Examples 170:

$\chi(K \times S^1) = 1-2+1 = 0$

$\chi(\mathbb{R}P^n) = \begin{cases} 1, & 2 \mid n \\ 0, & 2 \nmid n \end{cases}$

$\chi(M_g) = 1 - 2g + 1 = 2 - 2g$

$\chi(N_g) = 1 - g + 1 = 2 - g.$

using cells.

V 9. Mayer-Vietoris sequence

So far we had the following long exact sequences.

(I) For the quotient X/A where
(126) (X,A) is a good pair and $A \neq \emptyset$.

(II) For the pair (X,A)
(131) using $H_*(A)$, $H_*(X)$ and $H_*(X,A)$

(III) For the pair (X,A) using
(133) $\tilde{H}_*(A)$, $\tilde{H}_*(X)$ and $H_*(X,A)$.

(IV) For a triple (X,A,B) using rel. homology groups

The idea of excision gives another one.

Thm 171: (M-V) Suppose $A,B \subseteq X$ and $\mathring{A} \cup \mathring{B} = X$. Then we get a long exact sequence:

$$\hookrightarrow H_n(A\cap B) \longrightarrow H_n(A)\oplus H_n(B) \longrightarrow H_n(X) \xrightarrow{\delta_n}$$
$$\hookrightarrow \cdots \to H_0(A\cap B) \to H_0(A)\oplus H_0(B) \to H_0(X) \to 0$$

and

$$\hookrightarrow \tilde H_n(A\cap B) \to \tilde H_n(A)\oplus \tilde H_n(B) \to \tilde H_n(X)$$
$$\hookrightarrow \cdots \to \tilde H_0(A\cap B) \to \tilde H_0(A)\oplus \tilde H_0(B) \to \tilde H_0(X) \to 0$$

Proof: The sequence

$$0 \to C_\bullet(A\cap B) \xrightarrow{\varphi} C_\bullet(A)\oplus C_\bullet(B) \xrightarrow{\psi} C_\bullet^{\{A,B\}}(X) \to 0$$

with $\varphi(x) := (x, -x)$

$\psi(x_A, x_B) := x_A + x_B$.

is exact. By Prop 139 the map $C_\bullet^{\{A,B\}}(X) \hookrightarrow C_\bullet(X)$ is a chain homotopy equivalence. This provides the first long exact sequence.

For the second l.e.s. use the augmented chain complexes. □

Remark 172: The connecting morphism $\delta_n : H_n(X) \longrightarrow H_{n-1}(A\cap B)$ is given by $(n>0)$:

$$\delta_n([x_A + x_B]) = [\partial_n x_A] = [\partial_n(-x_B)]$$

Example 173: $(n\in\mathbb{N}_0)$ $X = S^n$

$A =$ open nbhd of upper hemisphere

$B =$ —"— lower hemisphere

We get

$$0\oplus 0 \to \tilde H_k(S^n)$$
$$\hookrightarrow \tilde H_{k-1}(S^{n-1}) \to 0\oplus 0 \to \tilde H_{k-1}(S^n)$$
$$\hookrightarrow \cdots \to \tilde H_0(S^{n-1}) \to 0\oplus 0 \to \tilde H_0(S^n) \to \begin{cases} 0, & n>0 \\ \mathbb{Z}, & n=0 \end{cases}$$

for all $k\in\mathbb{N}$.

So by induction, for $k, n \in \mathbb{N}_0$,

$$\tilde{H}_k(S^n) = \begin{cases} 0, & k \neq n \\ \mathbb{Z}, & k = n \end{cases}$$

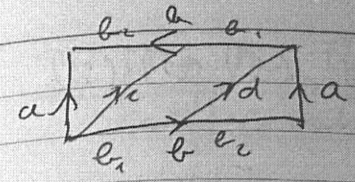
Remark 174: If (X, A) and (X, B) are good s.t. there are def. retracting nbhds $U \supseteq A$, $V \supseteq B$ s.t. $U \cap V$ def. retr. to $A \cap B \neq \emptyset$, then, if $A \cup B = X$, then we have an exact seq:

$$\hookrightarrow \tilde{H}_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \to H_n(X) \hookrightarrow \dots$$

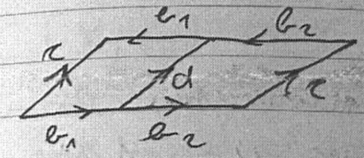
We call this condition: $(X, A, B, A \cap B)$ is good.

Example 175:

K Kleinian bottle is two Möbius strips glued at their boundary.



We cut along c to get



and  is a Möbius strip. Call them M_1, M_2.

$$\hookrightarrow \tilde{H}_k(S^1) \xrightarrow{\alpha_k} \tilde{H}_k(M_1) \oplus \tilde{H}_k(M_2) \xrightarrow{\beta_k} \tilde{H}_k(K) \hookrightarrow \tilde{H}_{k-1}(S^1)$$

for $k > 0$. Note M_i is homotopic to S^1.
So except for $k \in \{1, 2\}$ we get $\tilde{H}_k(K) = 0$ immediately.

$$\tilde{H}_1(S^1) \xrightarrow{\partial_1} \tilde{H}_1(S^1) \oplus \tilde{H}_1(S^1)$$

has the form

$$1 \longmapsto (2,-2)$$

(One time around the boundary induces 2 times around the core circle.)

So ∂_1 is injective and

$$\operatorname{coker}(\partial_1) \simeq \mathbb{Z}\oplus\mathbb{Z}/\mathbb{Z}(2,-2) \simeq \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}.$$

So $\tilde{H}_1(K) = \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ and

$\tilde{H}_2(K) = 0.$

V 10. Homology with coefficients.

Let G be an abelian group.

$$C_\cdot(X;G) := C_\cdot(X)\otimes_{\mathbb{Z}} G.$$

(We look at chains with coefficients in G.

$$\sum n_i\sigma_i \qquad n_i \in G.\ )$$

$C_\cdot(X;G)$ is called the <u>singular chain complex with coefficients in G.</u>

Analogously $C_\cdot(X,A;G) = C_\cdot(X;G)/C_\cdot(A;G)$

(Note: $C_\cdot(X,A;G) \simeq C_\cdot(X,A)\otimes_{\mathbb{Z}} G$ because

$$0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$$

splits for each $n\in\mathbb{N}_0$)

All the results from V.1 to V.9. carry over to homology with coefficients.

excision, l.e.s., M-V, ..

For the cellular complex we have

$$C_n^{CW}(X,G) = H_n(X^n, X^{n-1}) \otimes_{\mathbb{Z}} G$$

because we have for $n>0$:

$$H_n(X^n, X^{n-1}; G) \simeq \tilde{H}_n(X^n/X^{n-1}; G)$$

$$\simeq \tilde{H}_n\Big(\bigsqcup_{d_\alpha=n} D_\alpha \Big/ \bigsqcup_{d_\alpha=n} \partial D_\alpha ; G\Big)$$

$$= \bigoplus_{d_\alpha=n} \tilde{H}_n(D_\alpha/\partial D_\alpha ; G)$$

$$\simeq \bigoplus_{d_\alpha=n} H_n(D_\alpha, \partial D_\alpha ; G)$$

$$= \bigoplus_{d_\alpha=n} G\tilde{e}_\alpha \simeq \Big(\bigoplus_{d_\alpha=n} \mathbb{Z}\tilde{e}_\alpha\Big) \otimes_{\mathbb{Z}} G$$

Here we used that

$$H_n(\Delta_\alpha, \partial\Delta_\alpha ; G) = G[\mathrm{id}_{\Delta_\alpha}].$$

<u>Claim:</u>
The boundary maps have the form

$$d_n(n_\alpha e_\alpha) = \sum d_{\alpha\beta} \cdot n_\alpha e_\beta$$

$(n_\alpha \in G.)$

where $d_{\alpha\beta}$ is the degree of $\Delta_{\alpha\beta} : \partial D_\alpha \longrightarrow D_\beta/\partial D_\beta$.

<u>Proof:</u> Let $f : S^n \longrightarrow S^n$ of degree m.
Then we claim that

$$f_* : H_n(S^n; G) \to H_n(S^n; G)$$

is the multiplication with m.

Take $g \in G$. Consider

$$\begin{array}{ccc} [c] & \longmapsto & [mc] \\ \mathbb{Z}[c] = H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \\ \downarrow \wr \quad [c] \mapsto [gc] \quad \circlearrowleft G \quad \downarrow g & \circlearrowleft & \downarrow g \\ g\mathbb{Z}[c] \subseteq H_n(S^n;G) & \xrightarrow{f_*} & H_n(S^n;G) \quad [gmc] = m[gc] \end{array}$$

$$H_n(S^n;G) \cong H_n(D^n, \partial D^n; G) \cong G \quad (n>0)$$

(Note: that $f_*: H_n(S^n;G) \to H_n(S^n;G)$ is a G-module homomorphism. So the big square is commutative.)

$\square$

Example 176: In general we don't have

$$H_n(X;G) \simeq H_n(X) \otimes_{\mathbb{Z}} G$$

$X := \mathbb{R}P^2$.

$C^{CW}_\bullet(X): \quad 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

$$C^{CW}_\bullet(X;\mathbb{Z}/2\mathbb{Z}): 0 \to \underbrace{\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}}_{\cong\ \mathbb{Z}/2} \xrightarrow{2} \underbrace{\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2}_{\cong\ \mathbb{Z}/2} \xrightarrow{0} \mathbb{Z}/2 \to 0$$

$$\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/2$$

Here $d_2 \otimes_{\mathbb{Z}} \mathbb{Z}/2$ is 0.

So $H_2(\mathbb{R}P^2; \mathbb{Z}/2) \simeq \mathbb{Z}/2$.

and $H_2(\mathbb{R}P^2) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \simeq 0 \otimes_{\mathbb{Z}} \mathbb{Z}/2 = 0$.

For the relation between $H_n(X;G)$ and $H_n(X) \otimes_{\mathbb{Z}} G$ see the Universal coefficient Theorem (later).