Example 177: We given an example for a map $f: X \longrightarrow Y$ st.

$f_*: \tilde{H}_k(X) \longrightarrow \tilde{H}_k(Y)$ is zero for all $k \in \mathbb{N}_0$
and
$f_*: \tilde{H}_k(X, \mathbb{Z}_m) \longrightarrow \tilde{H}_k(Y, \mathbb{Z}_m)$

is non-zero for some $k, m \in \mathbb{N}$

Take $m \in \mathbb{N}$, $m \geq 2$.
$X := S^n \cup e^{n+1}$
e^{n+1} is attached to S^n via
$q: S^n \longrightarrow S^n$ of degree m.

So we have a cell complex
$X = e^0 \cup e^n \cup e^{n+1}$

(For $n=1$: $X = \mathbb{R}P^2$)

Consider $f: X \longrightarrow X/S^n \simeq S^{n+1}$

$\underline{\mathbb{Z}:}$ Now $\tilde{H}_k(S^{n+1}) \simeq \begin{cases} \mathbb{Z}, & k = n+1 \\ 0, & k \neq n+1 \end{cases}$

The cellular chain complex for X is

$$\underset{\text{degree } n+1}{0 \to \mathbb{Z}} \xrightarrow{m} \mathbb{Z} \to 0 \to \dots \to 0 \to \underset{\text{degree } 0}{\mathbb{Z}} \to 0$$

So $\tilde{H}_k(X) \simeq \begin{cases} \mathbb{Z}/m\mathbb{Z}, & k = n \\ 0, & k \neq n. \end{cases}$

So $f_* : \tilde{H}_k(X) \to \tilde{H}_k(S^{n+1})$ is zero.

<u>$\mathbb{Z}/m\mathbb{Z}$:</u> Using $\mathbb{Z}/m\mathbb{Z}$ coefficients we get

$$\tilde{H}_k(S^{n+1}; \mathbb{Z}/m\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}/m\mathbb{Z}, & k = n+1 \\ 0, & k \neq n+1 \end{cases}$$

and for X:

$$0 \to \mathbb{Z}/m\mathbb{Z} \xrightarrow[\text{0-map}]{m} \mathbb{Z}/m\mathbb{Z} \to 0 \to \dots \to 0 \to \mathbb{Z}/m\mathbb{Z} \to 0$$

So $\tilde{H}_k(X; \mathbb{Z}/m\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/m\mathbb{Z}, & k = n+1, n \\ 0, & \text{else.} \end{cases}$

In the long exact sequence for the quotient we get

$$0 = \tilde{H}_{n+1}(S^n; \mathbb{Z}/m\mathbb{Z}) \to \underbrace{\tilde{H}_{n+1}(X; \mathbb{Z}/m\mathbb{Z})}_{= \mathbb{Z}/m\mathbb{Z}} \xrightarrow{f_*} \underbrace{\tilde{H}_{n+1}(X/S^n; \mathbb{Z}/m\mathbb{Z})}_{= \mathbb{Z}/m\mathbb{Z}}.$$

By exactness f_* is injective, so non-zero on $\tilde{H}_{n+1}(X; \mathbb{Z}/m\mathbb{Z})$.

VI Cohomology

__Motivation 178:__ Given a simplicial complex X with only finitely many simplices we have two ways to define a map between

$$\Delta_n(X) \text{ and } \Delta_{n+1}(X).$$

To explain it we consider $\mathbb{Z}/2\mathbb{Z}$ coefficients.

① (Homology approach)

$$\Delta_{n+1}(X, \mathbb{Z}/2\mathbb{Z}) \to \Delta_n(X, \mathbb{Z}/2\mathbb{Z})$$

An $n+1$-simplex is mapped to the sum of its faces.

$$\sigma \longmapsto \sum_{i=0}^{n+1} \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_{n+1}]}$$

(2) (cohomology approach)

$$\Delta_n(X; \mathbb{Z}/_{2\mathbb{Z}}) \longrightarrow \Delta_{n+1}(X; \mathbb{Z}/_{2\mathbb{Z}})$$

An n-simplex is mapped to the sum of those having it as a face.

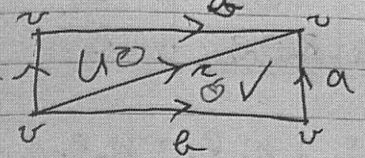
$$\varsigma \longmapsto \sum_{\sigma, \exists i} \sigma$$

$$\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_{n+1}]} = \varsigma$$

This increases the dimension.

This idea also applies to a Δ-complex and for $\mathbb{Z}$ coefficients we introduce signs $(-1)^i$.

Example 179: $X = T$



orientation by characteristic map (not an orientation of T!)

Δ-structure: $e_v, e_a, e_b, e_c, e_U, e_V$

with characteristic maps $\sigma_v, \sigma_a, \sigma_b, \sigma_c, \sigma_U, \sigma_V$

$$\sigma_U|_{[e_0, e_1]} = \sigma_a = \sigma_V|_{[e_1, e_2]}$$

$$\sigma_U|_{[e_0, e_2]} = \sigma_c = \sigma_V|_{[e_0, e_2]}$$

$$\sigma_U|_{[e_1, e_2]} = \sigma_b = \sigma_V|_{[e_0, e_1]}$$

$$e_U, e_V \longmapsto e_a + e_b - e_c$$

Chain complex

$$\cdot \xrightarrow{\partial_2} \cdot \xrightarrow{\partial_1 = 0} \cdot \xrightarrow{\partial_0} \cdot$$

$$0 \quad \mathbb{Z}e_U \oplus \mathbb{Z}e_V \xleftarrow{\partial^1} \mathbb{Z}e_a \oplus \mathbb{Z}e_b \oplus \mathbb{Z}e_c \xleftarrow{\partial^0 = 0} \mathbb{Z}e_v \xleftarrow{0} 0$$

$$e_U + e_V \longleftarrow\!\shortmid e_a, e_b$$

$$-e_U - e_V \longleftarrow\!\shortmid e_c$$

degree	2	1	0
$H_i^{\Delta}(T)$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$
$H^i_{\Delta}(T)$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$

$$\ker(\partial^1) = \{ n_a e_a + n_b e_b + n_c e_c \mid n_a + n_b - n_c = 0 \}$$

$$= \langle e_a + e_c , e_a - e_b \rangle$$

VI 1. Simplicial cochain complex and examples

Remark 180: How do we generalize the idea of 178 (2) to a Δ-complex with infinite branching.

E.g. $X :=$ 

∞-many

$$= \bigcup_{i=1}^{\infty} [P, Q_i]$$

Then $\partial^0(P) = \sum_{i=1}^{\infty} (-]P, Q_i[)$

a sum with ∞-many non-zero terms.

So instead of a direct sum the module of 1-cochains should be a direct product.

$\prod_{i=1}^{\infty} \mathbb{Z}$ or equivalently

$\mathrm{Map}(\mathbb{N}_0, \mathbb{Z})$ or equivalently

$\mathrm{Hom}_{\mathbb{Z}}(\Delta_1(X), \mathbb{Z})$.

We obtain a similar observation when we consider coefficients in G i.e. for the given X:

$$\prod_{i=1}^{\infty} G \simeq \mathrm{Hom}_{\mathbb{Z}}(\Delta_1(X), G)$$

$$\simeq \mathrm{Hom}_{G}(\Delta_1(X;G), G).$$

(G an abelian group).

Def 181: (simplicial cochain complex)

Let X be a Δ-complex and G be an abelian group. The simplicial cochain complex for X (with respect to the Δ-structure) is defined as follows:

$$\Delta^i(X;G) := \mathrm{Hom}_{\mathbb{Z}}(\Delta_i(X), G), \; i \geq 0$$

$$\Delta^i(X;G) = 0 \quad \forall_i < 0.$$

$$\partial^i : \Delta^i(X;G) \longrightarrow \Delta^{i+1}(X;G)$$

$$\psi$$

$$\partial^i(\psi)(\sigma) := \sum_{j=0}^{i+1} (-1)^j \psi(\sigma|_{[v_0, \dots, \hat{v}_j, \dots, v_{i+1}]})$$

Exercise 182: Check that the ∂^1 of Def 181 coincides with ∂^1 in Example 179 after replacing $\{e_a, e_b, e_c\}$ and $\{e_u, e_v\}$ with the dual bases.

<u>Def 183:</u> $H^i_\Delta(X;G) := \ker \partial^i \Big/ \operatorname{im} \partial^{i-1}$

$=$ "ith cocycles" / "ith coboundaries"

is called the <u>ith simplicial cohomology group with coefficients in G.</u>

<u>Example 184:</u> Cohomology groups for 1-dim Δ-complexes.

Let X be a 1-dim Δ-complex, i.e. an oriented graph (V, E)

$$\Delta^0(X;G) = \mathrm{Map}(V, G)$$

$$\Delta^1(X;G) = \mathrm{Map}(E, G)$$

$$0 \longrightarrow \Delta^0(X;G) \xrightarrow{\partial^0} \Delta^1(X;G) \longrightarrow 0$$

$H^0_\Delta(X;G)$: For $\varphi \in \Delta^0(X;G)$

we have $\partial^0\varphi = 0 \iff \underset{\substack{e \in E \\ \| \\ \overrightarrow{PQ}}}{\forall} : \varphi(Q) - \varphi(P) = 0$

$\iff$ φ is constant along each simplicial path

$\iff$ φ is constant on each path-connected component.

Thus $H^0_\Delta(X;G) \simeq \prod_{\substack{\text{path-connected} \\ \text{component} \\ \text{of } X}} G$

$H^1_\Delta(X;G)$: Suppose X is connected,

Let $T \subseteq X$ be a spanning tree, i.e.

$V_T = \{\text{vertices in } T\} = V$

and

$E_T = \{e \in E \mid e \subseteq T\}$

does not enable a closed path (ignoring the orientation) and T is connected.

Then $H^1_\Delta(T; G) = 0$.

(<u>Proof</u>: $\psi \in \Delta^1(T; G)$.

Fix $P \in V_T$, put $\varphi(P) = 0$ and define φ on $Q \in V_T = V$ along the path from P to Q using ψ, to obtain $\partial^0 \varphi = \psi$ □)

<u>Claim</u>: $$H^1_\Delta(X; G) \simeq \prod_{e \in E \setminus E_T} G.$$

<u>Proof</u>: We construct the map

$[\psi] \in H^1_\Delta(X; G)$

Take $\varphi_\psi \in \Delta^0(X, G)$ s.t. $\partial^0_T \varphi_\psi = \psi|_{E_T}$.

and we define $H^1_\Delta(X, G) \longrightarrow \prod_{e \in E \setminus E_T} G$

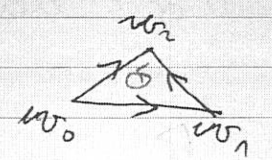
via $[\psi] \longmapsto (\psi(e) - \partial^0 \varphi_\psi(e))_{e \in E \setminus E_T}$.

This map is a group isomorphism (exercise). □

<u>Example 185</u>: Cohomology groups for 2-dim. Δ-complexes.

$X = (V, E, F)$ — oriented faces; vertices; oriented edges

<u>ex</u>:



The orientation of the face is not consistent with orientation on $[w_0, w_2]$

$$0 \to \Delta^0(X; G) \xrightarrow{\partial^0} \Delta^1(X; G) \xrightarrow{\partial^1} \Delta^2(X; G) \to 0$$

$$H^1_\Delta(X; G) = \ker(\partial^1) / \mathrm{im}(\partial^0).$$

$$\partial^1(\psi)([w_0, w_1, w_2]) = \psi([w_1, w_2]) - \psi([w_0, w_2]) + \psi([w_0, w_1])$$