ADVANCED ALGEBRA, FALL 2026
PROBLEM SHEET 4

PROF. DANIEL SKODLERACK

Problem 1 (10, quotient field)**.** Let x be an element transcendental over $\mathbb{Q}$. For which polynomials $f(X) \in \mathbb{Q}[X]$ is the field $\mathbb{Q}(f(x))$ equal to $\mathbb{Q}(x)$?

Problem 2 (10+10, Noether basis)**.** Let F be a field such that -1 is not a square in F. Consider the polynomial $f(X,Y) = X^2 + Y^2$. Let $\mathfrak{a}$ be the ideal in $\mathbb{Q}[X,Y]$ generated by f.

(i) Prove that f is irreducible.
(ii) Find a Noether basis of $B := F[X,Y]/\mathfrak{a}$ over F

Problem 3 (10+10, Hilbert's Nullstellensatz)**.** Let F be a field with algebraic closure L. Consider the polynomial ring $B_F = F[X_1, \dots, X_m]$. For a subset S of B_L, we write

$$V(S) = \{x \in F^m | \ \forall_{f \in S} f(x) = 0\}$$

for the set of zeros of S, called the algebraic set of S.

(i) Let $\mathfrak{a}$ be an ideal in B_F, such that $1 \notin \mathfrak{a}$. Prove that $V(\mathfrak{a})$ is non-empty and $\mathrm{Aut}(L|F)$-invariant.
(ii) Let W be an $\mathrm{Aut}(L|F)$-invariant subset of L^m such that $W = V(S)$ for some subset S of B_L. Is there an ideal $\mathfrak{a}$ of B_F such that $V(\mathfrak{a}) = W$?

Problem 4 (10, algebraically independent set)**.** Prove Remark 82 from the lecture notes, see Page 88.

Date: Please hand in your solution before the lecture on Tuesday, **October 27th 2026**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference.