ADVANCED ALGEBRA, FALL 2026
PROBLEM SHEET 1

PROF. DANIEL SKODLERACK

Problem 1 (10, compositum)**.** Suppose that two fields L and K are subfields of a field E. Let R be the set of all sums of the form

$$\Sigma_{i=1}^{t} k_i l_i,$$

where t is a positive integer, $k_i \in K$ and $l_i \in L$.

(i) Suppose that every element of K is algebraic over L. Prove that the compositum KL is equal to R.

(ii) Find two fields K and L contained in a common field such that the compositum KL is not equal to R.

Problem 2 (10, computation of a Galois group)**.** Compute the Galois group of $\mathbb{Q}[2^{\frac{1}{3}}, \xi]|\mathbb{Q}$ where ξ is a primitive 9^{th} root of unity in $\mathbb{C}$.

Problem 3 (10, cyclotomic field extension)**.** Prove that the nth cyclotomic field extension over $\mathbb{Q}$ has degree $\varphi(n)$ (Euler's totient function).

Problem 4 (10, existence of an algebraic closure of a field)**.** Let F be a field.

(i) Show that there is an algebraic field extension $E|F$ such that every non-constant polynomial over F has a root in E.

(ii) Show that there is an algebraically closed field E containing F such that $E|F$ is algebraic.

Date: Please hand in your solution before the lecture on Tuesday, **September 29$^{\text{th}}$ 2026**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference.