm{C}(\Delta)$ and $\mathrm{C}(\Delta')$. This construction does not need a fixed ambient vector space for both strata. And in this section we are going to generalize it to semisimple strata in a rigorous way.

Definition 6.6. Two semisimple Δ and Δ' (possibly on different vector spaces) which have the same group level and the same degree are called endo-equivalent if there is a bijection ζ from I to I' such that $\Delta_i \oplus \Delta'_{\zeta(i)}$ is equivalent to a simple stratum.

Proposition 6.7. (i) If Δ and Δ' are endo-equivalent then the map ζ from Definition 6.6 is uniquely determined, i.e. there is no other map ζ' from I to I' such that $\mathrm{Res}_F(\Delta_i) \oplus \mathrm{Res}_F(\Delta'_{\zeta'(i)})$ is equivalent to a simple stratum, for all indexes $i \in I$.

(ii) On semisimple strata endo-equivalence is an equivalence relation.

We will denote the map ζ in Definition 6.6 by $\zeta_{\Delta,\Delta'}$.

Proof. This result follows directly from the transitivity of endo-equivalence for simple strata: Corollary 4.31(ii) and from [5, 1.9], using a †-construction. □

6.2.1 Transfer for a fixed division algebra D

Definition 6.8. Given semisimple characters $\theta^j \in \mathrm{C}(\Delta^j)$, $j = 1, \ldots, l$, we define the tensor product $\theta^1 \otimes \ldots \otimes \theta^l$ to be the map from $H(\bigoplus_j \Delta^j)$ to $\mathbb{C}^\times$ which is defined under the Iwahori decomposition of $\bigoplus_j V^j$ via

$$(\otimes_j \theta^j)(u_- x u_+) = \prod_j \theta(x_j).$$

This definition depends on the choice of the strata.

Definition 6.9. Let Δ and Δ' be two endo-equivalent semisimple strata and take a semisimple character $\theta \in \mathrm{C}(\Delta)$. An element $\theta' \in \mathrm{C}(\Delta')$ is called a *(generalized) transfer* of θ from Δ to Δ' if $\theta \otimes \theta' \in \mathrm{C}(\Delta \oplus \Delta')$. It exists and is unique as it is the image of θ under $res_{\Delta\oplus\Delta',\Delta'} \circ res^{-1}_{\Delta\oplus\Delta',\Delta}$.

Proposition 6.10. Fix the assumptions of Definition 6.9. The transfer does only depend on the equivalence classes of Δ and Δ'. If V and V' coincide and Δ and Δ' intertwine then the transfer in Definition 6.9 coincides with the transfer in Definition 5.9.

Proof. Take a semisimple character $\theta \in \mathrm{C}(\Delta)$ and a semisimple stratum Δ'' equivalent to Δ and a semisimple stratum Δ''' equivalent to Δ'. The groups $H(\Delta \oplus \Delta')$ and $H(\Delta'' \oplus \Delta''')$ differ in general, and to avoid confusion we write subscripts on $\theta \otimes \theta'$. We have to show $(\theta \otimes \theta')_{\Delta'',\Delta'''} \in \mathrm{C}(\Delta'' \oplus \Delta''')$. Consider $\tilde{\theta} := \tau_{\Delta\oplus\Delta',\Delta''\oplus\Delta'''}((\theta \otimes \theta')_{\Delta,\Delta'})$. This is a character which decomposes under the Iwahori decomposition with respect to $V \oplus V'$ with restrictions $\tilde{\theta}|_V = \theta$ and $\tilde{\theta}_{V'} = \theta'$. Thus $(\theta \otimes \theta')_{\Delta'',\Delta'''}$ is a character equal to $\tilde{\theta}$ and we have finished the proof of the first assertion. To prove the second assertion we assume that $V = V'$ and that both strata intertwine. We apply $\tau_{\Delta\oplus\Delta,\Delta\oplus\Delta'}$ on $\theta \otimes \theta$ to obtain a semisimple character $\theta \otimes \theta' \in \mathrm{C}(\Delta \oplus \Delta')$ which is intertwined by $I(\Delta \oplus \Delta, \Delta \oplus \Delta')$. In particular θ' is the generalized transfer of θ and they are intertwined by $I(\Delta, \Delta')$, i.e. θ' is also the transfer of θ according to Definition 5.9. ◻

We denote the map $res_{\Delta\oplus\Delta',\Delta'} \circ res^{-1}_{\Delta\oplus\Delta',\Delta}$ by $\tau_{\Delta,\Delta'}$ and call it a *transfer map.*

Proposition 6.11. Transfer is an equivalence relation and external restrictions maps are transfer maps.

Proof. Suppose we are given three endo-equivalent semisimple strata Δ, Δ' and Δ''. Consider the following commutative diagram consisting of external restriction maps, which are here bijections by Proposition 6.1(ii).

$$\begin{array}{ccccc}
\mathrm{C}(\Delta) & \leftarrow & \mathrm{C}(\Delta \oplus \Delta') & \rightarrow & \mathrm{C}(\Delta') \\
\uparrow & \nwarrow & \uparrow & \nearrow & \uparrow \\
\mathrm{C}(\Delta \oplus \Delta'') & \leftarrow & \mathrm{C}(\Delta \oplus \Delta' \oplus \Delta'') & \rightarrow & \mathrm{C}(\Delta' \oplus \Delta'') \\
 & \searrow & \downarrow & \swarrow & \\
 & & \mathrm{C}(\Delta'') & &
\end{array}$$

The commutativity of this diagram proves that $\tau_{\Delta',\Delta''} \circ \tau_{\Delta,\Delta'}$ is equal to $\tau_{\Delta,\Delta''}$, and this finishes the proof of the first assertion.

For the second assertion, given semisimple strata Δ and Δ' the following map:

$$\mathrm{C}(\Delta \oplus \Delta') \xrightarrow{res^{-1}} \mathrm{C}(\Delta \oplus \Delta' \oplus \Delta) \xrightarrow{res} \mathrm{C}(\Delta),$$

is the restriction map, because the restriction map on the right does not depend on the copy of Δ which is used for the restriction, by Proposition 6.5. ◻

Remark 6.12. In [18] the authors extended the notion of transfer to simple strata over D. It is transitive and they proved in *loc.cit.* Theorem 2.17 that internal restrictions are transfer maps, according to their definition of transfer, and it extends to external restriction maps by *loc.cit.* Theorem 2.13. Thus their notion of transfer and the notion of (generalized) transfer in Definition 6.9 coincide, by Proposition 6.11.

6.2.2 Transfers between different Brauer classes

The key idea to move between sets of semisimple characters for different simple algebras over F with possibly different Brauer class is from [16] where he defined the transfer for simple characters by extension of scalers to L followed by the transfer map for the split case. We follow this procedure to generalize it to semisimple strata. But at first let us state an unramified extension theorem for semisimple characters whose proof is exactly as in part (i) of Proposition 5.6.

Proposition 6.13. Let Δ be a semisimple stratum and $\tilde{L}$ be an unramified extension of F then the restriction map from $\mathrm{C}(\Delta \otimes \tilde{L})$ into $\mathrm{C}(\Delta)$ is surjective.

Definition 6.14. Suppose now that Δ and Δ' for D and D', respectively, are two endo-equivalent semisimple strata and take an unramified field extension $\tilde{L}|F$ which splits D and D'. We define the (generalized) transfer map from $\mathrm{C}(\Delta)$ to $\mathrm{C}(\Delta')$ as follows: For $\theta \in \mathrm{C}(\Delta)$ we choose an extension $\theta_{\tilde{L}}$ of θ into $\mathrm{C}(\Delta \otimes \tilde{L})$. We define

$$\tau_{\Delta,\Delta'}(\theta) := \tau_{\Delta\otimes\tilde{L},\Delta'\otimes\tilde{L}}(\theta_{\tilde{L}})|_{H(\Delta')}.$$

Lemma 6.15. The map $\tau_{\Delta,\Delta'}$ is well defined, i.e. it does not depend on the choice of $\theta_{\tilde{L}}$ and the choice of $\tilde{L}$ and it is consistent with the Definition 6.9.

Proof. (i) We start with the consistence to Definition 6.9. Suppose $D = D'$ and take $\theta \in \mathrm{C}(\Delta)$ and an extension $\theta_{\tilde{L}} \in \mathrm{C}(\Delta\otimes\tilde{L})$. The transfer $\theta'_{\tilde{L}}$ of $\theta_{\tilde{L}}$ is the unique semisimple character of $\mathrm{C}(\Delta'\otimes\tilde{L})$ such that $\theta_{\tilde{L}} \otimes \theta'_{\tilde{L}}$ is an element of $\mathrm{C}((\Delta \oplus \Delta') \otimes \tilde{L})$. Thus, $\theta_{\tilde{L}} \otimes \theta_{\tilde{L}}|_{H(\Delta\oplus\Delta')}$ is an element of $\mathrm{C}(\Delta \oplus \Delta')$. Thus, $\tau_{\Delta,\Delta'}(\theta)$ the generalized transfer of Δ in the sense of Definition 6.9. In particular, we see in this case the independence from the choices made.

(ii) By part (i) we have for $\tilde{L} \subseteq \tilde{L}'$ that $\tau_{\Delta\otimes\tilde{L},\Delta'\otimes\tilde{L}}$ can be verified using $\tau_{\Delta\otimes\tilde{L}',\Delta'\otimes\tilde{L}'}$. And thus the definition of $\tau_{\Delta,\Delta'}$ does not depend on the choice of $\tilde{L}$.

(iii) In the case where Δ and Δ' are simple we can use a †-construction to reduce to the simple strict case. The statement is then [16, 3.53], more precisely it proves the independence of the choice of θ_L.

(iv) In the semisimple case we have a reduction to the simple case, by the formula

$$\tau_{\Delta\otimes\tilde{L},\Delta'\otimes\tilde{L}}(\theta_{\tilde{L}}) = \otimes_i \tau_{\Delta_i\otimes\tilde{L},\Delta'_{\zeta(i)}\otimes\tilde{L}}((\theta_{\tilde{L}})_i),$$

□

Thus, this defines the *generalized transfer* of θ from Δ to Δ'. The definition does only depend on the equivalence class of Δ as this is the case for $\tau_{\Delta\otimes\tilde{L},\Delta'\otimes\tilde{L}}$.

6.3 Potentially semisimple characters

Definition 6.16. For an endo-equivalence class $\mathfrak{E}$ of semisimple strata we denote by $\mathrm{C}(\mathfrak{E})$ the class of all semisimple characters θ for which there is a stratum $\Delta \in \mathfrak{E}$ such that $\theta \in \mathrm{C}(\Delta)$. A *potentially semisimple character (pss-character)* is a map Θ from $\mathfrak{E}$ to $\mathrm{C}(\mathfrak{E})$ such that

(i) $\Theta(\Delta)$ is an element of $\mathrm{C}(\Delta)$, for all $\Delta \in \mathfrak{E}$, and

(ii) the values of Θ are related by transfer, for all semisimple strata $\Delta, \Delta' \in \mathfrak{E}$, i.e. $\tau_{\Delta,\Delta'}(\Theta(\Delta))$ is equal to $\Theta(\Delta')$.

We write sometimes $\Theta_{\mathfrak{E}}$ to indicate the domain and we write $\mathfrak{E}(\Delta)$ for the endo-equivalence class of Δ. Two potentially semisimple characters $\Theta_{\mathfrak{E}}$ and $\Theta'_{\mathfrak{E}'}$ of the same group level and the same degree are called *endo-equivalent* if there are semisimple strata $\Delta \in \mathfrak{E}$ and $\Delta' \in \mathfrak{E}'$ such that $\Theta(\Delta)$ and $\Theta'(\Delta')$ intertwine.

The set of pss-characters defined on $\mathfrak{E}(\Delta)$ is in canonical bijection to $\mathrm{C}(\Delta)$. A pss-character $\Theta_{\mathfrak{E}}$ is called *potentially simple* character (ps-character) if $\mathfrak{E}$ is the endo-class of a simple stratum. We call two semisimple characters $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$ of the same group level and the same degree *endo-equivalent* if the corresponding pss-characters are endo-equivalent. The set of endo-classes of semisimple characters is canonically in one-to-one correspondence to the set of end-classes of pss-characters. And we identify these classes and write $\mathcal{E}(\theta)$ for the endo-class of θ.

Definition 6.17 (comparison pairs)**.** Let $\mathfrak{E}$ and $\mathfrak{E}'$ be endo-classes of semisimple characters of the same group level and the same degree. We identify all index sets of semisimple strata in $\mathfrak{E}$ to an index set $I_{\mathfrak{E}}$. Suppose there is a bijection ζ from $I_{\mathfrak{E}}$ to $I_{\mathfrak{E}'}$. We call a pair $(\Delta, \Delta') \in \mathfrak{E}\times\mathfrak{E}'$ a ζ*-comparison pair* if $D = D'$ and for all $i \in I_{\mathfrak{E}}$ the vector spaces V^i and $V'^{\zeta(i)}$ have the same D-dimension.

Theorem 6.18. Suppose $\Theta_{\mathfrak{E}}$ and $\Theta'_{\mathfrak{E}'}$ are two pss-characters. Then the following assertions are equivalent:

(i) $\Theta'_{\mathfrak{E}'}$ and $\Theta_{\mathfrak{E}}$ are endo-equivalent.

(ii) There is a bijection ζ from $I_{\mathfrak{E}}$ to $I_{\mathfrak{E}'}$ such that for every ζ-comparison pair (Δ, Δ') the semisimple characters $\Theta(\Delta)$ and $\Theta'(\Delta')$ intertwine.

By Theorem 5.44 the map ζ is uniquely determined and we call it the *matching* from Θ to Θ'. From Theorem 6.18 follows also that endo-equivalence is an equivalence relation.

Proof. This theorem is known for ps-characters by [4, 1.11] and Proposition 5.35. Suppose the pss-characters $\Theta'_{\mathfrak{E}'}$ and $\Theta_{\mathfrak{E}}$ are endo-equivalent, then there are semisimple strata $\Delta \in \mathfrak{E}$ and $\Delta' \in \mathfrak{E}'$ such that $\Theta(\Delta)$ and $\Theta'(\Delta')$ intertwine. Then there is a bijection ζ from $I_{\mathfrak{E}}$ to $I_{\mathfrak{E}'}$ such that $\Theta(\Delta)_i$ and $\Theta'(\Delta')_{\zeta(i)}$ intertwine. Thus for every $i \in I_{\mathfrak{E}}$ the ps-characters Θ_i and $\Theta'_{\zeta(i)}$ corresponding to $\Theta(\Delta)_i$ and $\Theta'(\Delta')_{\zeta(i)}$ are endo-equivalent and thus the case of ps-characters finishes the proof. □

7 Endo-parameters

We classify the intertwining classes of semisimple characters of G using endo-classes and numerical invariants. This data is called an endo-parameter. A semisimple character is called *full* if $r = 0$, similar we define full pss-characters. An endo-class of a ps-character is also called a *simple* endo-class. The *degree* $\deg(c)$ of a simple endo-class c is the degree of the field extension $E|F$, for a (or equivalently any) simple stratum Δ in the domain of some ps-character in c. We call the sum $\deg(f) := \sum_c f(c)\deg(c)$ the *degree* of an endo-parameter f.

Definition 7.1. An *endo-parameter* is a map f from the set $\mathcal{E}$ of endo-classes of full ps-characters to the set of non-negative integers with finite support. It is called an endo-parameter for G if

(i) the degree of f is equal to the degree of A, i.e. $\dim_D V \deg(D)$,

(ii) and $f(c)$ is divisible by $\frac{\deg(D)}{\gcd(\deg(c),\deg(D))}$.

The second main theorem of this article is

Theorem 7.2 (**2nd Main Theorem**)**.** There is a canonical bijection of the set of intertwining classes of full semisimple characters of G and the set of endo-parameters for G. The map has the following form: The intertwining class of a full semisimple character θ is mapped to the endo-parameter f_θ which is supported on the set of the endo-classes corresponding to the θ_i. The value of f_θ at $\mathcal{E}(\theta_i)$ is defined to be the degree of $\mathrm{End}_{E_i\otimes_F D}(V^i)$ over E_i, i.e. it is the square root of the E_i-dimension of $\mathrm{End}_{E_i\otimes_F D}(V^i)$.

The proof needs a proposition which shows that one can add up any two given semisimple characters to obtain a new semisimple characters.

Proposition 7.3 ([12, 13.5]). Let $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$ be two semisimple characters with the same period and $r = r'$. Then there is a semisimple stratum Δ'' split by $V \oplus V'$, such that $\Delta''|_V = \Delta$, $\mathrm{C}(\Delta''|_{V'}) = \mathrm{C}(\Delta')$, and $\theta \otimes \theta'$ is an element of $\mathrm{C}(\Delta'')$.

Proof. The proof is literally the same as in [12, 13.5]. For the translation principle one takes Theorem 5.43 and instead of Lemma [12, 13.4] we take Lemma 6.3. □

Proof of Theorem 7.2. We have to show that f_θ and $f_{\theta'}$ coincide for intertwining full semisimple characters θ and θ'. The characters θ_i and $\theta'_{\zeta_{\theta,\theta'}(i)}$ intertwine by a D-linear isomorphism g_i from V^i to $V^{\zeta_{\theta,\theta'}(i)}$. Thus θ_i and $\theta_{\zeta_{\theta,\theta'}(i)}$ define the same endo-class and therefore f_θ and $f_{\theta'}$ have the same support. The vector space V^i and $V^{\zeta_{\theta,\theta'}(i)}$ have the same dimension over D, and E_i and $E'_{\zeta_{\theta,\theta'}(i)}$ have the same degree over F and the same inertia degree by Corollary 5.48. Thus both f_θ and $f_{\theta'}$ coincide.

Surjectivity: Let us now take an endo-parameter f for G. For every simple endo-class c' in the support of f, we pick a full simple character $\theta_{c'}$ which occurs in the image of one of the ps-characters in c'. Proposition 7.3 provides a semisimple character $\theta \in \mathrm{C}(\Delta)$ with restrictions $\theta_{c'}$, $c' \in \mathrm{supp}(f)$. To get a semisimple character for G we need to find an appropriate lattice sequence. This is done as follows: Let $M_{c'}$ is a minimal $E_{c'} \otimes D$-module. It has dimension $\frac{\deg(c')\dim_F D}{\gcd(\deg(c'),\deg(D))}$ over F. Now take a vertex $\Gamma_{c'}$ in the Bruhat–Tits building of $C_{\mathrm{End}_D(M_{c'})}(E_{c'})$ and apply the inverse of $j_{E_{c'}}$ to obtain an $o_{E_{c'}}$-o_D-lattice sequence $\Lambda_{c'}$ in $M_{c'}$. The vector space V is as a D-vector space isomorphic to the direct sum of $M_{c'}$s, where $M_{c'}$ occurs with multiplicity $f(c')\frac{\gcd(\deg(c'),\deg(D))}{\deg(D)}$. Let Δ_f be the direct sum of the simple strata $[\Lambda_{c'}, n_{c'}, 0, \beta_{c'}]$ over $c' \in \mathrm{supp}(f)$. The stratum Δ_f is semisimple by a small argument using Corollary 4.31(ii), because Δ is semisimple. We take the transfer θ_f of θ from Δ to Δ_f. Then $f = f_{\theta_f}$ by construction, because

$$f_{\theta_f}(c_i)\deg(c_i) = \dim_D V^{c_i}\deg(D) = f(c_i)\gcd(\deg(c_i),\deg(D))\dim_D M_{c_i},$$

and M_{c_i} has dimension $\frac{\deg(c_i)}{\gcd(\deg(c_i),\deg(D))}$ over D as a simple $E_{c_i} \otimes_F D$-module. Thus $f_{\theta_f}(c_i) = f(c_i)$.

Injectivity: Take two full semisimple characters $\theta \in \mathrm{C}(\Delta)$ and $\theta' \in \mathrm{C}(\Delta')$ such that $f_\theta = f_{\theta'}$. Then there is a bijection $\zeta : I \to I'$ such that θ_i and $\theta_{\zeta(i)}$ define the same endo-class. The dimensions of V^i and $V^{\zeta(i)}$ coincide because $f_\theta(\mathcal{E}(\theta_i)) = f_{\theta'}(\mathcal{E}(\theta'_{\zeta(i)}))$. Take an isomorphism g_i between them. Thus $(g_i.\Delta_i, \Delta'_{\zeta(i)})$ is a-comparison pair for the ps-characters attached to θ and θ'. Thus $g_i.\theta_i$ and $\theta'_{\zeta(i)}$ intertwine by Theorem 6.18. Thus θ and θ' intertwine. □

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