# ADVANCED ALGEBRA, FALL 2026
# PROBLEM SHEET 5

PROF. DANIEL SKODLERACK

**Problem 1** (10+5+5+10, radical of an ideal)**.** Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}_i, i \in I$, be ideals in a ring $A$. Prove the following assertions:

(i) $\sqrt{\mathfrak{a}+\mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}}+\sqrt{\mathfrak{b}}}$, $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$

(ii) $\sqrt{\bigcap_{i\in I}\mathfrak{a}_i} = \bigcap_{i\in I}\sqrt{\mathfrak{a}_i}$

(iii) Suppose all the $\mathfrak{a}_i$ are prime ideals, then $\bigcap_{i\in I}\mathfrak{a}_i$ is a radical ideal.

(iv) Prove that $\sqrt{\mathfrak{a}}$ is the intersection of all prime ideals $\mathfrak{p}$ such that $\mathfrak{p} \supseteq \mathfrak{a}$. *Hint: For the more difficult direction Take an element $x \notin \sqrt{a}$ and a radical ideal $\mathfrak{p}$ which is maximal with the property that it doesn't contain $x$. Show that $\mathfrak{p}$ is a prime ideal.*

**Problem 2** (10, Hilbert's Nullstellensatz for infinitely many variables(?))**.** Let $k$ be an algebraically closed field and $\mathfrak{a}$ be an ideal of $k[X_1, X_2, X_3, ...]$ (countable many variables) such that $1 \notin \mathfrak{a}$. Is there a sequence $x = (x_i)_{i\in\mathbb{N}}$, $x_i \in k$, such that $x$ is a root of $\mathfrak{a}$? If yes give a proof, otherwise give a counter example, i.e. an algebraically closed field $k$ and a proper ideal which has no zero.

**Problem 3** (10, field of definition)**.** Is the following set an algebraic set in $\mathbb{C}^3$?

$$V = \{(t^2, s^3, st) |\ s, t \in \mathbb{C}\}$$

If yes, find the smallest field $k$ such that $V$ is $k$-algebraic. And then find the ideal in $k[X, Y, Z]$ associated to $V$.

*Date*: Please hand in your solution before the lecture on Tuesday, **November 3rd 2026**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference.