V.6. Equivalence of simplicial and singular homology.

Def 148: (relative simplicial homology)
Let X be a Δ-complex and A a subcomplex.
We put $\Delta_n(X, A) := \Delta_n(X) / \Delta_n(A)$

and $H_n^{\Delta}(X, A) := H_n\left(\Delta_\bullet(X) / \Delta_\bullet(A) \right)$,

$n \in \mathbb{N}_0$.

Thm 149: The canonical homomorphism $H_n^{\Delta}(X,A) \longrightarrow H_n(X,A)$ is an isomorphism for all $n \in \mathbb{N}_0$.

Proof: Case 1: $X = X^m$ (the m-skeleton w.r.t. the Δ-complex structure)
So we have

$$X^0 \subseteq X^1 \subseteq \dots \subseteq X^{m-1} \subseteq X^m = X.$$

We prove by induction that
$$C^\Delta_\bullet(X^k) \hookrightarrow C_\bullet(X^k)$$
induces isomorphisms
$$H_n^\Delta(X^k) \xrightarrow{\sim} H_n(X^k), \quad k \in \mathbb{N}_0.$$

<u>base case ($k=0$):</u>

$$\begin{array}{ccc} H_n^\Delta(X^0) & \longrightarrow & H_n(X^0) \\ \wr\!\! & & \wr\!\! \\ \bigoplus_{\substack{\alpha\in A \\ d_\alpha=0}} H_n^\Delta(e_\alpha) & \longrightarrow & \bigoplus_{\substack{\alpha\in A \\ d_\alpha=0}} H_n(e_\alpha) \end{array}$$

For $Y = \{y\}$ we have

$$\begin{array}{l} C_\bullet(Y): \cdots \xrightarrow{\sim} \mathbb{Z}\sigma_3 \xrightarrow{0} \mathbb{Z}\sigma_2 \xrightarrow{\sim} \mathbb{Z}\sigma_1 \xrightarrow{0} \mathbb{Z}\sigma_0 \to 0 \\ \quad\uparrow \qquad\qquad \uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow \mathrm{id} \\ \Delta_\bullet(Y): \cdots \to 0 \to 0 \to 0 \to \mathbb{Z}\sigma_y \to 0 \end{array}$$

Thus $H_n^\Delta(\{y\}) \xrightarrow{\sim} H_n(\{y\})$ for all $n \in \mathbb{N}_0$.

<u>(IS) $k>0$:</u> Consider the diagram

$$\begin{array}{ccccccc} H_{n+1}^\Delta(X^k, X^{k-1}) & \to & H_n^\Delta(X^{k-1}) & \to & H_n^\Delta(X^k) & \to & H_n^\Delta(X^k, X^{k-1}) \\ \downarrow & & \wr\downarrow & & \downarrow & & \downarrow \\ H_{n+1}(X^k, X^{k-1}) & \to & H_n(X^{k-1}) & \to & H_n(X^k) & \to & H_n(X^k, X^{k-1}) \end{array}$$

We show that $H_n^\Delta(X^k, X^{k-1}) \to H_n(X^k, X^{k-1})$ is an isomorphism for all $n \in \mathbb{N}_0$. The the 5-lemma finishes the proof of Case 1.

We compute $H_n^\Delta(X^k, X^{k-1})$.

$$\Delta_\bullet(X^k, X^{k-1}) = \Delta_\bullet(X^k) / \Delta_\bullet(X^{k-1})$$

$$\cdots \to 0 \to 0 \to \bigoplus_{d_\alpha = k} \mathbb{Z}\sigma_\alpha \to 0 \to 0 \cdots \to 0$$

Thus
$$H_n^\Delta(X^k, X^{k-1}) = \begin{cases} 0, & n \neq k \\ \bigoplus_{d_\alpha=k} \mathbb{Z}\sigma_\alpha, & n = k. \end{cases}$$

We compute $H_n(X^k, X^{k-1})$.

Put $Y_k := \coprod_{\substack{\alpha\in A\\ d_\alpha = k}} \Delta^k_\alpha$

$Z_k := \coprod_{\substack{\alpha\in A\\ d_\alpha = k}} \partial\Delta^k_\alpha$

Then $X^k/X^{k-1} \simeq Y_k/Z_k$

and

$$\begin{tikzcd}
\tilde{H}_n(X^k/X^{k-1}) \arrow[d, dash, "142\ \cong"'] & \tilde{H}_n(Y_k/Z_k) \arrow[l, "\sim"'] \arrow[d, dash, "\cong"'] \\
H_n(X^k, X^{k-1}) \arrow[r, phantom, "\circlearrowleft"] & \bigoplus_{\substack{\alpha\in A\\ d_\alpha=k}} \tilde{H}_n(\Delta^k_\alpha/\partial\Delta^k_\alpha) \arrow[d, dash, "142\ \cong"'] \\
 & \bigoplus_{\substack{\alpha\in A\\ d_\alpha=k}} H_n(\Delta^k_\alpha, \partial\Delta^k_\alpha) \arrow[ul]
\end{tikzcd}$$

The top and the bottom map are induced by the

characteristic maps $(\varphi_\alpha)_{\substack{\alpha\in A\\ d_\alpha=k}}$:

$$\Phi : (Y_k, Z_k) \xrightarrow{\sqcup\varphi_\alpha} (X^k, X^{k-1})$$

. By Example 146 $H_k(\Delta^k_\alpha, \partial\Delta^k_\alpha)$

$= \mathbb{Z}\, i_{k,\alpha}$ (free $\mathbb{Z}$ module, and

$i_{k,\alpha}$ is represented by $id_{\Delta^k_\alpha}$)

Thus $H_k(X^k, X^{k-1})$ is a free $\mathbb{Z}$-module with basis represented by φ_α, $\alpha\in A$ with $d_\alpha = k$.

Thus $H^\Delta_k(X^k, X^{k-1}) \xrightarrow{\sim} H_k(X^k, X^{k-1})$.

. For $k \neq n$ we have
$H_n(\Delta^k_\alpha, \partial\Delta^k_\alpha) \simeq \tilde{H}_n(S^k) = 0$.

So $\underbrace{H^\Delta_n(X^k, X^{k-1})}_{=0} \xrightarrow{\sim} \underbrace{H_n(X^k, X^{k-1})}_{=0}$

$\square$ (Case 1).

<u>Case 2: ($\dim X = \infty$)</u>:

· We show that $H_n^{\Delta}(X) \to H_n(X)$ is <u>surjective</u>: Take $[c] \in H_n(X)$

$\Rightarrow \exists k \geq 0$: $c \in C_n(X^k)$.

$\underset{\text{Case 1}}{\Rightarrow}$ $\exists c' \in Z_n^{\Delta}(X^k)$: c' and c represent the same class in $H_n(X^k)$

$\Rightarrow$ c' and c represent the same class in $H_n(X)$.

· <u>injective</u>: Exercise. □ (Case 2)

□

<u>Corollary 150</u>: Suppose X is a Δ-complex with only finitely many simplices, i.e. $|\mathcal{A}| < \infty$. Then $H_n(X)$ is finitely generated, $n \in \mathbb{N}_0$.

<u>Def. 151</u>: $\beta_i := \operatorname{rank}(H_i(X))$ is called the ith <u>Betti number</u> of X.

(Recall the rank of an abelian group G: $\operatorname{rank}(G) := \dim_{\mathbb{Q}} G \otimes_{\mathbb{Z}} \mathbb{Q}$.)

<u>Example 152</u>: (a) $X = T = S^1 \times S^1$

$H_0(X) \cong \mathbb{Z}$ $\quad \beta_0(X) = 1$
$H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ $\quad \beta_1(X) = 2$
$H_2(X) \cong \mathbb{Z}$ $\quad \beta_2(X) = 1$

(b) $X = K$ Kleinian bottle

$H_0(K) \cong \mathbb{Z}$ $\quad \beta_0(X) = 1$
$H_1(K) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ $\quad \beta_1(X) = 1$
$H_2(K) = 0$ $\quad \beta_2(X) = 0$

(c) $X = \mathbb{R}P^2$
$H_0(X) \cong \mathbb{Z}$ $\quad \beta_0(X) = 1$
$H_1(X) \cong \mathbb{Z}/2\mathbb{Z}$ $\quad \beta_1(X) = 0$
$H_2(X) = 0$ $\quad \beta_2(X) = 0$.

V.7. Degree of a map.

Def 153: $(n>0)$ Let $f: S^n \to S^n$ be continuous and fix a generator w of $H_n(S^n) = \mathbb{Z}w$.
Consider $f_*: \mathbb{Z}w \to \mathbb{Z}w$.
Then $\exists\, d \in \mathbb{Z}$ $f_*(w) = dw$.
We call d the degree of f and write $\deg(f)$.

Example 154: (a) $\deg(id_{S^n}) = 1$.

(b) Let f be the restriction of an orthogonal reflection $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$.
Claim: $\deg(f) = -1$.

Proof: $S^n \simeq \Delta_1^n \sqcup \Delta_2^n \big/$ boundary identified

(This also gives a Δ-complex structure.)

Then $H_n(S^n) \simeq H_n(\Delta_1^n \sqcup \Delta_2^n/\sim)$
$\simeq H_n^\Delta(\Delta_1^n \sqcup \Delta_2^n/\sim)$

$= \mathbb{Z}(\Delta_1^n - \Delta_2^n)$

We view the linear simplices Δ_1, Δ_2 (in $\Delta_1^n \sqcup \Delta_2^n/\sim$) as singular simplices in S^n.
(upper and lower hemisphere.)
Suppose $\partial\Delta_1$ and $\partial\Delta_2$ are the intersection of S^n with the reflection hyperplane.
Then
$$f_*(\Delta_1 - \Delta_2) = \Delta_2 - \Delta_1.$$

Thus $\deg(f) = -1$.

(c) $\deg(f \circ g) = \deg(f)\deg(g)$.

Thm 155: $(n>0)$ S^n has a nowhere vanishing continuous tangent field iff $2 \nmid n$.

Proof: "if": Suppose $2 \nmid n = 2k-1$

Put $v(x) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2k}, -x_{2k-1})$

"only if": Suppose $v : S^n \to \mathbb{R}^{n+1}$ is a nowhere 0 continuous tangent field. Replace $v(x)$ by $\frac{v(x)}{|v(x)|_2}$, so that we can w.l.o.g. assume $|v(x)|_2$ for all $x \in S^n$.

$F : S^n \times [0, \pi] \longrightarrow S^n$,
$F(x, t) := \cos(t) x + \sin(t) v(x)$,
is a homotopy from id_{S^n} to $-id_{S^n}$.

Thus $1 = \deg(id_{S^n}) = \deg(-id_{S^n})$

Now $-id_{S^n} = r_{e_1} \circ r_{e_2} \circ \ldots \circ r_{e_{n+1}}$

is a composition of $n+1$ orthogonal

reflections. So $\deg(-id_{S^n}) = (-1)^{n+1}$, by 154(b)(c).
Thus $1 = (-1)^{n+1}$ and $2 \nmid n$ □

The degree is the sum of local degrees.

Def 156: (n ≥ 0) Given $f : S^n \longrightarrow S^n$ continuous, $x \in S^n$ and U an open nbhd of x in S^n, such that $f^{-1}(f(x)) \cap U = \{x\}$.
We get:

$$\begin{tikzcd}
H_n(U, U \setminus \{x\}) \arrow[r, "f_*"] \arrow[d, "\text{excision}"', "\cong"] & H_n(V, V \setminus \{f(x)\}) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus \{x\}) \arrow[d, "\text{long ex. seq.}"', "\cong"] & H_n(S^n, S^n \setminus \{f(x)\}) \arrow[d, "\cong"] \\
H_n(S^n) \arrow[r, phantom, "\circlearrowright"] \arrow[d, equal] & H_n(S^n) \arrow[d, equal] \\
\mathbb{Z}w \arrow[r] & \mathbb{Z}w \\
w \arrow[r, mapsto] & d_x w.
\end{tikzcd}$$

Here $V \ni f(x)$ is an open nbhd s.t. $f(U) \subseteq V$.

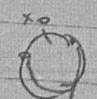
d_x is called the local degree of f at x.
Write $\deg f|_x$.

<u>Example 157:</u> $f : S^1 \longrightarrow S^1$
$f(x) := x^2$ in $\mathbb{C}$.
Take $x_0 = e^{i\varphi_0} \in S^1$.

$x_0 \in U$ = small open intervall of arclength δ.
$x_0^2 \quad V = f(U) = $ — " — 2δ

$\sigma : \Delta^1 \longrightarrow e^{i[\varphi_0 - \frac{\delta}{3}, \varphi_0 + \frac{\delta}{3}]}$

$$\begin{array}{ccc} \mathbb{Z}[\sigma] = H_1(U, U \setminus \{x_0\}) & \xrightarrow{f_*} & H_1(V, V \setminus \{x_0^2\}) \\ \wr| & & \wr| \\ H_1(S^1, S^1 \setminus \{x_0\}) & & H_1(S^1, S^1 \setminus \{x_0^2\}) \\ \wr| & & \wr| \\ \mathbb{Z}[\sigma - \rho] = H_1(S^1) & & H_1(S^1) \\ & & \shortparallel \\ & & \mathbb{Z}[\sigma - \rho] \\ & & \shortparallel \\ & & \mathbb{Z}[\sigma^2 - \tau] \end{array}$$

$\rho : \Delta^1 \longrightarrow$ 

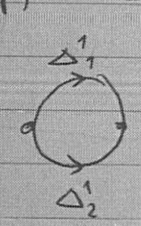
<u>Claim:</u> $\sigma^2 - \sigma + \rho - \tau$ is a 1-boundary in $C_*(S^1)$.

<u>Proof:</u> exercise. $\square$

Thus $[\sigma - \rho] = [\sigma^2 - \tau]$.
Thus $\deg f|_{x_0} = 1$.

But $\deg(f) = 2$.

<u>Proof:</u>



$f_*([\Delta_1^1 - \Delta_2^1]) = [\overline{\Delta}_2^1 + \Delta_1^1 - (\overline{\Delta}_1^1 + \Delta_2^1)]$

$= 2[\Delta_1^1 - \Delta_2^1]$ $\square$

<u>Prop 158:</u> Suppose $f^{-1}(y) = \{x_1, \dots, x_k\}$ is finite. Then
$\deg(f) = \sum_{i=1}^{k} \deg f|_{x_i}$

__Proof:__ Choose nbhds $U_i \ni x_i$ s.t. $U_1, \ldots, U_k$ are p.w. disjoint.

We get the following diagram:

$$\begin{tikzcd}
 & H_n(U_i, U_i \setminus \{x_i\}) \arrow[rr, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] \arrow[drr, phantom, "\circlearrowleft"] & & H_n(V, V \setminus \{y\}) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus \{x_i\}) \arrow[ddr, equal] \arrow[dr, phantom, "\circlearrowleft"] & \bigoplus_{j=1}^{k} H_n(U_j, U_j \setminus \{x_j\}) \arrow[l, "pr_i"'] \arrow[rr, "\Sigma (f|U_j)_*"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & & H_n(S^n, S^n \setminus \{y\}) \arrow[dd, equal] \\
 & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[urr, "f_*"] \arrow[drr, phantom, "\circlearrowleft"] & & \\
 & H_n(S^n) \arrow[u] \arrow[rr, "f_*"] & & H_n(S^n)
\end{tikzcd}$$

Warning:

$$\begin{tikzcd}
 & {} \arrow[r] \arrow[dl, "\cong"'] & {} \arrow[d, "\cong"] \\
{} \arrow[dr, equal] & & {} \arrow[d, equal] \\
 & {} \arrow[r] & {}
\end{tikzcd}$$

is not commutative, but all the small diagrams are.

Under k_i: $w \in H_n(S^n)$ is mapped to $(0, \ldots, w, \ldots, 0)$.

Note we have

$$\begin{tikzcd}
 & {} \arrow[r] \arrow[dl, "\cong"'] & {} \arrow[d, "\cong"] \\
{} \arrow[d, equal] & & {} \arrow[d, equal] \\
w \arrow[rr, bend left=80] & & d_i \cdot w
\end{tikzcd}$$

.

So $H_n(S^n, S^n \setminus f^{-1}(y)) \longrightarrow H_n(S^n, S^n \setminus \{y\})$ has the form

$$(z_1 w, \ldots, z_k w) \mapsto \Big(\sum_{j=1}^{k} z_j d_j\Big) w$$

On the other hand the bottom square has the form

$$\begin{tikzcd}
(w, w, \ldots, w) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[r] & H_n(S^n, S^n \setminus \{y\}) \arrow[d, equal] \\
{} \arrow[u, mapsto] & H_n(S^n) \arrow[u] \arrow[r] & H_n(S^n) \\
 & w \arrow[r, mapsto] & dw
\end{tikzcd}$$

So $\Big(\sum_{j=1}^{k} d_j\Big) w = dw$ $\square$