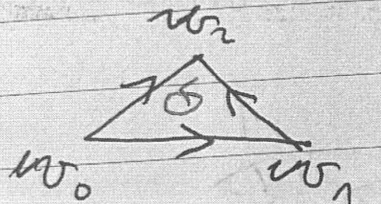
This map is a group isomorphism (exercise). $\square$

<u>Example 185:</u> Cohomology groups for 2-dim. Δ-complexes.

$$X = (V, E, F)$$

vertexes, oriented edges, oriented faces.

<u>Ex:</u>



The orientation of the face is not consistent with orientation on $[w_0, w_2]$

$$0 \to \Delta^0(X;G) \xrightarrow{\partial^0} \Delta^1(X;G) \xrightarrow{\partial^1} \Delta^2(X;G) \to 0$$

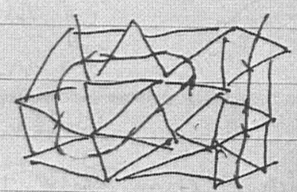
$$H^1_\Delta(X;G) = \ker(\partial^1) / \operatorname{im}(\partial^0).$$

$$\partial^1(\psi)([w_0, w_1, w_2]) = \psi([w_1, w_2]) - \psi([w_0, w_2]) + \psi([w_0, w_1])$$

- $\partial^1(\psi) = 0 \iff \psi$ is additive on a path on the boundary of a 2-simplex.

- Take $G = \mathbb{Z}/2$ and X simplicial complex.

  $\partial^1(\psi) = 0 \iff \forall [w_0, w_1, w_2] \in \mathcal{F}$:
  $\psi^{-1}(1) \cap \{[w_0,w_1],[w_0,w_2],[w_1,w_2]\}$
  has 0 or 2 elements.



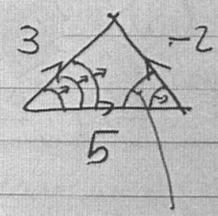
$\partial^0 \varphi = \psi$ has a solution $\varphi \in \Delta^0(X, \mathbb{Z}/2)$
iff $\partial^1\psi = 0$ and we can find a partition

$$Y := X \setminus \underset{\text{blue curve}}{\bigcup C} = X_0 \cup X_1$$

s.t. X_0, X_1 are unions of path connected components of Y

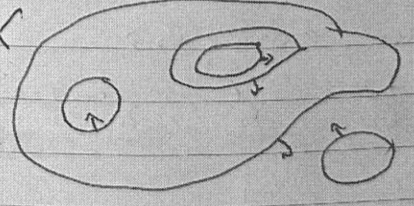
- $\forall$ triangle Δ: $X_0 \cap \Delta$ and $X_1 \cap \Delta$ are connected.

<u>Exercise 186:</u> Study the example for $H^1_\Delta(X; \mathbb{R}) = 0$ for a 2-dim simplicial complex.



$\circlearrowleft$ $(5 + (-2) - 3) = 0$

X



$\partial^0\varphi = \psi$ has sol^n
$\iff$ $\partial^1\psi = 0$ and blue curves give a system of level curves.

VI 2. Universal Coefficient Theorem for cohomology groups.

The construction of cohomology groups is a purely algebraic process starting from a chain complex.

Let $C_\bullet$ be a chain complex of free abelian groups.

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{0} 0$$

and G be an abelian group. We put $C^n := \mathrm{Hom}_{\mathbb{Z}}(C_n, G)$, $n \in \mathbb{Z}$ and we obtain a cochain complex

$$0 \longrightarrow C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \cdots \longrightarrow C^{n-1} \xrightarrow{\delta^{n-1}} C^n \longrightarrow$$

where δ^n is the dual of ∂_{n+1}.

$$\delta^n(\varphi) = \varphi \circ \partial_{n+1}$$

The cohomology groups are defined via

$$H^n(C;G) := \ker(\delta^n) / \operatorname{im}(\delta^{n-1})$$

$$= \text{"cocycles"} / \text{"coboundaries"}$$

$H^\bullet(C;G)$ only depends on $H_\bullet(C)$ and G in the following way.

Theorem 187: (UCT for cohomology)

We have
$$H^n(C;G) \simeq \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(C), G) \oplus \mathrm{Hom}_{\mathbb{Z}}(H_n(C), G),$$

more precisely there is a canonical split exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(C), G) \to H^n(C;G) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(C), G) \to 0$$

<u>Proof:</u> $\forall n \in \mathbb{Z}$ we have a split exact sequence

$$0 \to Z_n \longrightarrow C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0$$

Thus the dual

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(B_{n-1}, G) \xrightarrow{\delta^{n-1}} \mathrm{Hom}_{\mathbb{Z}}(C_n, G) \to \mathrm{Hom}_{\mathbb{Z}}(Z_n, G) \to 0$$

is split exact.

So we get a short exact sequence of cochain complexes

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(B_{\bullet-1}, G) \xrightarrow{\delta} \mathrm{Hom}_{\mathbb{Z}}(C_\bullet, G) \xrightarrow{r} \mathrm{Hom}_{\mathbb{Z}}(Z_\bullet, G) \to 0$$

that is

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathrm{Hom}(B_{n-1}, G) & \xrightarrow{\delta^{n-1}} & \mathrm{Hom}(C_n, G) & \xrightarrow{r} & \mathrm{Hom}(Z_n, G) & \to 0 \\
 & 0\downarrow & & \delta^n\downarrow & & 0\downarrow & \\
0 \to & \mathrm{Hom}(B_n, G) & \xrightarrow{\delta^n} & \mathrm{Hom}(C_{n+1}, G) & \xrightarrow{r} & \mathrm{Hom}(Z_{n+1}, G) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

Thus we obtain a long exact sequence in cohomology

$$\hookrightarrow \mathrm{Hom}(B_{n-1}, G) \xrightarrow[\text{res}]{\delta_*^{n-1}} H^n(C; G) \xrightarrow{r_*} \mathrm{Hom}(Z_n, G)$$

$$\hookrightarrow \mathrm{Hom}(B_n, G) \xrightarrow{\delta_*^n} H^{n+1}(C; G) \xrightarrow{r_*} \mathrm{Hom}(Z_{n+1}, G)$$

$$\hookrightarrow \cdots$$

and we obtain the exact sequence

$$0 \to \frac{\mathrm{Hom}(B_{n-1}, G)}{\mathrm{res}(\mathrm{Hom}(Z_{n-1}, G))} \to H^n(C; G) \to \mathrm{im}(r_*) \to 0$$

- $\mathrm{im}(r_*) = \{ f \in \mathrm{Hom}(Z_n, G) \mid f|_{B_n} = 0 \}$

$$\simeq \mathrm{Hom}_{\mathbb{Z}}(H_n(C), G).$$

- $0 \to B_{n-1} \to Z_{n-1} \to H_{n-1}(C) \to 0$ is a free resolution of $H_{n-1}(C)$.

Take cohomology grps of

$$0 \to \underset{\deg 0}{\mathrm{Hom}(Z_{n-1}, G)} \to \underset{\deg 1}{\mathrm{Hom}(B_{n-1}, G)} \to 0. -$$

(The ith homology group is called $\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(C), G)$.)

The 1^{st} homology group is

$$\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(C), G) = \mathrm{Hom}(B_{n-1}, G) \,/\, \mathrm{res}(\mathrm{Hom}(Z_{n-1}, G))$$

<u>To show:</u> The sequence is split.

We give a splitting for

$$\begin{tikzcd} H^n(C; G) \arrow[r, two heads, "r_*"] & \mathrm{Hom}_{\mathbb{Z}}(H_n(C), G) \arrow[l, bend left, "\lambda"] \end{tikzcd}$$

We have $C_n \simeq B_{n-1} \oplus Z_n$

Take $f \in \mathrm{Hom}_{\mathbb{Z}}(H_n(C), G)$.

We inflate f to Z_n:

$$\begin{tikzcd} Z_n \arrow[rr, "\tilde{f}"] \arrow[d, two heads, "q"'] & & G \\ H_n(C) \arrow[urr, "f"'] \arrow[ur, phantom, "\circlearrowleft"] & & \end{tikzcd}$$

We extend $\tilde{f}$ to C_n in putting it 0 on B_{n-1}, still called $\tilde{f}$.

Then

$$\delta^n(\tilde{f}) = \tilde{f} \circ \partial_{n+1} = f \circ q \circ \partial_{n+1}$$

$$= f \circ 0 = 0.$$

We put $\lambda(f) := [\tilde{f}] \in H^n(C; G)$.

$\square$

If H is a f.g. abelian group then $\mathrm{Ext}^1_{\mathbb{Z}}(H, G)$ is easy to compute.

<u>Prop 188:</u> Let H and H' be two abelian groups and G be an abelian group. Then

(a) $\mathrm{Ext}^1_{\mathbb{Z}}(H \oplus H', G) \simeq \mathrm{Ext}^1_{\mathbb{Z}}(H, G) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H', G)$

(b) $\mathrm{Ext}^1_{\mathbb{Z}}(H, G) = 0$ if H is free

(c) $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, G) \simeq G/mG$

<u>Proof:</u> (a) Let $0 \to F_1 \to F_0 \to H \to 0$
and $0 \to F_1' \to F_0' \to H' \to 0$
be two free resolutions.
Then

$$0 \to F_1 \oplus F_1' \to F_0 \oplus F_0' \to H \oplus H' \to 0$$

is a free resolution of $H \oplus H'$
and

$$\begin{aligned}\mathrm{Ext}^1_{\mathbb{Z}}(H \oplus H', G) &\simeq H^1(F \oplus F', G)\\ &\simeq H^1(F, G) \oplus H^1(F', G)\\ &\simeq \mathrm{Ext}^1_{\mathbb{Z}}(H, G) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H', G)\end{aligned}$$

(b) $0 \to \underset{\deg 1}{H} \to \underset{\deg 0}{H} \to 0$ is a free resolution of H.
So we have the 0 group at degree 1.

$\Rightarrow \quad \mathrm{Ext}^1_{\mathbb{Z}}(H, G) = 0$

(c) $0 \to \underset{\deg 1}{\mathbb{Z}} \xrightarrow{m} \underset{\deg 0}{\mathbb{Z}} \to \mathbb{Z}/m\mathbb{Z} \to 0$
is a free resolution of $\mathbb{Z}/m\mathbb{Z}$.
We take

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$$

Apply $\mathrm{Hom}(-, G)$:

$$0 \to \underset{\deg 0}{\mathrm{Hom}(\mathbb{Z}, G)} \xrightarrow{m} \underset{\deg 1}{\mathrm{Hom}(\mathbb{Z}, G)} \to 0$$

⌈ Note m^* is just the multiplication with m.

$$\begin{aligned}m^*(\varphi)(z) &= \varphi(mz) = m\,\varphi(z)\\ &= (m\varphi)(z)\end{aligned}$$ ⌋

$\Rightarrow \quad 0 \to \underset{\deg 0}{G} \xrightarrow{m} \underset{\deg 1}{G} \to 0$

$\mathrm{Hom}(\mathbb{Z}, G) \simeq G$

$\Rightarrow \quad \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, G) \simeq G/mG$.

Compute H^1

□

<u>Example 18.9:</u> $H = \mathbb{Z}^r \oplus \mathbb{Z}/12\mathbb{Z}$

Then $\mathrm{Ext}^1_{\mathbb{Z}}(H, G) \simeq \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/12\mathbb{Z}, G)$

$$\simeq \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/3, G) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/4, G) \simeq G/3G \oplus G/4G.$$

$\boxed{\mathbb{Z}/12 \underset{CRT}{\simeq} \mathbb{Z}/3 \oplus \mathbb{Z}/4}$

If $G = \mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$.

$$\Rightarrow \operatorname{Ext}^1_{\mathbb{Z}}(H, \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/\gcd(3,n)\mathbb{Z} \oplus \mathbb{Z}/\gcd(4,n)\mathbb{Z}.$$

$$\left(\text{as } \ell G = \gcd(\ell, n)\mathbb{Z}/n\mathbb{Z}\right)$$

<u>Cor 190.</u> Suppose the homology groups $H_n(C)$, $H_{n-1}(C)$ are f.g. abelian groups with torsion parts $T_{n-1} \subseteq H_{n-1}(C)$ and $T_n \subseteq H_n(C)$.
Then

$$H^n(C; \mathbb{Z}) \simeq T_{n-1} \oplus H_n(C)/T_n$$

<u>Proof:</u> $\operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}, G)$

$$\underset{H_{n-1} \simeq \mathbb{Z}^r \oplus T_{n-1}}{\simeq} \operatorname{Ext}^1_{\mathbb{Z}}(T_{n-1}, G) \overset{G=\mathbb{Z}}{=} \operatorname{Ext}^1_{\mathbb{Z}}(T_{n-1}, \mathbb{Z})$$

$$\simeq T_{n-1}$$

$$\operatorname{Hom}_{\mathbb{Z}}(H_n, G) \overset{G=\mathbb{Z}}{=} \operatorname{Hom}_{\mathbb{Z}}(H_n, \mathbb{Z})$$

$$\simeq \operatorname{Hom}_{\mathbb{Z}}(H_n/T_n, \mathbb{Z}) \simeq \mathbb{Z}^{\operatorname{rk}(H_n)}$$

$$\simeq H_n/T_n$$

□

VI 3. Cohomology of spaces

<u>Def 191</u>: (singular cochain complex)

Let X be a top. space. We define

$$C^n(X;G) := \mathrm{Hom}_{\mathbb{Z}}(C_n(X), G)$$

$$0 \to C^0(X;G) \xrightarrow{\delta^0} C^1(X;G) \xrightarrow{\delta^1} C^2(X;G) \xrightarrow{\delta^2} \cdots$$

with $\delta^n(\varphi) := \varphi \circ \partial_{n+1}$.

$$H^n(X;G) := \ker \delta^n / \operatorname{im} \delta_{n-1}$$

$$= \text{"}n\text{-cocycles"} / \text{"}n\text{-coboundaries"}.$$

"nth singular cohomology gp with coefficients in G".

<u>Rk 192</u>: The UCT gives a split exact seq.

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X), G) \to H^n(X;G) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(X), G) \to 0$$

for all $n \in \mathbb{Z}$.

<u>Example 193</u>:

(1) <u>$n=0$</u>: $H_{-1}(X) = 0 \Rightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_{-1}(X), G) = 0$.

$$\underset{192}{\Longrightarrow} H^0(X;G) \simeq \mathrm{Hom}(H_0(X), G).$$

<u>$n=1$</u>: $H_0(X)$ is free $\Rightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_0(X), G) = 0$

$$\underset{192}{\Longrightarrow} H^1(X;G) \simeq \mathrm{Hom}(H_1(X), G)$$

(2) The proof of UCT (Thm 187) only used the fact that:
"submodules of free $\mathbb{Z}$-modules are free"
Any PID has a similar property.

Let F be a field. Then we have the analogue UCT for F:

$$0 \to \mathrm{Ext}^1_F(H_{n-1}(X;F), F) \to H^n(X;F) \to \mathrm{Hom}_F(H_n(X;F), F) \to 0.$$

$H_{n-1}(X;F)$ is an F-vector space and

therefore free and projective.
$\Rightarrow \operatorname{Ext}^1_F(H_{n-1}(X;F),F)=0$.
So
$$H^n(X;F) \simeq \operatorname{Hom}_F(H_n(X;F),F)$$
$$\cancel{\simeq \operatorname{Hom}_{\mathbb{Z}}(H}$$

Over a field $H^n(X;F)$ is just the F-dual of $H_n(X;F)$.

(3) Look at $F=\mathbb{Z}/p$.
Then
$$\operatorname{Ext}^1_{\mathbb{Z}}(F,F) \simeq F/pF \simeq F$$
and
$$\operatorname{Ext}^1_{\mathbb{Z}/p}(F,F) \simeq 0 \text{ as}$$

$F=\mathbb{Z}/p$ is a projective F-module.

(4) $G=\mathbb{R}$ and X a manifold
$\Rightarrow H^n(X;\mathbb{R}) \simeq$ de Rham cohomology group.

(de Rham's Theorem)

We now collect the similar terminology for singular cochain complexes.

Def 19.4: (reduced cohomology groups)
Here we start with the augmented chain complex
$$\cdots \to C_2(X) \to C_1(X) \to C_0(X) \to \mathbb{Z} \to 0$$
Now apply $\operatorname{Hom}_{\mathbb{Z}}(-,G)$ and take cohomology groups, denoted by
$$\tilde{H}^n(X;G).$$

Then $\tilde{H}^n(X;G)=H^n(X;G)\ \forall_{n\geq 1}$.

If $X\neq\emptyset$ then $\tilde{H}_{-1}(X)=0$ and by UCT
$$\tilde{H}^0(X;G)=\operatorname{Hom}_{\mathbb{Z}}(\tilde{H}_0(X),G).$$

Rk 195: We give a geometric explaination of $H^0(X;G)$ and $\tilde{H}^0(X;G)$ for $X \neq \emptyset$.

$H^0(X;G)$:

$$H^0(X;G) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_0(X), G)$$
$$\simeq \{ f \in \mathrm{Hom}_{\mathbb{Z}}(C_0(X), G) \mid f|_{B_0(X)} = 0 \}$$
$$\simeq \{ f : X \to G \mid f \text{ is constant on path-components} \}$$

$\tilde{H}^0(X;G)$:

$$\tilde{H}^0(X;G) \simeq \mathrm{Hom}_{\mathbb{Z}}(\tilde{H}_0(X), G)$$
$$\simeq \{ f \in \mathrm{Hom}_{\mathbb{Z}}(\tilde{Z}_0(X), G) \mid f|_{B_0(X)} = 0 \}$$
$$\simeq \{ f \in \mathrm{Hom}_{\mathbb{Z}}(C_0(X), G) \mid f|_{B_0(X)} = 0 \} / \sim$$

(with $f \sim g \overset{\text{def.}}{\iff} f - g|_{\tilde{Z}_0} = 0$
$\iff f - g|_X$ is constant)

$$\simeq \{ f : X \to G \mid f \text{ is constant on path-components of } X \} / \text{constant maps.}$$

Note: $\tilde{H}^{-1}(\emptyset, G) \simeq \mathrm{Hom}_{\mathbb{Z}}(\tilde{H}_{-1}(\emptyset), G)$
$\simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \simeq G.$
and
$\tilde{H}^{-1}(X, G) = 0$ for $X \neq \emptyset$.

Def 196: (relative groups)
Given a pair (X, A) we put
$$C^n(X, A; G) := \mathrm{Hom}_{\mathbb{Z}}(C_n(X, A), G)$$
$$\delta^n := \text{dual of } \partial_{n+1} \text{ (of } C_n(X, A))$$

with relative cohomology gp.
$$H^n(X, A; G).$$

Rk 197:

We have

$$0 \to C_n(A) \xrightarrow{\iota} C_n(X) \xrightarrow{q} C_n(X,A) \to 0$$

(split exact)

$\Rightarrow$ We get

$$0 \to C^n(X,A;G) \to C^n(X;G) \to C^n(A;G) \to 0$$

split exact.

$$\Rightarrow \quad C^n(X,A;G) \simeq \{ f \in \mathrm{Hom}_{\mathbb{Z}}(C_n(X),G) \mid f|_{C_n(A)} = 0 \}$$

Rk 198: The short exact $\overset{\text{split}}{\vee}$ seq.

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X,A) \to 0$$

induces a short ex. seq.

$$0 \to C^\bullet(X,A;G) \to C^\bullet(X;G) \to C^\bullet(A;G) \to 0$$

And we get a long exact seq. in cohomology

$$\cdots \to H^n(X,A;G) \to H^n(X;G) \to H^n(A;G) \to H^{n+1}(X,A;G) \to H^{n+1}(X;G) \to H^{n+1}(A,G) \to \cdots$$

Exercise: (a) The connecting map $\partial : H^n(A;G) \to H^{n+1}(X,A;G)$ has the form

$$\partial([\varphi]) = \varphi \circ \partial_{n+1,X}.$$

Note: $Z^n(X,A;G) \simeq \{ f \in C^n(X;G) \mid f \circ \partial_{n+1,X} \in C^n(A;G) \}$

(b) $H^n(X,\emptyset;G) \simeq H^n(X;G)$

$H^n(X,\{x_0\},G) \simeq \tilde{H}^n(X;G)$.