Rk 195: We give a geometric explaination of $H^0(X;G)$ and $\tilde{H}^0(X;G)$ for $X \neq \emptyset$

$H^0(X;G)$:

$$H^0(X;G) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_0(X), G)$$
$$\simeq \{ f \in \mathrm{Hom}_{\mathbb{Z}}(C_0(X), G) \mid f|_{B_0(X)} = 0 \}$$
$$\simeq \{ f : X \longrightarrow G \mid f \text{ is constant on path-components} \}$$

$\tilde{H}^0(X;G)$:

$$\tilde{H}^0(X;G) \simeq \mathrm{Hom}_{\mathbb{Z}}(\tilde{H}_0(X), G)$$
$$\simeq \{ f \in \mathrm{Hom}_{\mathbb{Z}}(\tilde{Z}_0(X), G) \mid f|_{B_0(X)} = 0 \}$$
$$\simeq \{ f \in \mathrm{Hom}_{\mathbb{Z}}(C_0(X), G) \mid f|_{B_0(X)} = 0 \} / \sim$$

(with $f \sim g \overset{\text{def.}}{\Longleftrightarrow} f - g|_{\tilde{Z}_0} = 0$

$\Leftrightarrow f - g|_X$ is constant)

$$\simeq \{ f : X \longrightarrow G \mid f \text{ is constant on path-components of } X \} / \text{constant maps.}$$

Note: $\tilde{H}^{-1}(\emptyset, G) \simeq \mathrm{Hom}_{\mathbb{Z}}(\tilde{H}_{-1}(\emptyset), G)$

$$\simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \simeq G.$$

and
$\tilde{H}^{-1}(X, G) = 0$ for $X \neq \emptyset$.

Def 196: (relative groups)
Given a pair (X,A) we put

$$C^n(X,A;G) := \mathrm{Hom}_{\mathbb{Z}}(C_n(X,A), G)$$

$$\delta^n := \text{dual of } \partial_{n+1}$$
(of $C_n(X,A)$)

with relative cohomology gp.

$$H^n(X,A;G).$$

<u>Rk 197:</u>

We have

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{q} C_n(X,A) \to 0$$

(split exact)

$\Rightarrow$ We get

$$0 \to C^n(X,A;G) \to C^n(X;G) \to C^n(A;G) \to 0$$

split exact.

$$\Rightarrow \quad C^n(X,A;G) \simeq \{ f \in \mathrm{Hom}_{\mathbb{Z}}(C_n(X),G) \mid f|_{C_n(A)} = 0 \}$$

<u>Rk 198:</u> The short exact$\overset{\text{split}}{\vee}$ seq.

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X,A) \to 0$$

induces a short ex. seq.

$$0 \to C^\bullet(X,A;G) \to C^\bullet(X;G) \to C^\bullet(A;G) \to 0$$

And we get a long exact seq. in cohomology

$$\cdots \to H^n(X,A;G) \to H^n(X;G) \to H^n(A;G) \to H^{n+1}(X,A;G) \to H^{n+1}(X;G) \to H^{n+1}(A;G) \to \cdots$$

<u>Exercise:</u> (a) The connecting map $\partial : H^n(A;G) \to H^{n+1}(X,A;G)$ has the form

$$\partial([\varphi]) = \varphi \circ \partial_{n+1,X}.$$

<u>Note:</u> $Z^n(X,A;G) \simeq \{ f \in C^n(X;G) \mid f|_{C_n(A)} = 0, \ f \circ \partial_{n+1,X} = 0 \}$

(b) $H^n(X,\emptyset;G) \simeq H^n(X;G)$

$H^n(X,\{x_0\};G) \simeq \tilde{H}^n(X;G)$.

In some arguments that follow we need the following lemma:

<u>Lem 199:</u> Let $0 \to M \xrightarrow{\beta} N \xrightarrow{\alpha} F \to 0$ be an exact sequence of abelian groups and let F be free, then the sequence splits, i.e. $\exists\, s : F \to N$ homom. such that $\alpha \circ s = id_F$.

<u>Proof:</u> Take a basis B of F and choose $n_b \in \alpha^{-1}(b)$ for all $b \in B$.
Put $s(b) := n_b$ and extend linearly. Then $\alpha \circ s = id_F$ □

The next lemma states a consequence of the naturality of the UCT exact sequence

<u>Lem 200:</u> Let $C_\cdot$ and $C'_\cdot$ be two chain complexes of free abelian groups and $\alpha : C_\cdot \to C'_\cdot$ be a chain map.
Suppose $\alpha_{*,n} : H_n(C) \to H_n(C')$ is an isom. for all $n \in \mathbb{Z}$.
Let G be an ab. group. Then

$$\alpha^{*,n} : H^n(C'; G) \longrightarrow H^n(C; G)$$

is an isomorphism for all $n \in \mathbb{Z}$.

<u>Proof:</u> We have the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(C), G) \arrow[r] & H^n(C; G) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(H_n(C), G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(C'), G) \arrow[r] \arrow[u, "(\alpha_*)^* \wr"] & H^n(C'; G) \arrow[r] \arrow[u, "\alpha^*"] & \mathrm{Hom}_{\mathbb{Z}}(H_n(C'), G) \arrow[r] \arrow[u, "(\alpha_*)^* \wr"] & 0
\end{tikzcd}$$

So the middle α^* is an isom. □

Def 201: Given a continuous $f: X \to Y$

$\Rightarrow$ We get $f_\#: C_n(X) \longrightarrow C_n(Y)$

$\Rightarrow$ We get $f^\#: C^n(Y;G) \to C^n(X;G)$

which induces

$$f^*: H^n(Y;G) \longrightarrow H^n(Y;G)$$

Rk 202: Suppose $f, g: X \to Y$ are homotopic. Then $\forall n \in \mathbb{N}_0$: $f^* = g^*: H^n(Y;G) \longrightarrow H^n(X;G)$.

Pf: $f \sim g \overset{\text{Thm 122}}{\Longrightarrow} \exists$ chain homotopy P such that $f_\# - g_\# = \partial P + P\partial$

$\underset{\substack{\uparrow \\ \text{dual w.r.t. } G}}{\Longrightarrow}$ $f^\# - g^\# = P^\# \delta + \delta P^\#$

$\Rightarrow$ $f^* = g^*$ on cohomology groups. $\square$

Thm 203 (Excision): Given $Z \subseteq A \subseteq X$ s.t. $\overline{Z}^X \subseteq \mathring{A}$

Then $\iota: (X \setminus Z, A \setminus Z) \longrightarrow (X, A)$ induces an isomorphism

$$H^n(X, A; G) \longrightarrow H^n(X \setminus Z, A \setminus Z; G)$$

for all $n \in \mathbb{Z}$.

Proof: Use the UCT for cohomology of (X, A) and $(X \setminus Z, A \setminus Z)$.

The naturality implies the assertion using the excision for homology groups $\square$

Thm 204: (simplicial cohom. vis-a-vis singular cohom.)
Let X be a Δ-complex and A be a subcomplex.
Then
$$H^n_\Delta(X,A;G) \xrightarrow{\sim} H^n(X,A;G)$$

Proof: The chain map
$$\Delta_\circ(X,A) \hookrightarrow C_\circ(X,A)$$
induces an isomorphism on homology.
The UCT for relative cohomology and Lemma 200 give the assertion $\square$

As in the case of homology groups, there is formally much more to do for cellular cohomology groups.

Note that by UCT we have
$$H^k(X^n,X^{n-1};G) \cong \mathrm{Hom}_{\mathbb{Z}}(H_k(X^n,X^{n-1}),G)$$
for all $n \in \mathbb{N}_0$, see Lemma 162.

Def 205: (Cellular cochain complex)
Let X be a CW complex. $C^\bullet_{CW}(X;G)$ is the following cocomplex $H^\bullet(X^\bullet, X^{\bullet-1};G)$:

$$\begin{tikzcd}
 & H^{n-1}(X^{n-1};G) \arrow[r] \arrow[dr, "\delta^{n-1}"] & 0 & & \\
\cdots \arrow[r] & H^{n-1}(X^{n-1},X^{n-2};G) \arrow[u, "i^{n-1}"] \arrow[r, "d^{n-1}"] & H^n(X^n,X^{n-1};G) \arrow[r, "d^n"] \arrow[dr, "i^n"] & H^{n+1}(X^{n+1},X^n;G) \arrow[r] & \cdots \\
 & & & H^n(X^n;G) \arrow[u, "\delta^n"] \arrow[r] & 0 \\
 & H^n(X;G) \cong & H^n(X^{n+1};G) \arrow[ur] & & \\
 & & 0 \arrow[u] & &
\end{tikzcd}$$

UCT and 162
$\Rightarrow H^k(X,X^{n+1};G)=0$
$\forall k \le n+1$.
l.e.s.
$\Rightarrow H^k(X;G) \cong H^k(X^{n+1};G)$
$\forall k \le n$.

Thm 206. (a) $H^n(X;G) \cong \ker(d^n) / \mathrm{im}(d^{n-1})$

(b) $(H^n(X^n,X^{n-1};G), d^n)_{\mathbb{N}_0}$

is the dual of
$(H_n(X^n,X^{n-1};G), d_n)_{\mathbb{N}_0}$

i.e. $H^n(X^n, X^{n-1}; G) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_n(X^n, X^{n-1}), G)$

and $d^n = (d_{n+1})^*$

<u>Proof:</u> We have $d^n = \delta^n \circ \iota^n$.

(a) At first we show that the diagonal sequences in the defining diagram are exact.

$$H^k(X^n, X^{n-1}; G) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_k(X^n, X^{n-1}), G) \underset{162}{\simeq} 0 \quad \text{for } k \neq n.$$

Long exact sequence for (X^n, X^{n-1}, G)

$\Rightarrow H^k(X^n; G) \simeq H^k(X^{n-1}; G) \ \forall_{k \neq n, n-1}$

$\Rightarrow \forall_{k > n \geq 0}: \ H^k(X^n; G) \simeq H^k(X^0; G) \underset{UCT}{\simeq} \mathrm{Hom}_{\mathbb{Z}}(H_k(X^0), G) \underset{k>0}{\simeq} 0$

Note also $H^0(X^0, \overset{\emptyset}{\overset{\shortparallel}{X^{-1}}}; G) \xrightarrow[\text{l.e.s.}]{\simeq} H^0(X^0; G)$

So, all diagonal sequences are exact, i.e. the "zeros" are correct.

Then $H^n(X, G) \simeq \ker \delta^n$

$$\simeq \ker d^n / \mathrm{im}\, \delta^{n-1}$$

$$\underset{\iota^{n-1} \text{ is surjective}}{=} \ker d^n / \mathrm{im}\, d^{n-1}$$

$\forall_{n \geq 0}$

(b) We consider

$$\begin{tikzcd}
H^k(X^k, X^{k-1}; G) \arrow[r] \arrow[d, "hs"] \arrow[rr, bend left, "d^k"] \arrow[dr, phantom, "\mathrm{I}"] & H^k(X^k; G) \arrow[r, "\delta^k"] \arrow[d, "h"] \arrow[dr, phantom, "\mathrm{II}"] & H^{k+1}(X^{k+1}, X^k; G) \arrow[d, "hs"] \\
\mathrm{Hom}(H_k(X^k, X^{k-1}), G) \arrow[r] \arrow[rr, bend right, "(\partial_{k+1})^*"'] & \mathrm{Hom}(H_k(X^k), G) \arrow[r, "\partial^*"] & \mathrm{Hom}(H_{k+1}(X^{k+1}, X^k), G)
\end{tikzcd}$$

We need to show that the red squares commute.

II commutes, because of Exercise (a) after Remark 198

I commutes, because of the naturality of h (from the UCT exact sequence)

So $d^k \triangleq (\partial_{k+1})^*$ under h $\square$

<u>Thm 207</u> (M-V sequence)

Suppose $X = \mathring{A} \cup \mathring{B}$, for some $A, B \subseteq X$.
Then we have a long exact sequence

$$\cdots \to H^n(X;G) \to H^n(A;G) \oplus H^n(B;G) \to H^n(A \cap B;G) \to H^{n+1}(X;G) \to \cdots$$

<u>Proof:</u>

$$0 \to C_n(A \cap B) \xrightarrow{x \mapsto (x,-x)} C_n(A) \oplus C_n(B) \xrightarrow{(x,y) \mapsto x+y} C_n(A+B) \to 0$$

is exact and split (as $C_n(A+B)$ is free, see Lem 199)

$$\Rightarrow 0 \to C^n(A+B,G) \to C^n(A;G) \oplus C^n(B;G) \to C^n(A \cap B;G) \to 0$$

is exact and split
Take the long exact seq. in cohomology and use that $C_\bullet(A+B) \hookrightarrow C_\bullet(X)$ is a homotopy equiv. $\square$