VI 4. The cup product.

Let $(R, +, \cdot)$ be a ring.
Note that $(R, +)$ is an abelian group, so it plays the role of G, and $(R, \cdot)$ is just a semigroup.

Rk 208: (a) In addition we have the two distributivity laws
$$(r+s)t = rt + st \text{ and}$$
$$t(r+s) = tr + ts.$$

(b) If $(R, \cdot)$ is commutative then $(R, +, \cdot)$ is called a commutative ring

(c) If R has an elt. 1 s.t.
$$1r = r1 = r \ \forall r \in R$$
we call R a unital ring.

Ex 209: $(\mathbb{Z}, +, \cdot)$ is a unital commutative ring

Other examples
$(\mathbb{Q}, +, \cdot)$, $(\mathbb{Z}/_m, +, \cdot)$

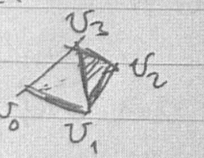
$(M_2(\mathbb{R}) = \mathbb{R}^{2\times 2}, +, \circ)$ is
a unital ring, but not
a commutative ring.

__Def 210:__ (cup product)
$k, l \geq 0$:

$$C^k(X;R) \times C^l(X;R) \xrightarrow{\cup} C^{k+l}(X;R)$$
$$(\varphi, \psi) \longmapsto \varphi \cup \psi$$

$$(\varphi \cup \psi)(\sigma) := \varphi(\sigma|_{[v_0,\dots,v_k]})\, \psi(\sigma|_{[v_k,\dots,v_{k+l}]})$$

Picture:



$\xrightarrow{\sigma} X$

By the following lemma $\cup$ induces to

$$H^k(X;R) \times H^l(X;R) \longrightarrow H^{l+k}(X;R)$$

__Lem 211:__ $\delta(\varphi \cup \psi) = (\delta\varphi) \cup \psi + (-1)^k \varphi \cup \delta\psi$

__Proof:__ This is done by direct computation.

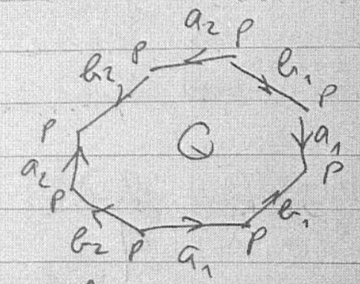
Take $\sigma : \Delta^{k+l+1} \longrightarrow X$.

$$\delta(\varphi \cup \psi)(\sigma) = (\varphi \cup \psi)(\partial\sigma)$$
$$= \sum_{i=0}^{k+l+1} (-1)^i (\varphi \cup \psi)(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+l+1}]})$$
$$= \sum_{i=0}^{k} (-1)^i \varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+1}]})\, \psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$$
$$+ \sum_{i=k+1}^{k+l+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,v_k]})\, \psi(\sigma|_{[v_k,\dots,\hat{v}_i,\dots,v_{k+l+1}]})$$
$$\left\{ \begin{array}{l} + (-1)^{k+1} \varphi(\sigma|_{[v_0,\dots,v_k]})\, \psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]}) \\ + (-1)^k \varphi(\sigma|_{[v_0,\dots,v_k]})\, \psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]}) \end{array} \right.$$
$$= (\delta\varphi \cup \psi) + (-1)^k (\varphi \cup \delta\psi) \qquad \square$$

Example 2.12: M_g, $g \geq 1$, $R = \mathbb{Z}$

Let us first compute the cohomology groups.

$(g=2)$



cellular chain complex

$$0 \to \underset{\deg 2}{\mathbb{Z}} \xrightarrow[\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}]{} \underset{\deg 1}{\mathbb{Z}^{2g}} \xrightarrow{0} \underset{\deg 0}{\mathbb{Z}} \longrightarrow 0$$

$$\Rightarrow \quad H_i(M_g) \simeq \begin{cases} \mathbb{Z}, & i = 0, 2 \\ \mathbb{Z}^{2g}, & i = 1 \\ 0, & i \geq 3. \end{cases}$$

are all free abelian groups.

$$\text{UCT} \Rightarrow \quad H^i(M_g) \overset{h}{\simeq} \mathrm{Hom}_{\mathbb{Z}}(H_i(M_g), \mathbb{Z})$$

$$\simeq \begin{cases} \mathbb{Z}, & i = 0, 2 \\ \mathbb{Z}^{2g}, & i = 1 \\ 0, & i \geq 3. \end{cases}$$

$$H_1(M_g) = \bigoplus_{i=1}^{g} (\mathbb{Z} a_i \oplus \mathbb{Z} b_i)$$

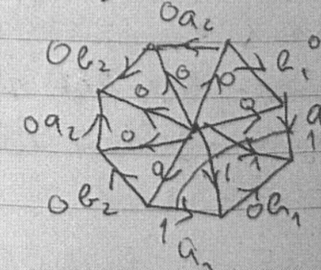
As a basis of $H^1(M_g)$ we take the basis $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g\}$ dual to $\{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\}$.

Goal: Compute $\alpha_i \cup \alpha_j$ and $\alpha_i \cup \beta_j$.

We need to find representatives φ_i, ψ_j of α_i, β_j, i.e.

$$\alpha_i = [\varphi_i] \in H^1(M_g)$$
$$\beta_j = [\psi_j] \in$$

We choose a Δ-structure and use the geometry of $H^1_\Delta(M_g)$.

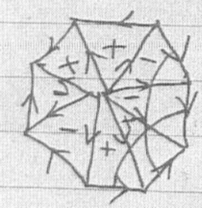


dim	0	1	2
# cells	1	3g	2g

We need to find φ_1 s.t.
$\varphi_1(a_1) = 1$ and $\varphi_1(a_j) = 0$ $\varphi_1(b_\ell)$
for $j \geq 2$ and $\ell \geq 1$.
To get a cocycle we draw
a closed curve crossing a_1,
see Example 186.

See the blue curve. The
blue numbers are the values
of φ_1.

For ψ_1 we draw a red curve.



$[\tilde{c}] = c \in H_2(M_g)$ is indicated
by the sign, i.e.
c is represented by the signed
sum of $2g$ singular triangles.

$\alpha_1 \cup \beta_1 \in H^2(M_g) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_2(M_g), \mathbb{Z}).$
$\shortparallel$
$[\varphi_1 \cup \psi_1] \longmapsto \left([\tilde{c}] \mapsto (\varphi_1 \cup \psi_1)(\tilde{c})\right)$

$(\varphi_1 \cup \psi_1)(\tilde{c}) = (\varphi_1 \cup \psi_1)\left(\quad \right)$

other terms
give 0

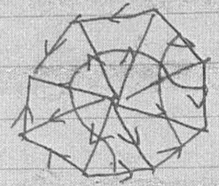
$= \varphi_1\left(\quad \right) \cdot \psi_1\left(\quad \right) = 1 \cdot 1 = 1$

Thus c must be a generator of
$H_2(M_g) \simeq \mathbb{Z}$, because otherwise

c would generate a subgroup $m\mathbb{Z}$
with $m > 1$ and then
$m \mid (\varphi_1 \cup \psi_1)(\tilde{c})$.

Further
$$H^2(M_g) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_2(M_g), \mathbb{Z}) \simeq \mathbb{Z} \qquad f \longmapsto f(c)$$
$$[\varphi_1 \cup \psi_1] \longmapsto 1$$
So $[\varphi_1 \cup \psi_1] = \alpha_1 \cup \beta_1$ is a generator of
$H^2(M_g)$.

<u>Exercise</u>: Compute $\varphi \cup \psi$ for



, i.e. we take another representative of α_1 and one is asked to perform the direct computation of $(\varphi \cup \psi)(\tilde{\tau})$.

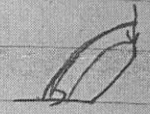
A similar argument gives

$$\alpha_i \cup \beta_j = \begin{cases} \gamma, & i = j \\ 0, & i \neq j \end{cases} = -(\beta_j \cup \alpha_i)$$

where γ is the generator of $H^2(M_g)$ satisfying $\gamma(\tau) = 1$.

We get a non-zero $\cup$-product exactly when the curves inter-sect.

So $\alpha_1 \cup \alpha_1 = 0$, as we can consider



<u>Theorem 2.13</u>: Suppose R is commutative, $\alpha \in H^k(X;R)$ and $\beta \in H^\ell(X;R)$. Then

$$\alpha \cup \beta = (-1)^{k\ell} \beta \cup \alpha.$$

<u>Proof</u>: Consider $rev: C_\cdot(X) \to C_\cdot(X)$

$$rev(\sigma) := \varepsilon_k \bar{\sigma} \quad (\sigma \text{ being } k\text{-simplex})$$

with $\bar{\sigma} = \sigma|_{[v_k, \dots, v_0]}$ and

$$\varepsilon_k = (-1)^{k(k+1)/2}$$

<u>Claim</u>: rev is a chain map homotopic to id.

We prove the claim later.

Suppose the claim is true.

Take $\varphi \in C^k(X;R)$, $\psi \in C^l(X;R)$

Then, for a sing. $k+l$ simplex $\sigma : \Delta^{k+l} \longrightarrow X$, we get.

$$rev^*(\varphi \cup \psi)(\sigma) = \varepsilon_{k+l}\, \varphi\left(\sigma|_{[k+l,\dots,l]}\right) \cdot \psi\left(\sigma|_{[l,\dots,0]}\right)$$

and

$$\left((rev^*\psi) \cup (rev^*\varphi)\right)(\sigma)$$

$$= \varepsilon_k \varepsilon_l\, \psi\left(\sigma|_{[v_l,\dots,v_0]}\right) \varphi\left(\sigma|_{[v_{l+k},\dots,v_l]}\right)$$

and $\varepsilon_{k+l} = (-1)^{kl} \varepsilon_k \varepsilon_l$ as

$$(k+l)(k+l+1) = k^2 + l^2 + 2kl + k + l$$
$$= k(k+1) + l(l+1) + 2kl.$$

Inducing to cohomology we get for classes $\alpha \in H^k(X;R)$, $\beta \in H^l(X;R)$:

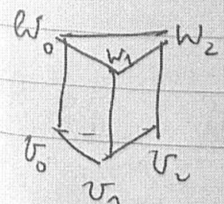
$$\begin{array}{ccc} rev^*(\alpha \cup \beta) & = & (-1)^{kl}\left(rev^*(\beta) \cup rev^*(\alpha)\right) \\ \text{claim} \| & & \| \text{claim} \\ \alpha \cup \beta & & (-1)^{kl}(\beta \cup \alpha) \end{array}$$

<u>Proof of the claim:</u>

- The chain map property $\partial \circ rev = rev \circ \partial$ is an exercise

- As for the homotopy theorem we use the prism operator.

$\Delta^n \times I$ with vertices $v_0, \dots, v_n$ on the bottom and $w_0, \dots, w_n$ on the top.



We define

$$P : C_n(X) \longrightarrow C_{n+1}(X) \quad \text{via}$$

$$P(\sigma) := \sum_{i=} (-1)^i (\sigma \circ \pi)\Big|_{[v_0,\dots,v_i,w_n,\dots,w_i]}$$

$\pi : \Delta \times I \to \Delta$
projection along I

Then $(\partial P + P\partial)(\sigma) = \mathrm{rev}(\sigma) - \sigma$.

(The computations are similar to the — " — for the usual prism operator, but we get $\mathrm{rev}(\sigma)$ as we have reversed the order of the vertices w_i.)

$\square$

VII Poincaré Duality

R is a commutative unital ring.

Theorem 214: Let M be a closed R-orientable n-manifold. Then there is a canonical isomorphism

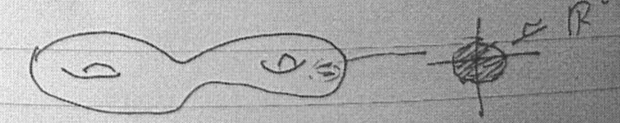
$$H^k(M;R) \xrightarrow{D} H_{n-k}(M;R)$$

for all $k \in \mathbb{Z}$.

(Poincaré Duality)

Def 215: (n-manifold) ($n \geq 0$)

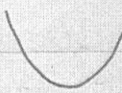
(a) A Hausdorff space M s.t. around each point there is a nbhd $\cong \mathbb{R}^n$ is called n-manifold.



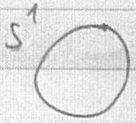
(b) An n-manifold is called

<u>closed</u> if it is compact.

<u>Eg.</u>:  and

 $y = x^2$ in $\mathbb{R}^2$

are not closed.

- S^1  is a closed 1-mf.

as well as S^n and $\mathbb{RP}^n$ are closed n-mf.

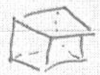
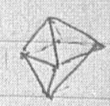
- M_g and N_g are closed 2-mfs.
If you remove a closed disc then you get some non-closed mf.

VII 1. First idea of Poincaré duality

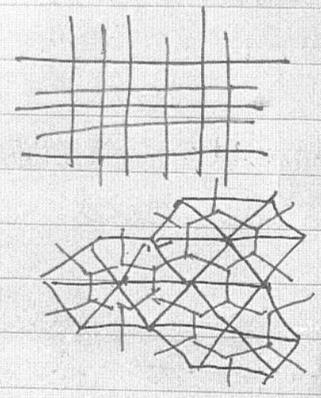
$X = S^2$ or M_g. $(g \geq 1)$

Give S^2 a cell-structure given by a Platonic body. There is a dual cell-structure.

CW		CW'
2-cell	$\xmapsto{\text{baryc.}}$	0-cells
1-cell	$\longmapsto$ edge between barycentres of adj. 2-cells	1-cells
0-cell	$\longmapsto$ connected comp. after removing the dual 1-skeleton	2-cells.

For S^2 cube  octahedron 

general



 for T^2

This gives a bijection

$$C_i^{CW}(X) \xrightarrow{\sim} C_{2-i}^{CW'}(X) \xrightarrow{\sim} C_{CW'}^{2-i}(X)$$

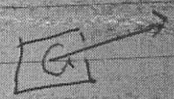
dual basis in case of finitely many cells

$$\begin{tikzcd}
0 \arrow[r] & C_2^{CW}(X) \arrow[r, "\partial_2"] \arrow[d, "\cong"] & C_1^{CW}(X) \arrow[r, "\partial_1"] \arrow[d, "\cong"] & C_0^{CW}(X) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & C^0_{CW'}(X) \arrow[r, "\delta^0"] & C^1_{CW'}(X) \arrow[r, "\delta^1"] & C^2_{CW'}(X) \arrow[r] & 0
\end{tikzcd}$$

<u>Note</u>: For getting a cell-structure we need to choose orientations on the cells. This is needed to get the differentials.

We do this as follows:

- 2-cells Take the outside normal vector-field to orientate the 2-cells

 (RHR)

- For each 1-cell choose an orientation on the dual cells by rotation by $90°$.

Ex 216: Then $\delta^i \triangleq \pm \partial_{2-i}$.

Thus $H^i(X) \simeq H_{2-i}(X)$.

To get a first glimpse, try to understand the case of $\mathbb{Z}/2$ coefficients.

So we have to go through the following steps.

- orientation and fundamental class
- cap product
- cohomology with compact supports and M-V-sequences
- Proof of PD theorem (the version for non-closed and closed mf.

$$H_c^i(M;R) \simeq H_{n-i}(M;R).$$

VIII 2. Orientation and fundamental class.

Notation 217: We will use the following notation for relative homology and cohomology groups.

$$H_i(X|A;G) := H_i(X, X \setminus A; G)$$

(classes represented by "relative cycles which intersect A")

$$H^i(X|A;G) := H^i(X, X \setminus A; G)$$

(— " — by "cocycles which vanish outside A".
If A is closed we would say "cocycles with support in A")

<u>Example 218:</u> M n-mf and $x \in M$

$$H_i(M \mid x; G) \underset{\substack{\uparrow \\ \text{excision}}}{\cong} H_i(\mathbb{R}^n \mid x; G)$$

$$\overset{(*)}{\cong} H_i(\mathbb{R}^n \mid B; G)$$

B bounded open ball around x ($\mathbb{R}^n \setminus B$ is a def. retract of $\mathbb{R}^n \setminus \{x\}$)

$$\cong \tilde{H}_i\left(\mathbb{R}^n / \mathbb{R}^n \setminus B ; G \right)$$

$$\cong \tilde{H}_i(S^n; G)$$

$$\cong \begin{cases} 0, & i \neq n \\ G, & i = n. \end{cases}$$

<u>Remark 219:</u> To see $\overset{(*)}{\cong}$ needs several steps.

1st l.e.s for $(\mathbb{R}^n \setminus \{x\}, \mathbb{R}^n \setminus B)$

$$\hookrightarrow H_i(\mathbb{R}^n \setminus B; G) \underset{\substack{\uparrow \\ \text{def. retract}}}{\cong} H_i(\mathbb{R}^n \setminus \{x\}; G) \to H_i(\mathbb{R}^n \setminus \{x\}, \mathbb{R}^n \setminus B; G)$$

So each $H_i(\mathbb{R}^n \setminus \{x\}, \mathbb{R}^n \setminus B; G) = 0$

2nd l.e.s. for $(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}, \mathbb{R}^n \setminus B)$

$$\hookrightarrow \underset{\substack{\| \\ 0}}{H_i(\mathbb{R}^n \setminus \{x\}, \mathbb{R}^n \setminus B; G)} \to H_i(\mathbb{R}^n, \mathbb{R}^n \setminus B; G) \overset{(*)}{\cong} H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}; G) \to$$

So we get the isomorphism $(*)$.

<u>Def 220:</u> (a) Let M be an n-mf. A <u>local orientation</u> at $x \in M$ is a generator μ_x of $H_n(M \mid x)$.

(b) A <u>local R-orientation</u> at x is a generator (as an R-module) of $H_n(M \mid x; R)$.

(c) An orientation of M is a map $\mu = (\mu_x)_{x \in M}$ with $\mu_x \in H_n(M \mid x)$ a local orientation, s.t. $\forall x_0 \in M$ $\exists$ open ball U around x_0 (bounded) $\exists \mu_U \in H_n(M \mid U)$

p.t.

$$H_n(M|U) \xrightarrow{\sim} H_n(M|x)$$
$$\mu_U \longmapsto \mu_x$$

for all $x \in U$.

(d) Similarly we define an R-orientation.

<u>Rk 221</u>: (orientation manifold)

(a) $\tilde{M} := \{(\nu, x) \mid x \in M,\ \nu$ is a local orientation of M at $x\}$

charts: Take U a bounded open ball of M and $\mu_U \in H_n(M|U)$.

$U(\mu_U) := \{(\nu, x) \mid x \in U,$ ν a local orientation of M at x s.t. ν is the image of μ_U under $H_n(M|U) \to H_n(M|x)\}$

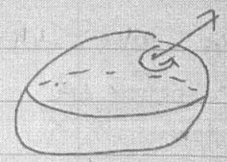
<u>Exercise</u>: $\tilde{M}$ is an n-manifold and $\tilde{M} \xrightarrow{p} M$ is
$(\nu, x) \longmapsto x$
a two sheeted covering space of M.

(b) <u>Then</u> $\mu = (\mu_x)_M$ is an orientation of M iff μ is a continuous section of p, i.e.
$\mu : M \longrightarrow \tilde{M}$ satisfies
μ is continuous and
$p \circ \mu = id_M$.

<u>Example 222</u>: $S^n \simeq \partial\Delta^n \ni [v_0, v_1, \dots, v_n]$
$\subseteq \mathbb{R}^n$. (top. boundary)

Here consider $v_0 = 0,\ v_1 = e_1, \dots, v_n = e_n$

Take $x \in \partial\Delta^n$. Then the class of ∂id_{Δ^n} in $H_n(\partial\Delta^n|x)$ is the orientation given by the outside pointing normal vector field.



Thm 223 Let M be a closed connected n-mf.

(a) Then are equivalent

1° M is R-orientable

2° $\forall x \in M: R \cong H_n(M|x;R) \longleftarrow H_n(M;R)$ is an R-isomorphism.

(b) $\forall i > n: \quad H_i(M;R) = 0.$

(c) Suppose M is not R-orientable. Then $\forall x \in M$

$$H_n(M;R) \hookrightarrow H(M|x;R) \cong R$$

with image

$$\{ r \in R \mid 2r = 0 \}.$$

Pf: The following lemma for $A = M$. □

Def 224: A class $\mu \in H_n(M;R)$ s.t. $\mu|_x \in H_n(M|x;R)$ is a generator for all $x \in M$. is called a *fundamental class for M with coefficients in R*.

Lem 225: Let $M_R = \{(\nu, x) \mid x \in M, \nu \in H_n(M|x;R)\}$ with the usual mf. structure and $A \subseteq M$ be compact. Then

(a) Let $x \longmapsto \alpha_x$ be a section of $M_R \to M$ on M. Then $\exists\, \alpha_A \in H_n(M|A;R)$ such that $\alpha_A|_x = \alpha_x \ \forall x \in A$.

(b) $H_i(M|A;R) = 0 \quad \forall i > n.$

Pf: Step 1: If (a), (b) hold for compact sets A, B and $A \cap B$, then also for $A \cup B$.

Pf: Use the M-V-sequence.

$$\hookrightarrow H_{n+1}(M|A\cup B) \to \underset{0}{H_{n+1}(M|A)} \oplus \underset{0}{H_{n+1}(M|B)} \to \underset{0}{H_{n+1}(M|A\cap B)}$$

$$\hookrightarrow H_n(M|A\cup B) \to H_n(M|A) \oplus H_n(M|B) \to H_n(M|A\cap B)$$

$\hookrightarrow$

So $H_i(M|A\cup B)=0 \ \forall i>n$.
Note $\alpha_A|_{A\cap B} = \alpha_B|_{A\cap B} = \alpha_{A\cap B}$

So $\exists\, \alpha_{A\cup B} \in H_n(M|A\cup B)$: $\alpha_{A\cup B} \longmapsto (\alpha_A, -\alpha_B)$,
i.e. $\alpha_{A\cup B}|_A = \alpha_A$ and $\alpha_{A\cup B}|_B = \alpha_B$.

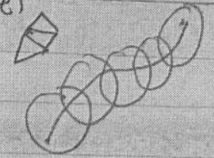
Uniqueness is obvious.

<u>Step 2</u>: By Step 1 we can reduce to the case $M=\mathbb{R}^n$.

<u>Step 3</u>: If $A\subseteq \mathbb{R}^n$ is convex and compact. Then $\forall x\in A$: $\forall i\in\mathbb{Z}$
$$H_i(M|A) \xrightarrow{\cong} H_i(M|x)$$
as $M\setminus A$ and $M\setminus\{x\}$ def. retract to an $(n-1)$-sphere.

<u>Step 4</u>: Now A is an arbitrary non-empty compact set in $\mathbb{R}^n$.
<u>For (b)</u>: Take $\underset{[z]_A}{\beta} \in H_i(M|A)$, $i>n$,
and K a finite union of compact balls s.t. $K\cap(\mathrm{im}(\partial z))=\emptyset$ and $A\subseteq \mathring{K}$.

$\mathrm{im}(\partial z)$



$$\beta_K := [z]_K \in H_i(M|K) = 0$$

$$\Rightarrow \quad \beta = \beta_K|_A = 0.$$

<u>For (a)</u>: Existence. Take a compact ball $B\supseteq A$ and put
$$\alpha_A := \alpha_B|_A$$

Uniqueness: Suppose $\beta\in H_n(M|A)$ satisfies $\beta|_x = 0 \ \forall x\in A$.
Choose $[z]_A=\beta$ and K as above.
For all closed balls B in K we have $H_n(M|B)\cong H_n(M|x) \ \forall x\in B$.
K finite union of compact balls with center in A.
So for such balls we have for $\beta_K := [z]_K$
$$\beta_K|_B = 0.$$
$$\Rightarrow \beta_K|_x = 0 \ \forall x\in K \overset{\text{Step 3/1}}{\Longrightarrow} \beta_K = 0. \quad \square$$

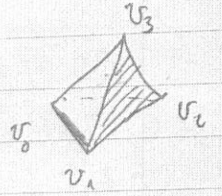
VIII3. Cap product

Def 226: (cap product)

X a space and $k \geq \ell \geq 0$

$$\cap : C_k(X;R) \times C^\ell(X;R) \longrightarrow C_{k-\ell}(X;R)$$

$$\sigma \cap \varphi := \varphi(\sigma|_{[v_0,\dots,v_\ell]})\, \sigma|_{[v_\ell,\dots,v_k]}$$



$\ell = 1$

$$\varphi(\sigma|_{[v_0,v_1]})\, \sigma|_{[v_1,v_2,v_3]}.$$

Exercise 227: For $c \in C_k(X;R)$ and $\varphi \in C^\ell(X;R)$ we have

$$\partial(c \cap \varphi) = (-1)^k\big((\partial c) \cap \varphi - c \cap \delta\varphi\big)$$

for $k \geq \ell \geq 0$.

So we get an R-bilinear map

$$H_k(X;R) \times H^\ell(X;R) \longrightarrow H_{k-\ell}(X;R)$$

Rk 228: (PD map for Thm 214)

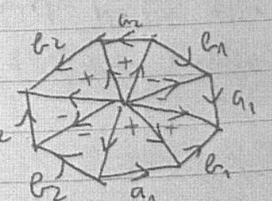
Let M be a closed n-mf with R-orientation. Let $[M]$ be the corresponding fundamental class $\in H_n(M;R)$.

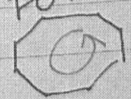
We obtain a map

$$H^k(M;R) \xrightarrow{D} H_{n-k}(M;R)$$

$$D(\alpha) := [M] \cap \alpha.$$

Example 229: M_g $\quad g = 2$



A generator of $H_2(M)$ is indicated by the signs for the triangles. This is the fundamental class $[M]$ for the orientation given by .

$H^1(M) \cong \mathrm{Hom}(H_1(M), \mathbb{Z})$
Let $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ be the dual basis of $\{a_1, b_1, a_2, b_2\}$.

$$H^1(M) \xrightarrow{D} H_1(M)$$

$$D(\alpha) = [M] \cap \alpha$$

$$D(\alpha_1) = b_1, \quad D(\beta_1) = -a_1.$$

We will prove the PD (Thm 214) by induction starting from open sets in $\mathbb{R}^n$. So we need PD for non-compact oriented mf.

VII 4. Cohomology with compact supports

Def 230: X Hausdorff.

$$C^i_c(X;R) := \{\varphi \in C^i(X;R) \mid \exists K \subseteq X \text{ compact such that } \varphi|_{C^i(X \setminus K)} = 0\}$$

We obtain a complex

$$0 \to C^0_c(X;R) \xrightarrow{\delta^0} C^1_c(X;R) \xrightarrow{\delta^1} C^2_c(X;R) \xrightarrow{\delta^2} \cdots$$

$H^i_c(X;R)$ "ith cohomology gp with compact supports".

Rk 231: (a) $C^i_c(X;R) = \bigcup_K C^i(X|K;R)$

(b) For $K \subseteq L$ both compact in X we have
$$C^i(X|K;R) \subseteq C^i(X|L;R)$$
and
$$\begin{tikzcd} H^i(X|K;R) \arrow[rr] \arrow[dr] & & H^i(X|L;R). \arrow[dl] \\ & H^i_c(X;R) & \end{tikzcd}$$

Thus

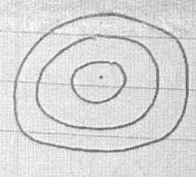
$$\varinjlim_{K} H^i(X|K;R) \longrightarrow H^i_c(X;R).$$

<u>Exercise</u>: This map is an R-module isomorphism.

<u>Example 232</u>: $X = \mathbb{R}^n$ $(n \geq 1)$

$B_m = \overline{B}_m(0)$ closed ball around 0 with radius m.

$B_1 \subseteq B_2 \subseteq B_3 \subset B_4 \dots$



It is a cofinal system for $(\{K \subseteq \mathbb{R}^n \mid K \text{ compact}\}, \subseteq)$

$$\Rightarrow H^i_c(\mathbb{R}^n) = \varinjlim_{B_m} H^i(\mathbb{R}^n | B_m) \underset{218}{=} \begin{cases} 0, & i \neq n \\ \mathbb{Z}, & i = n \end{cases}$$

<u>Exercise 233</u>: Compute $H^i_c(\mathbb{R}^n)$ using the direct definition of $H^i_c(\mathbb{R}^n)$ using the complex $C^\bullet_c(\mathbb{R}^n)$.

<u>Def 234</u>: (PD map for the non-compact case.)
Let M be an R-oriented n-mf.
and $\mu_K \in H_n(M|K;R)$
the relative fundamental class for the R-orientation, see Lem 225, for $K \subseteq M$ compact.

Then for $K \subseteq L$ compact: $\mu_L|_K = \mu_K$.

and we get, for $0 \leq k \leq n$,

$$\begin{tikzcd}
H^k(M|K;R) \arrow[r, "D|_K"] & H_{n-k}(M;R) \\
H^k(M|L;R) \arrow[u] \arrow[ur, "D|_L"'] \arrow[r, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

$$D|_K(\alpha) := \mu_K \cap \alpha$$

This defines

$$D_M : H_c^k(M;R) \longrightarrow H_{n-k}(M;R)$$
$$\qquad\quad \parallel$$
$$\quad \varinjlim_K H^k(M|K;R).$$

<u>Thm 235:</u> D is an R-isomorphism.

<u>Rk 236:</u> (M-V-sequence for H_c)
Let X be locally compact, $U, V \subseteq X$ open.
Then $H_c^i(U;R) = \varinjlim_{\substack{K \subseteq U \\ \text{compact}}} H^i(U|K;R)$

$$\underset{\text{excision}}{=} \varinjlim_{K \subseteq U} H^i(M|K;R)$$

So we obtain a long exact sequence

$$\hookrightarrow H_c^i(U \cap V;R) \to H_c^i(U;R) \oplus H_c^i(V;R) \to H_c^i(U \cup V;R)$$
$$\hookrightarrow H_c^{i+1}(U \cap V;R) \to \dots$$

(exercise.)
Mayer-Vietoris sequence for cohom. with comp. supports.

<u>Lem 237:</u> The following diagram commutes up to a sign depending on k: (M orientable, $U, V \subseteq M$ open)

$$\begin{CD}
\to H_c^k(U \cap V) @>>> H_c^k(U) \oplus H_c^k(V) @>>> H_c^k(U \cup V) \to \cdots \\
@V{D_{U \cap V}}VV @V{(D_U, D_V)}VV @V{D_{U \cup V}}VV \\
\cdots \to H_{n-k}(U \cap V) @>>> H_{n-k}(U) \oplus H_{n-k}(V) @>>> H_{n-k}(U \cup V) \to \cdots
\end{CD}$$

<u>Idea of the proof of Thm 235:</u>

<u>Case 1:</u> $M = \mathbb{R}^n$ Take $\ell >> m$

$$\begin{array}{ccc} & & u \in R^{\times} \\ \beta & \longmapsto & u[\sigma] \cap \beta \\ H^n(B_\varepsilon, \partial B_\varepsilon; R) & \xrightarrow[\sim]{} & H_0(B_\varepsilon; R) \simeq R \\ \wr\vert & \text{Ⓐ} & \wr\vert \\ H^n(\mathbb{R}^n \vert B_m; R) & \xrightarrow{D_m} & H_0(\mathbb{R}^n; R) \\ \alpha & \longmapsto & \mu_m \cap \alpha \end{array}$$

⌈ Note:

$H_n(\mathbb{R}^n \vert B_m; R) \simeq H_n(B_\varepsilon, \partial B_\varepsilon; R)$

$= R[\sigma]$

$\sigma : \Delta^n \xrightarrow{\simeq} B_\varepsilon$ ⌋

The top arrow is an iso by

$H^n(B_\varepsilon, \partial B_\varepsilon; R) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_n(B_\varepsilon, \partial B_\varepsilon), R)$

$= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\sigma], R)$

$\simeq R$

So all D_m are isom.
Thus $D_{\mathbb{R}^n}$ is an isom.

<u>Case 2:</u> $\mathcal{M} := \{ U \subseteq M \mid U \text{ open and } D_U \text{ is an iso.} \}$

Case 1 $\Rightarrow$ $\mathcal{M} \neq \emptyset$.
$\mathcal{M}$ is inductively ordered.

Zorn's lemma $\Rightarrow$ $\mathcal{M}$ has a maximal element $\hat{U}$.

<u>Claim</u> $\hat{U} = M$.

<u>Pf:</u> Assume $\hat{U} \subsetneq M$. Take $x \in M \setminus \hat{U}$ and V around x open s.t. $V \simeq \mathbb{R}^n$.
Case 1 and Lemma 237 and 5-lemma
$\Rightarrow$ $\hat{U} \cup V \in \mathcal{M}$ ↯ $\square$

$\square$