<u>Remark 159:</u> ($n>0$) (a) Suppose $f,g: S^n \to S^n$ are homotopic. Then $\deg f = \deg g$.

(b) Suppose $f,g: S^n \to S^n$ satisfy $\deg f = \deg g$. Then f is homotopic to g. (Theorem by Hopf)

<u>Proof:</u> (b) later.

(a) $f \sim g \Rightarrow f_* = g_*$ on $H_n(S^n)$

$\Rightarrow \deg f = \deg g$. $\square$

<u>Examples 160:</u> (a) $f: S^1 \to S^1$, $f(x) := x^k$ $(k \in \mathbb{Z})$

Then $\deg f = k$.

$k=1$: $f = id_{S^1} \Rightarrow \deg f = 1$

$k=-1$: f = reflection about x-axis$|_{S^1}$

$\Rightarrow \deg(f) = -1$.

$k=0$: $f \equiv 1$.

$H_1(S^1) \cong \mathbb{Z}(\Delta^1_+ - \Delta^1_-)$

$\Rightarrow f_\#(\Delta^1_+ - \Delta^1_-) = 1 - 1 = 0$

$\Rightarrow f_*: H_1(S^1) \to H_1(S^1)$ is the zero map.

$\Rightarrow \deg f = 0$.

<u>$k>1$:</u> $y_0 = 1$. $f^{-1}(1) = \{1, \zeta, \zeta^2, \ldots, \zeta^{k-1}\}$

$\zeta = e^{\frac{2\pi i}{k}}$.

Locally around $x_i := \zeta^i$ f looks like a rotation followed by a stretching around 1. Both have degree 1, so

$\deg f|_{x_i} = 1$.

$168 \Rightarrow \deg(f) = \sum_{i=0}^{k-1} \deg f|_{x_i} = k$

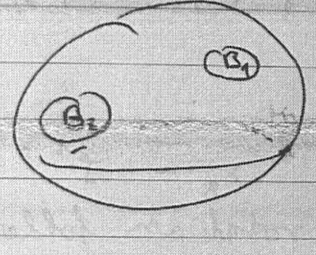
<u>$k<-1$:</u> $f_k = f_{-k} \circ f_{-1}$

$\Rightarrow \deg(f_k) = \deg(f_{-k}) \cdot \deg(f_{-1})$

$= (-k) \cdot (-1) = k$.

(iv) Let $d \in \mathbb{Z}$. Then
$\exists f: S^n \to S^n$ continuous
with $\deg(f) = d$.

To see the strategy, let us
construct
$$f: S^2 \longrightarrow S^2$$
of degree 2.



B_1, B_2 open
balls in S^2
(of same radius)
$x_i :=$ center of B_i

s.t. $\overline{B_1} \cap \overline{B_2} = \emptyset$.

(B_i is the intersection of a ball
in $\mathbb{R}^3$ with S^2 with center
in S^2)

$$S^2 \xrightarrow{q} S^2 / S^2 \setminus (B_1 \cup B_2)$$

$$= \overline{B_1} / \partial \overline{B_1} \vee \overline{B_2} / \partial \overline{B_2} \xrightarrow{r} \overline{B_1} / \partial \overline{B_1}$$
$$\cong S^2 \vee S^2$$
$$\xrightarrow[\sim]{\varphi} S^2$$

Here $\cdot$ $r|_{B_1} = id_{B_1}$

$\cdot$ $r|_{B_2}$ is given by a
rotation of S^2 which
maps B_2 onto B_1.

$\cdot$ φ is chosen to be id
near x_1.

$f := \varphi \circ r \circ q$

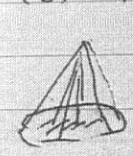
$f^{-1}(x_1) = \{x_1, x_2\}$.
$\deg f|_{x_1} = \deg id|_{x_1} \overset{158}{=} \deg id_{S^2} = 1$.

$\deg f|_{x_2} = \deg r|_{x_2} \underset{\uparrow}{=} \deg R|_{x_2} = \deg R = 1$.
r is restr. of a rotation R
near x_2.

So $\deg f = 2$.

(Note: $\deg R = 1$ as R is a composition of two reflections.)

(c) Recall for X:



$$CX = X \times [0,1] \big/ X \times \{1\} \quad \text{cone of } X$$

$$SX := CX \big/ X \times \{0\} \quad \text{suspension of } X$$

Then for $f: S^n \to S^n$ cont. we have

$$\deg f = \deg(Sf)$$

for the suspension $Sf : \underbrace{SS^n}_{S^{n+1}} \to \underbrace{SS^n}_{S^{n+1}}$

<u>Proof:</u> Consider the long exact seq. for $(C.S^n, S^n)$

$$\Rightarrow \begin{array}{ccc} H_{n+1}(S^{n+1}) & \xrightarrow[\cong]{\partial} & H_n(S^n) \\ {\scriptstyle Sf_*}\downarrow & \circlearrowleft & \downarrow{\scriptstyle f_*} \\ H_{n+1}(S^{n+1}) & \xrightarrow[\cong]{\partial} & H_n(S^n) \end{array}$$

$\square$

V.8. Cellular homology groups

The idea is to compute singular homology groups using a CW structure.

Given a CW structure we construct a chain complex:

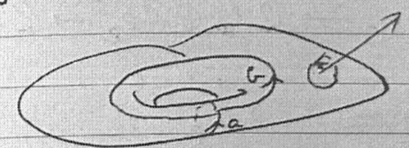
$$C_n^{cell}(X) := \bigoplus_{d_\alpha = n} \mathbb{Z} e_\alpha$$

The problem is to define the boundary maps.
For Δ-complexes we used an order of the set of vertexes of the standard simplex inherited to the faces.
For cell complexes this is more complicated. The main idea is to use the degree of a map.

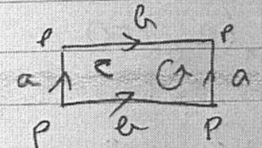
Before we give the rigorous definition we start with intuitive examples.

Example 161:

(a)



(right hand rule)



cells: e_p, e_a, e_b, e_c.

$$0 \to \mathbb{Z}e_c \xrightarrow{d_2} \mathbb{Z}e_a \oplus \mathbb{Z}e_b \xrightarrow{d_1} \mathbb{Z}e_p \xrightarrow{0} 0$$

$$d_2(e_c) = e_a - e_b - e_a + e_b = 0$$

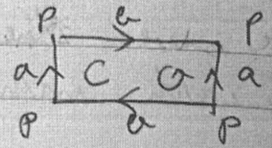
$$d_1(e_a) = d_1(e_b) = e_p - e_p = 0.$$

So $H_2^{cell}(T) = \mathbb{Z}$

$H_1^{cell}(T) = \mathbb{Z} \oplus \mathbb{Z}$

$H_0^{cell}(T) = \mathbb{Z}$.

(b) K



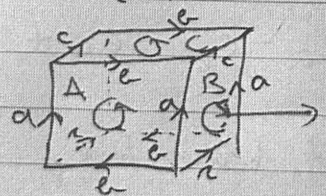
$$0 \to \mathbb{Z}e_C \xrightarrow{d_2} \mathbb{Z}e_a \oplus \mathbb{Z}e_b \xrightarrow{d_1} \mathbb{Z}e_p \xrightarrow{0} 0$$

$d_2(e_C) = -2e_b$

$d_1(e_a) = d_1(e_b) = 0.$

$H_2^{cell}(K) = 0$
$H_1^{cell}(K) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
$H_0^{cell}(K) = \mathbb{Z}.$

(c) $X = K \times S^1$



Q is the interior of the cube.
cells: $e_p, e_a, e_b, e_c, e_A, e_B, e_C$
and e_Q

$$0 \to \mathbb{Z}e_Q \xrightarrow{d_3} \mathbb{Z}e_A \oplus \mathbb{Z}e_B \oplus \mathbb{Z}e_C \xrightarrow{d_2} \mathbb{Z}e_a \oplus \mathbb{Z}e_b \oplus \mathbb{Z}e_c \xrightarrow{d_1} \mathbb{Z}e_p \xrightarrow{0} 0$$

$d_3(e_Q) = e_A + e_B + e_C - e_A - e_B + e_C$
$= 2e_C$

$d_2(e_A) = -2e_b$

$d_2(e_B) = d_2(e_C) = 0$

$d_1(e_a) = d_1(e_b) = d_1(e_c) = 0.$

So $H_3^{cell}(X) = 0$
$H_2^{cell}(X) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$H_1^{cell}(X) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$
$H_0^{cell}(X) = \mathbb{Z}.$

Check with Künneth formula that $H_\cdot^{cell}(X) \cong H_\cdot(X)$ for $X = K \times S^1$.

We now work with singular homology groups to construct $C_\bullet^{cell}(X)$.

<u>Lemma 162:</u> Let X be a CW complex. Then

(a) · $\forall_{n \in \mathbb{N}_0}$: $H_n(X^n, X^{n-1}) \simeq \bigoplus_{d_\alpha = n} \mathbb{Z} e_\alpha$.

· $\forall_{n \in \mathbb{N}_0} \forall_{k \neq n}$: $H_k(X^n, X^{n-1}) = 0$.

(b) $\forall_{k > n \geq 0}$ $H_k(X^n) = 0$

(c) $\forall_{0 \leq k < n}$: $i_*: H_k(X^n) \xrightarrow{\sim} H_k(X)$.

<u>Proof:</u> (a) Note: $X^{-1} := \emptyset$.

(X^n, X^{n-1}) is a good pair, so we get

$$H_k(X^n, X^{n-1}) \simeq H_k\left(X^n / X^{n-1}\right)$$

$$\simeq H_k\left(\bigsqcup_{d_\alpha = n} D_\alpha^n \Big/ \bigsqcup_{d_\alpha = n} \partial D_\alpha^n \right)$$

$$\underset{\text{good pair}}{\simeq} H_k\left(\bigsqcup_{d_\alpha = n} D_\alpha^n, \bigsqcup_{d_\alpha = n} \partial D_\alpha^n \right)$$

$$\simeq \bigoplus_{d_\alpha = n} H_k(D_\alpha^n, \partial D_\alpha^n)$$

$$\simeq \begin{cases} \bigoplus_{\substack{\alpha \in A \\ d_\alpha = n}} \mathbb{Z}, & k = n \\ 0, & k \neq n \end{cases}$$

(b) Consider the long exact sequence for (X^n, X^{n-1})

We get

$$H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1}).$$

By (a), for $k > n$, we get $H_k(X^{n-1}) \simeq H_k(X^n)$.

So for $k > n \geq 0$:

$$H_k(X^n) \simeq H_k(X^{n-1}) \simeq H_k(X^{n-2})$$
$$\simeq \dots \simeq H_k(X^0) \underset{\uparrow}{=} 0.$$

$k > 0$. and X^0 is just a discrete set.

(c) <u>Case</u>: $\dim X < \infty$, say $X = X^m$:

Then for $0 \leq k < n$: We use the long exact sequence for (X^{n+1}, X^n) to obtain

$$H_k(X^n) \simeq H_k(X^{n+1}) \simeq \dots \simeq H_k(X^m)$$

$$= H_k(X).$$

<u>Case</u>: $\dim X = \infty$:

To show $i_*: H_k(X^n) \xrightarrow{\simeq} H_k(X)$ for $k < n$.

surjective: Take $[c] \in H_k(X)$.

By compactness of "$\operatorname{im}(c)$": $c \in C_k(X^m)$ for m big enough.

$\Rightarrow$ $[c]$ is in the image of the map

$$H_k(X^n) \xrightarrow{\sim} H_k(X^m) \longrightarrow H_k(X).$$

injectivity: exercise. $\square$

<u>Def 163</u>: (cellular chain complex of a CW complex)

$$\begin{array}{c} C^{CW}_{n+1}(X) \\ \overset{\text{def}}{=} \end{array}$$

$$\to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2})$$

$$\dots \xrightarrow{d_2} H_1(X^1, X^0) \xrightarrow{d_1} H_0(X^0) \xrightarrow{0} 0$$

Where d_n is given through the l.e.s. for (X^n, X^{n-1}) and (X^{n-1}, X^{n-2}).

$$\begin{tikzcd}
 & H_{n-1}(X^{n-1}) \arrow[dr, "j_{n-1}"] & \\
H_n(X^n, X^{n-1}) \arrow[ur, "\partial_n"] \arrow[rr, "d_n"'] \arrow[rr, phantom, shift left=3, "\circlearrowleft"] & & H_{n-1}(X^{n-1}, X^{n-2})
\end{tikzcd}$$

$$d_n := j_{n-1} \circ \partial_n.$$

<u>Note:</u> $d_{n-1} \circ d_n = j_{n-2} \circ \underbrace{\partial_{n-1} \circ j_{n-1}}_{=0} \circ \partial_n$

$$= 0.$$

We need to know how to compute d_n in terms of cells.

<u>Remark 164:</u> $d_1 = \partial_1$, as

$$\begin{tikzcd}
 & H_0(X^0) \arrow[d, equal] \\
H_1(X^1, X^0) \arrow[ur, "\partial_1"] \arrow[r, "d_1"'] & H_0(X^0, \emptyset) \arrow[d, equal] \\
 & H_0(X^0)
\end{tikzcd}$$

<u>Theorem 165:</u> For the homology groups of $C_\bullet^{CW}(X)$ we have

$$H_n^{CW}(X) \simeq H_n(X)$$

for all $n \in \mathbb{N}_0$.

<u>Proof:</u>

<u>$n=0$:</u>

$$\begin{tikzcd}
 & & H_0(X^1) \arrow[r] \arrow[d, equal] & H_0(X^1, X^0) \arrow[d, equal] \\
 & H_0(X^0) \arrow[ur] \arrow[d, equal] & H_0(X) & 0 \\
H_1(X^1, X^0) \arrow[ur, "\partial_1"] \arrow[r, "d_1"'] \arrow[r, phantom, shift left=3, "\circlearrowleft"] & H_0(X^0, \emptyset) \arrow[r] & 0 &
\end{tikzcd}$$

$$\Rightarrow \quad H_0(X) \simeq H_0(X^0) / \mathrm{im}(\partial_1)$$

$$= H_0^{CW}(X).$$

<u>$n>0$:</u>

$$\begin{tikzcd}
 & & H_n(X^{n+1}) \arrow[r] \arrow[d, equal] & 0 \\
 & H_n(X^n) \arrow[ur] \arrow[d, "j_n"] & H_n(X) & \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[r, "d_{n+1}"] & H_n(X^n, X^{n-1}) \arrow[r, "d_n"] \arrow[dr, "\partial_n"'] & H_{n-1}(X^{n-1}, X^{n-2}) & \\
 & & H_{n-1}(X^{n-1}) \arrow[u, "j_{n-1}"'] &
\end{tikzcd}$$

So $\ker(d_n) = \ker(\partial_n) = \mathrm{im}(j_n)$

and $j_n(\mathrm{im}(\partial_{n+1})) = \mathrm{im}(d_{n+1})$.

$$\Rightarrow \quad j_n(H_n(X^n)) / j_n(\mathrm{im}(\partial_{n+1}))$$

$$= \ker(d_n) / \mathrm{im}(d_{n+1}) = H_n^{CW}(X)$$

Further $H_n(X^n) / \mathrm{im}(\partial_{n+1}) \simeq H_n(X)$.

$\square$

<u>Recall</u>: $\forall_\alpha \in \mathcal{A}$ we have

- attaching map
  $\varphi_\alpha : \partial D_\alpha \longrightarrow X^{n-1} \subseteq X$

  $(d_\alpha = n)$
- characteristic map
  $\Phi_\alpha : D_\alpha \longrightarrow X^n$

$$e_\alpha := \Phi_\alpha(\mathring{D}_\alpha)$$

Choose a generator of $H_n(D_\alpha, \partial D_\alpha)$ for $(n > 0)$:

$$\sigma_\alpha : \Delta^n \xrightarrow{\sim} D_\alpha$$

Then $[\sigma_\alpha] \in H_n(D_\alpha, \partial D_\alpha)$ is a generator

We have

$$\begin{array}{ccccc} [q\sigma_\alpha] & & [\sigma_\alpha] & & [\partial\sigma_\alpha] \\ H_n(D_\alpha/\partial D_\alpha) & \xleftarrow[\sim]{q_\alpha} & H_n(D_\alpha, \partial D_\alpha) & \xrightarrow[\sim]{\partial_n} & H_{n-1}(\partial D_\alpha) \\ & & \downarrow \Phi_{\alpha *} & & \\ & & H_n(X^n, X^{n-1}) \ni e_\alpha := \Phi_{\alpha *}([\sigma_\alpha]) & & \end{array}$$

(fix $D_\alpha/\partial D_\alpha \simeq S^n =: S_\alpha^n$)

Because of

$$H_n(X^n, X^{n-1}) \simeq \bigoplus_{\substack{\alpha \in \mathcal{A} \\ d_\alpha = n}} H_n(D_\alpha, \partial D_\alpha)$$

we also write $\tilde{e}_\alpha$ for $[\sigma_\alpha]$.

$$\begin{array}{ccccc} \tilde{q}_*(\tilde{e}_\alpha) & & \tilde{e}_\alpha & & \partial\tilde{e}_\alpha \\ H_n(S_\alpha) & \xleftarrow{\tilde{q}_*} & H_n(D_\alpha, \partial D_\alpha) & \longrightarrow & H_n(\partial D_\alpha) \\ & & \downarrow \Phi_{\alpha *} & & \\ & & H_n(X^n, X^{n-1}) \ni e_\alpha & & \end{array}$$

<u>Prop. 166:</u> (cellular boundary formula)

(a) $(n=1)$ For $d_\alpha = 1$ and $\Phi_\alpha \circ \sigma_\alpha : [v_0, v_1] \longrightarrow \overset{\bullet}{P} \longrightarrow \overset{\bullet}{Q} \subset X^1$

we have
$$d_1(e_\alpha) = e_Q - e_P$$

(b) $(n>1)$. For $d_\alpha = n$ we have
$$d_n(e_\alpha) = \sum_{\substack{\beta \in A \\ d_\beta = n-1}} d_{\alpha\beta}\, e_\beta$$

where $d_{\alpha\beta}$ is the degree of the map

$$\Delta_{\alpha\beta} : \partial D_\alpha \longrightarrow X^{n-1} \longrightarrow X^{n-1} / X^{n-1} \setminus e_\beta = S^{n-1}_\beta$$

with respect to the generators

$\partial \tilde{e}_\alpha$ and $\tilde{q}_*(\tilde{e}_\beta)$.

<u>Proof:</u> (a) follows from $d_1 = \partial_1$.

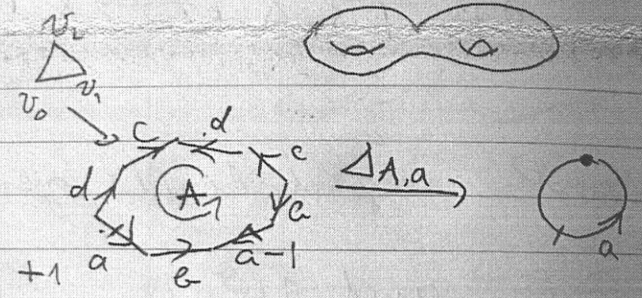
(b) $n>1$:

$$\begin{array}{ccccc} \tilde{e}_\alpha & & \partial_n \tilde{e}_\alpha & \longmapsto & d_{\alpha\beta}\tilde{q}_*(\tilde{e}_\beta) \\ H_n(D_\alpha, \partial D_\alpha) & \xrightarrow[\sim]{\partial_n} & H_{n-1}(\partial D_\alpha) & \xrightarrow{\Delta_{\alpha\beta *}} & H_{n-1}(S_\beta) \\ \downarrow \Phi_{\alpha *} & \circlearrowleft & \downarrow \varphi_{\alpha *} & \circlearrowleft & \uparrow q_{\beta *} \\ H_n(X^n, X^{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(X^{n-1}) & \xrightarrow{q_*} & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \\ e_\alpha & \searrow^{d_n} & \circlearrowleft \quad \downarrow j_{n-1} & \circlearrowleft & \nearrow_{\sim}\, q_* \\ & & H_{n-1}(X^{n-1}, X^{n-2}) & & \\ & & \sum d_{\alpha\gamma} e_\gamma & & \end{array}$$

$\square$

Examples 167:

(a) M_g: $(g \geq 1)$. Consider M_2:



So $\deg(\Delta_{A,a}) = 0$.

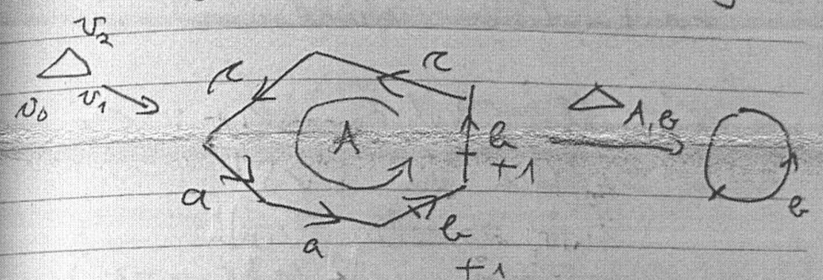
Thus, by Prop 166, we have $d_2 = 0$.

We get

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2 = 0} \mathbb{Z}^{2g} \xrightarrow{d_1 = 0} \mathbb{Z} \longrightarrow 0$$

So
$$H_i^{CW}(M_g) = \begin{cases} 0, & i \geq 3 \\ \mathbb{Z}, & i = 0, 2 \\ \mathbb{Z}^{2g}, & i = 1. \end{cases}$$

(b) N_g $(g \geq 1)$ Like for N_3:



So $\deg(\Delta_{A,b}) = 2$.

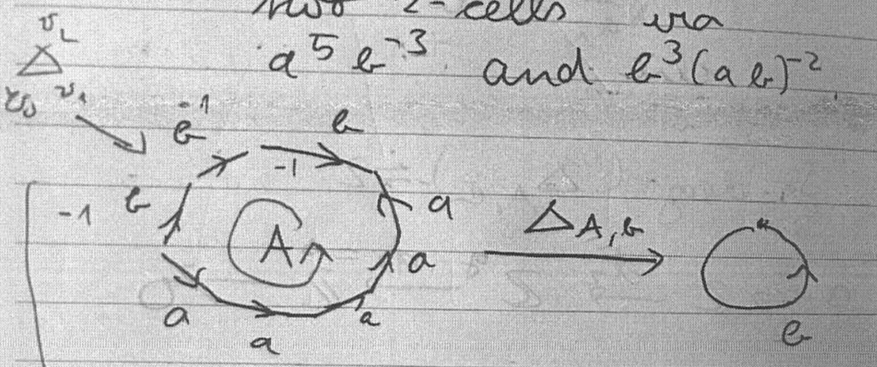
$$0 \rightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^g \xrightarrow{d_1 = 0} \mathbb{Z} \rightarrow 0$$

$$d_2(z) = (2z, 2z, \dots, 2z).$$

So
$$H_i^{CW}(N_g) = \begin{cases} 0, & i \geq 2 \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}, & i = 1 \\ \mathbb{Z}, & i = 0. \end{cases}$$

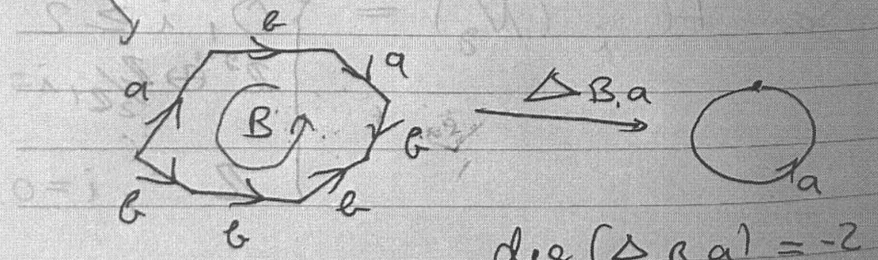
(c) An acyclic, but non-contractible space.

X is the CW complex: attach on $S^1 \vee S^1$ two 2-cells via $a^5 b^{-3}$ and $b^3 (ab)^{-2}$.



$\deg(\Delta_{A,b}) = -3$

$\deg(\Delta_{A,a}) = 5$



$\deg(\Delta_{B,a}) = -2$

$\deg(\Delta_{B,b}) = 1$

$$0 \to \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1 = 0} \mathbb{Z} \to 0$$

$$d_2 \triangleq \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$$

$\det \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} = -1 \in \mathbb{Z}^*$, so d_2 is invertible over $\mathbb{Z}$.

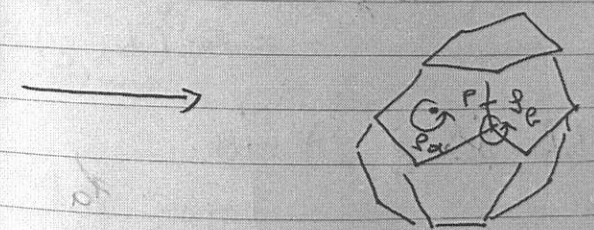
$$\Rightarrow \quad H_i^{CW}(X) = \begin{cases} 0, & i \geq 1 \\ \mathbb{Z}, & i = 0. \end{cases}$$

(acyclic: $\tilde{H}_i = 0 \ \forall i$)

But X is not contractible, as

$$\pi_1(X) \underset{\substack{\uparrow \\ \text{van Kampen}}}{=} \langle a, b \mid a^5 b^{-3},\ b^3 (ab)^{-2} \rangle$$

$\longrightarrow$



Dodecahedron

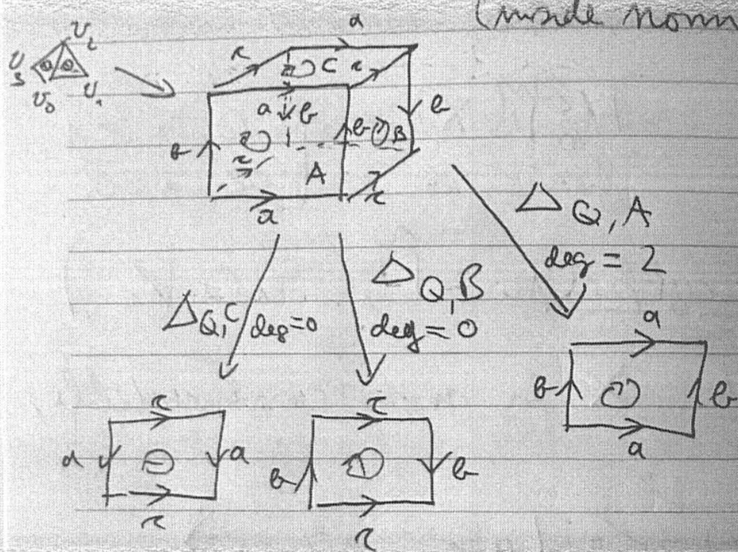
$a \longmapsto \varrho_a$ is a non-trivial

$b \longmapsto \varrho_b$ group homomorph.

($\varrho_a^5 = id = \varrho_b^3$, $(\varrho_a \varrho_b) = R(\pi)$ about p

$(\varrho_a \varrho_b)^2 = id = \varrho_b^3$)

(d) $X = K \times S^1$ (inside normal)



Further $\deg(\Delta_{A,a}) = \deg(\Delta_{A,b})$
$= \deg(\Delta_{A,c}) = 0$

$\deg(\Delta_{C,*}) = 0$

$\deg(\Delta_{B,a}) = \deg(\Delta_{B,c}) = 0$
$\deg(\Delta_{B,b}) = 2$

So we get

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{pmatrix}2\\0\\0\end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix}0&0&0\\0&2&0\\0&0&0\end{pmatrix}} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0$$

So $H_i^{CW}(X) = \begin{cases} 0, & i \geq 3 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}, & i=2 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}, & i=1 \\ \mathbb{Z}, & i=0. \end{cases}$

(e) $X = \mathbb{R}P^n \quad (n \geq 1)$

$$\overset{\shortparallel}{e^0 \cup e^1 \cup \ldots \cup e^n}$$

$$0 \to \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \to \ldots \xrightarrow{d_1} \mathbb{Z} \to 0$$

We need the degree of

$(k \geq 2)$

$$\begin{array}{ccc} S^{k-1} = \partial D^k & \overset{f}{\dashrightarrow} & S^{k-1} \\ \downarrow & & \uparrow \\ X^{k-1} & \longrightarrow & X^{k-1}/X^{k-2} \\ \shortparallel & & \\ \mathbb{R}P^{k-1} & & \end{array}$$

Take $y \in e_{k-1}$.

$f^{-1}(y) = \{x, -x\}$

$\deg(f)|_x = \deg(\mathrm{id}_{S^{k-1}})|_x = 1$

$\deg(f)|_{-x} = \deg(\text{point rot.})$

$= (-1)^k$

$$\Rightarrow \deg(f) = \begin{cases} 0, & 2 \nmid k \\ 2, & 2 \mid k. \end{cases}$$

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

So $$H_i^{CW}(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & n > i > 0, 2 \mid i \\ \mathbb{Z}/2\mathbb{Z}, & n > i > 0, 2 \nmid i \end{cases}$$

$$H_n^{CW}(\mathbb{R}P^n) = \begin{cases} 0, & 2 \mid n \\ \mathbb{Z}, & 2 \nmid n. \end{cases}$$

Def 168: (Euler characteristic)

Let X be a top. space, s.t. $\exists\, n \in \mathbb{N}\ \forall_{m \geq n}\ H_m(X) = 0$.

$$\chi(X) := \sum_{i=0}^{n} (-1)^i \operatorname{rank}(H_i(X))$$

is called the Euler characteristic of X.

Remark 169: Suppose X is a CW complex with finitely many cells.
Then
$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \#(i \text{ cells})$$

Pf: Tensor $C_\cdot^{CW}(X)$ with $\mathbb{Q}$

Then $$\chi(X) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{Q}} (H_i(X) \otimes_{\mathbb{Z}} \mathbb{Q})$$

$$= \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{Q}} (C_i^{CW}(X) \otimes_{\mathbb{Z}} \mathbb{Q})$$

□

Examples 170:

$$\chi(K \times S^1) = 1-2+1 = 0$$

$$\chi(\mathbb{RP}^n) = \begin{cases} 1, & 2 \mid n \\ 0, & 2 \nmid n \end{cases}$$

$$\chi(M_g) = 1-2g+1 = 2-2g$$

$$\chi(N_g) = 1-g+1 = 2-g.$$

using cells.