

ADVANCED ALGEBRA, FALL 2026
PROBLEM SHEET 3

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Problem 1 (10+10, valuation ring in a function field). Let F be a field and $K = F(X)$ be the quotient field of the polynomial ring $F[X]$.

- (i) Find the algebraic closure of F in K .
- (ii) Let $b \in K$ be transcendental over F . Find a valuation ring B of K which does not contain b .

Problem 2 (10, extension theorem 72). Prove the modification of Theorem 72, where we do not assume $\ker(\phi)$ to be maximal.

Problem 3 (10*+10, valuation rings described by norms, Part 1). Let K be a field and $|\cdot| : K \rightarrow \mathbb{R}^{\geq 0}$ be a norm on K , i.e. the map is definite, multiplicative and satisfies the triangle inequality. Let $\mathcal{O}_K$ be the set of all elements of K with norm ≤ 1 . Prove that the following assertions are equivalent.

- (i) $\mathcal{O}_K$ is a valuation ring of K .
- (ii) $|\cdot|$ satisfies the ultrametric inequality, i.e.

$$|x + y| \leq \max\{|x|, |y|\}, \quad \forall x, y \in K.$$

(We say that $|\cdot|$ is non-Archimedean.)

Problem 4 (10, valuation rings described by norms, Part 2). Can every valuation ring of a field K be described by some non-Archimedean norm $|\cdot| : K \rightarrow \mathbb{R}^{\geq 0}$? If yes, give a proof. If no, then give a counter example (B, K) .

Problem 5 (10, proper valuation rings). Is there a field K with two proper valuation rings B_1, B_2 such that $B_1 \subsetneq B_2$?