<u>Example 134:</u> Take $A = \{x_0\} \subseteq X$.

Then $\tilde{H}_n(A) = 0 \ \forall n \in \mathbb{N}_0$.
Rk 133 $\Rightarrow$ $\tilde{H}_n(X) \cong H_n(X, \{x_0\}) \ \forall_{n \in \mathbb{N}_0}$

We write $f: (X, A) \longrightarrow (Y, B)$ for maps $f: X \to Y$ with $f(A) \subseteq B$.

<u>Thm 135:</u> Suppose two continuous maps $f, g: (X, A) \longrightarrow (Y, B)$ are homotopic through maps
$(X, A) \longrightarrow (Y, B)$.
Then $f_* = g_*$ on $H_n(X, A) \ \forall_{n \in \mathbb{N}_0}$.

<u>Proof:</u> exercise □

The main property for relative homology groups is excision.

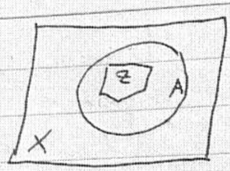
<u>Theorem 136</u> (Excision Theorem)
(1) Let $Z \subseteq A \subseteq X$ be subspaces such that $\overline{Z} \subseteq \mathring{A}$. Then $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism
$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\sim} H_n(X, A)$
for all $n \in \mathbb{N}_0$.

(2) Let $A, B \subseteq X$ be subspaces s.t. $\mathring{A} \cup \mathring{B} = X$. Then $(B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism
$H_n(B, A \cap B) \xrightarrow{\sim} H_n(X, A)$
for all $n \in \mathbb{N}_0$.

<u>Remark 137:</u> At first we observe that (1) $\Leftrightarrow$ (2).

<u>Pf:</u> ($\Rightarrow$): Given A, B as in (2), put $Z := X \setminus B \subseteq X \setminus \mathring{B} \subseteq \mathring{A}$.
Then $B = X \setminus Z$ and
$A \cap B = A \cap (X \setminus Z) = A \setminus Z$

($\Leftarrow$): exercise □



Before we start the proof we give a first application.

<u>Theorem 138:</u> Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be non-empty open subsets. Suppose $U \cong V$ (homeom.). Then $n = m$.

<u>Proof:</u> Take $x \in U$.

Then $H_k(U, U \setminus \{x\}) \underset{\substack{\uparrow \\ \text{excision} \\ B = U,\ A = \mathbb{R}^n \setminus \{x\}}}{\cong} H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$

$\underset{\substack{\uparrow \\ \text{long exact} \\ \text{sequence}}}{\cong} H_{k-1}(\mathbb{R}^m \setminus \{x\})$

$\cong H_{k-1}(S^n) \cong \begin{cases} \mathbb{Z}, & k = n+1 \\ 0, & k \neq n+1 \end{cases}$

for $k > 0$

and $H_0(U, U \setminus \{x\}) \cong H_0(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) = 0$.

1st: If $n = 0$ then $|U| = 1$, so $|V| = 1$, so $m = 0$, as V is open.

2nd: Suppose $n, m > 0$. Comparing relative homology groups we obtain $n+1 = m+1$. □

To prove Thm 136(2) we need to show that we can compute $H_n(X,A)$ just using singular simplices in A and in B.

$$\overset{\mathfrak{U}}{\parallel}$$

<u>Prop 139:</u> Let $(U_\alpha)_{\alpha \in \mathcal{A}}$ be a family of subsets of X s.t. $X = \bigcup_{\mathcal{A}} \mathring{U}_\alpha$, and let $C_\bullet^{\mathfrak{U}}(X)$ be the subcomplex of $C_\bullet(X)$, given by

$$C_n^{\mathfrak{U}}(X) := \sum_{\alpha \in \mathcal{A}} C_n(U_\alpha).$$

= subgroup generated by those sing n-simplices whose image is in some U_α.

Then the inclusion

$$C_\bullet^{\mathfrak{U}}(X) \xrightarrow{\iota} C_\bullet(X)$$

is a chain homotopy equivalence,

i.e. $\exists\ \rho : C_\bullet(X) \longrightarrow C_\bullet^{\mathfrak{U}}(X)$ (chain map) s.t. $\iota \circ \rho$ and $\rho \circ \iota$ are chain homotopic to the identity.

<u>Remark 140:</u> In fact we will construct ρ s.th. $\rho \circ \iota = id_{C_\bullet^{\mathfrak{U}}(X)}$

and $\iota \circ \rho \sim id_{C_\bullet(X)}$

<u>Proof:</u> The morphism $\rho : C_\bullet(X) \to C_\bullet^{\mathfrak{U}}(X)$ will be defined using iterated barycentric subdivision.

We have 4 steps:

Step 1: barycentric subdivision for a simplex and an estimate for the diameter

Step 2: barycentric subdivision for a linear chain

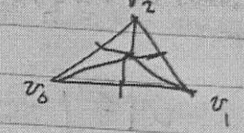
Step 3: barycentric subdivision for a general chain
(This uses Step 2)

Step 4: iterated barycentric sub-division.
(This uses Step 1 and Step 3)

Step 1: (bsd for Δ^n)
The bsd for Δ^n is the collection of all n-simplices given by inductively subdividing using barycenters.

$n=0$: $bsd([v_0]) = \{[v_0]\}$
$n>0$: $bsd([v_0,\dots,v_n]) = \{[b, w_0, \dots, w_{n-1}] \mid \exists_{0\le i\le n} : [w_0,\dots,w_{n-1}] \in bsd([v_0,\dots,\hat{v}_i,\dots,v_n])\}$

where $b = \sum_{j=0}^{n} \frac{1}{n+1} v_j$ barycenter of $[v_0,\dots,v_n]$

(Example: $n=2$: )

Claim: Let Δ be an n-simplex in $\mathbb{R}^n$ and $\Delta' \in bsd(\Delta)$. Then
$$diam(\Delta') \le \frac{n}{n+1} diam(\Delta).$$

Pf: $n=0$: $diam(\Delta') = 0 \le \frac{0}{1} diam(\Delta)$.

$n>0$: Let u, u' be two vertexes of Δ' such that $|u-u'| = diam(\Delta')$.

Δ' is the convex hull of b and some $\Delta'' \in bsd([v_0,\dots,\hat{v}_i,\dots,v_n])$ for some i.

- Case 1: u, u' vertexes of Δ''
Then $|u-u'| \le diam\,\Delta''$
$\underset{(IH)}{\le} \frac{n-1}{n} diam([v_0,\dots,\hat{v}_i,\dots,v_n])$
$\le \frac{n-1}{n} diam(\Delta)$
$\le \frac{n}{n+1} diam(\Delta)$.

• Case 2: $u = b = \sum_{j=0}^{n} \frac{1}{n+1} v_j$ and $u' \in \Delta^n$.

$$|u-u'| \leq \left| \frac{n}{n+1}\Big(\sum_{j\neq i} \frac{1}{n} v_j - u'\Big) \right| + \left| \frac{1}{n+1} (v_i - u') \right|$$

$$\underset{(IH)}{\leq} \frac{n}{n+1} \cdot \frac{n-1}{n} \operatorname{diam}([v_0, \dots, \hat{v}_i, \dots, v_n]) + \frac{1}{n+1} \operatorname{diam}(\Delta)$$

$$\leq \frac{n}{n+1} \operatorname{diam}(\Delta).$$

$\square$ (claim)

<u>Step 2</u>: A singular n-chain $\sigma : \Delta^n \to X$ ($X \subseteq \mathbb{R}^m$) is called linear if σ is an affine map. Such σ is determined by $\sigma(e_0), \sigma(e_1), \sigma(e_2), \dots, \sigma(e_n)$.
(We write $[\sigma(e_0), \dots, \sigma(e_n)]$ for a linear σ.)

$LC_\bullet(X) \subseteq C_\bullet(X)$ subcomplex generated by linear singular simplices.

We put $LC_{-1}(X) := \mathbb{Z}[\emptyset]$.
($[\emptyset]$ empty simplex)

We define a chain map $S : LC_\bullet(X) \to LC_\bullet(X)$

$$\begin{array}{ccccccccc}
\cdots \to & LC_2(X) & \to & LC_1(X) & \to & LC_0(X) & \to & LC_{-1}(X) & \to 0 \\
 & \downarrow S_2 & & \downarrow S_1 & & \downarrow S_0 & & \downarrow S_{-1} & \\
\to & LC_2(X) & \to & LC_1(X) & \to & LC_0(X) & \to & LC_{-1}(X) & \to 0
\end{array}$$

by induction: <u>$n=-1$</u>: $S([\emptyset]) := [\emptyset]$
<u>$n > -1$</u>: $S(\sigma) := b_\sigma(S(\partial\sigma))$

Here b_σ is the barycenter of σ and $b([w_0, \dots, w_{n-1}]) := [b, w_0, \dots, w_{n-1}]$

<u>Idea</u>: $S(\sigma)$ = sum of the restrictions of σ to $\operatorname{bd}(\Delta^n)$ with signs.

Example: We identify a linear $\sigma : \Delta^n \longrightarrow \mathbb{R}^m$ with $[\sigma(e_0), \dots, \sigma(e_n)]$
$$\overset{\shortparallel}{[w_0, \dots, w_n]}$$

$\underline{n=0:}$ $S([w_0]) = b_{[w_0]}\, S(\partial[w_0])$

$$= b_{[w_0]}\, S([\emptyset])$$

$$\underset{\substack{\uparrow \\ S([\emptyset]) = [\emptyset]}}{=} b_{[w_0]}([\emptyset])$$

$$= [b_{[w_0]}] \underset{\substack{\uparrow \\ b_{[w_0]} = w_0}}{=} [w_0]$$

$\underline{n=1:}$ $S([w_0, w_1]) = b_{[w_0, w_1]}\left(S(\partial[w_0, w_1])\right)$

$$\underset{\substack{\uparrow \\ \text{Case } n=0:}}{=} b_{[\,]}([w_1] - [w_0]) \underset{\substack{\uparrow \\ \text{Def of } b}}{=} [b_{[\,]}, w_1] - [b_{[\,]}, w_0]$$

$\underline{n=2:}$ $S([w_0, w_1, w_2]) = b_\sigma\left(S(\partial[w_0, w_1, w_2])\right)$

$$= b_\sigma\Big([b_{[w_1,w_2]}, w_2] - [b_{[w_1,w_2]}, w_1]$$

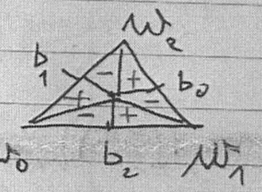
$$- [b_{[w_0,w_2]}, w_2] + [b_{[w_0,w_2]}, w_0]$$

$$+ [b_{[w_0,w_1]}, w_1] - [b_{[w_0,w_1]}, w_0]\Big)$$

$$= [b_\sigma, b_0, w_2] - [b_\sigma, b_0, w_1]$$

$$- [b_\sigma, b_1, w_2] + [b_\sigma, b_1, w_0]$$

$$+ [b_\sigma, b_2, w_1] - [b_\sigma, b_2, w_0]$$

$\mathbb{R}^m \supseteq$ 

$\lrcorner$

Claim: S is a chain map.

Pf: $n = -1$: $S(\partial[\emptyset]) = S(0) = 0$
$\partial S([\emptyset]) = \partial[\emptyset] = 0$.

$n > -1$: $\partial(S(\sigma)) = \partial(b_\sigma(S(\partial\sigma)))$

$$\underset{\substack{\uparrow \\ \partial b_\sigma = id - b_\sigma \circ \partial}}{=} S(\partial\sigma) - b_\sigma(\partial(S(\partial\sigma)))$$

$$\underset{\substack{(IH) \\ \partial S = S\partial \text{ for } n-1}}{=} S(\partial\sigma) - b_\sigma(S\partial\partial\sigma) = S(\partial\sigma).$$

$\square$

<u>Claim:</u> S is chain homotopic to id_{LC}.

<u>Proof:</u> We consider the following chain homotopy T:

$$\begin{tikzcd}
\cdots \arrow[r] & LC_2(X) \arrow[r] \arrow[dl, "T"'] & LC_1(X) \arrow[r] \arrow[dl, "T"'] & LC_0(X) \arrow[r] \arrow[dl, "T"'] & LC_{-1}(X) \arrow[r] \arrow[dl, "T_0"'] & 0 \\
\cdots \arrow[r] & LC_2(X) \arrow[r] & LC_1(X) \arrow[r] & LC_0(X) \arrow[r] & LC_{-1}(X) \arrow[r] & 0
\end{tikzcd}$$

$T_{-1} = 0$

$T_n := b_\sigma(\sigma - T(\partial\sigma))$ for $n > -1$.

We want $\partial \circ T + T \circ \partial = id - S$.

<u>Example:</u>

$$T([w_0]) = w_0([w_0] - \overbrace{T([\emptyset])}^{0})$$
$$= [w_0, w_0]$$

$$T([w_0, w_1]) = b_{[w_0,w_1]}([w_0, w_1] - T([w_1] - [w_0]))$$

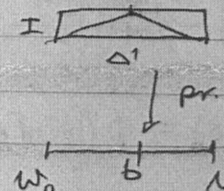
$$= [b, w_0, w_1] - [b, w_1, w_1] + [b, w_0, w_0].$$

(exercise: Compute $T([w_0, w_1, w_2])$.)

<u>Idea:</u> In $\Delta^n \times I$ Inductively join all

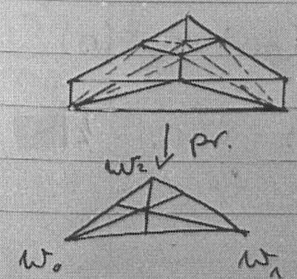
all simplices in $\Delta^n \times \{0\}$ and $(\partial\Delta^n) \times I$ with the barycenter of $\Delta^n \times \{1\}$.

<u>$n = 1$:</u>



$$T([w_0, w_1]) = pr|_{\triangle} - pr|_{\triangleleft} + pr|_{\triangleright}$$

<u>$n = 2$:</u>



10 restrictions.

To show: $\partial\circ T + T\circ\partial = id - S$

$n=-1$: ✓, as $S_{-1} = id_{LC_{-1}(X)}$ and $T_{-1} = 0 = T_{-2}$.

$n=0$: $id = S_0$ and

$$(\partial\circ T + T\circ\partial)([w_0])$$

$$= \partial([w_0, w_0]) + \underbrace{T([\emptyset])}_{0}$$

$$= [w_0] - [w_0] = 0.$$

$n>0$: $(\partial\circ T)(\sigma) = \partial(b_\sigma(\sigma - T(\partial\sigma)))$

$$= \sigma - T(\partial\sigma) - b_\sigma(\underbrace{\partial\sigma - \partial(T(\partial\sigma))}_{\substack{T(\partial\partial\sigma) + S(\partial\sigma) \\ \text{by (IH)}}})$$

$$\underset{\substack{\uparrow \\ b_\sigma S(\partial\sigma) = S(\sigma) \\ \text{by Def.}}}{=} \sigma - T(\partial\sigma) - S(\sigma)$$

□

$\partial\circ T + T\circ\partial = id - S$ is also true if we replace $LC_{-1}(X)$ by 0.

<u>Step 3:</u> X is again arbitrary.

$S: C_\circ(X) \to C_\circ(X)$ is defined by $S\sigma := \sigma_\#(S\Delta^n)$

where $S(\Delta^n)$ is defined in Step 2 and $\sigma_\#: C_n(\Delta^n) \longrightarrow C_n(X)$ via $\sigma: \Delta^n \longrightarrow X$.

Then $\partial\circ S = S\circ\partial$, because for $n\geq 1$:

$$\partial(S(\sigma)) = \partial(\sigma_\#(S\Delta^n)) \overset{\substack{\text{Step 2} \\ \downarrow}}{=} (\sigma_\#\circ S)(\partial\Delta^n)$$

$$= \sigma_\#(S(\sum_{i=0}^{n}(-1)^i \Delta^n_i)) \quad i\text{th face of } \Delta^n$$

$$= \sum_{i=0}^{n}(-1)^i \sigma_\#(S\Delta^n_i)$$

$$\underset{\substack{\uparrow \\ \text{Def.}}}{=} \sum_{i=0}^{n}(-1)^i S(\sigma|_{\Delta^n_i}) = S(\partial\sigma)$$

$T : C_\bullet(X) \to C_{\bullet+1}(X)$

$T(\sigma) := \sigma_\#(T(\Delta^n))$
satisfies $\partial\circ T + T\circ\partial = id - S$

using Step 2.

Step 4: We need a map

$$\rho : C_\bullet(X) \longrightarrow C^{\mathcal{U}}_\bullet(X).$$

Idea: Given σ, subdivide σ as much such that $S^m(\sigma) \in C^{\mathcal{U}}_n(X)$.

Note: S^m is homotopic to id via $D_m := \sum_{0\le i<m} T S^i$

Pf: $\partial\circ D_m + D_m\circ\partial = \sum_{0\le i<m} (\partial\circ T + T\circ\partial) S^i$

$= \sum_{0\le i<m} (id - S) S^i = id - S^m$ □

Problem: Different σ needs different m.

Put $m(\sigma) := \min\{m \in \mathbb{N}_0 \mid S^m(\sigma) \in C^{\mathcal{U}}_n(X)\}$.

Put $D : C_\bullet(X) \longrightarrow C_{\bullet+1}(X)$

$$D(\sigma) := D_{m(\sigma)}(\sigma)$$

Then
$\partial(D(\sigma)) + D(\partial\sigma) = \partial(D_{m(\sigma)}\sigma) + D(\partial\sigma)$

$= \sigma - \underbrace{\left(S^{m(\sigma)}(\sigma) + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)\right)}_{=: \rho(\sigma)}$

(exercise: $\rho : C_\bullet(X) \to C^{\mathcal{U}}_\bullet(X)$ is a chain map.)

Then $\iota\circ\rho \sim id_{C_\bullet(X)}$ via D □

<u>Proof</u> (Excision theorem 136)

We prove (2). $\mathcal{U} = \{A, B\}$

$$C_n(B) \big/ C_n(A \cap B) \cong \bigoplus_{\substack{\sigma : \Delta^n \to B \\ \operatorname{im}(\sigma) \nsubseteq A \cap B}} \mathbb{Z}\sigma \cong C_n^{\mathcal{U}}(X) \big/ C_n(A)$$

and $C_\bullet^{\mathcal{U}}(X) \big/ C_\bullet(A)$ is homotopic

to $C_\bullet(X) \big/ C_\bullet(A)$ by Prop. 139.

Thus $H_n(B, A \cap B) \cong H_n(X, A)$
induced by $(B, A \cap B) \hookrightarrow (X, A)$.

$\square$

<u>Def 141</u>: A pair (X, A) is called a <u>good pair</u>, if $A \neq \emptyset$ is closed and $\exists U$ nbhd of A s.t. A is a deformation retract of U.

We now prove Thm 126.

<u>Prop. 142:</u> Let (X, A) be a good pair. Then $q : (X, A) \to (X/A, A/A)$ induces an isomorphism

$$H_n(X, A) \xrightarrow{\sim} H_n(X/A, A/A) = \tilde{H}_n(X/A)$$

for all $n \in \mathbb{N}_0$.

For the proof we need a generalization of Thm 131.

<u>Lemma 143:</u> Given subspaces $U \subseteq V \subseteq X$ we obtain an exact sequence

$$0 \to C_\cdot(V)/C_\cdot(U) \to C_\cdot(X)/C_\cdot(U) \to C_\cdot(X)/C_\cdot(V) \to 0$$

and a long exact sequence.

$$\cdots \to H_n(V, U) \to H_n(X, U) \to H_n(X, V) \to \cdots \to H_0(V, U) \to H_0(X, U) \to H_0(X, V) \to 0.$$

Proof (of Prop. 142):

Let U be a nbhd of A which deformation retracts to A.

Then $(A,A) \hookrightarrow (U,A)$ is a homotopy equivalence.

$\Rightarrow H_n(U,A) \simeq H_n(A,A) = 0.$

Lemma 143 $\Rightarrow H_n(X,A) \simeq H_n(X,U)$.

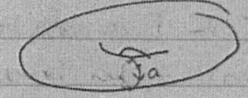
Consider the following diagram

$$\begin{array}{ccccc} H_n(X,A) & \xrightarrow{\sim} & H_n(X,U) & \xleftarrow[\text{excision}]{\sim} & H_n(X \setminus A, U \setminus A) \\ q_* \downarrow & \circlearrowleft & q_* \downarrow & \circlearrowleft & \downarrow q_* \\ H_n(X/A, A/A) & \xrightarrow{\sim} & H_n(X/A, U/A) & \xleftarrow[\text{excision}]{\sim} & H_n(X/A \setminus A/A, U/A \setminus A/A) \end{array}$$

$(X \setminus A, U \setminus A) \overset{q}{\cong} (X/A \setminus A/A, U/A \setminus A/A)$ is a homeomorphism.

$\Rightarrow$ The right q_* is bijective. □

Example 144:

$T = S^1 \times S^1$ 

$C :=$ circle given by a.

Compute: $\tilde{H}_n(T/C)$ $\forall n \in \mathbb{N}_0$

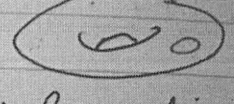
We have a long exact sequence

$$\to \underset{0}{\tilde{H}_3(C)} \to \underset{0}{\tilde{H}_3(T)} \to \underset{0}{\tilde{H}_3(T/C)}$$
$$\hookrightarrow \underset{0}{\tilde{H}_2(C)} \to \underset{\mathbb{Z}}{\tilde{H}_2(T)} \to \tilde{H}_2(T/C)$$
$$\hookrightarrow \underset{\mathbb{Z}\sigma_a}{\tilde{H}_1(C)} \xrightarrow{\sigma_a \mapsto \sigma_a} \underset{\mathbb{Z}\sigma_a \oplus \mathbb{Z}\sigma_b}{\tilde{H}_1(T)} \to \tilde{H}_1(T/C)$$
$$\hookrightarrow \underset{0}{\tilde{H}_0(C)} \to \underset{0}{\tilde{H}_0(T)} \to \underset{0}{\tilde{H}_0(T/C)} \to 0.$$

$\Rightarrow \tilde{H}_1(T/C) \simeq \mathbb{Z}\sigma_a \oplus \mathbb{Z}\sigma_b / \mathbb{Z}\sigma_a \simeq \mathbb{Z}\sigma_b$

and $\tilde{H}_2(T/C) \simeq \mathbb{Z}$.

Exercise 145:



$C \subseteq T$ a circle which cuts out an open disc.
Compute $\tilde{H}_n(T/C)$.

Example 146: We know $H_n(D^n, \partial D^n) \simeq \mathbb{Z}$.

Claim: In $H_n(\Delta^n, \partial\Delta^n) \simeq H_n(D^n, \partial D^n)$ the class represented by id_{Δ^n} generates $H_n(\Delta^n, \partial\Delta^n)$.

Proof: **$n=0$:** $(\underset{\{v\}}{\overset{\shortparallel}{\Delta^0}}, \partial\Delta^0 = \emptyset)$.

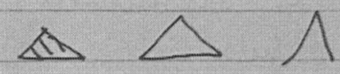
$$C_\cdot(\{v\}) / C_\cdot(\emptyset) = \to 0 \to \cdots \to 0 \to \underset{\mathbb{Z}\, id_{\Delta^0}}{\overset{\shortparallel}{C_0(\{x\})}} \to 0$$

(as $C_\cdot(\emptyset) = 0$)

$$\Rightarrow H_0(\Delta^0, \partial\Delta^0) \cong C_0(\Delta^0) = \mathbb{Z}\, id_{\Delta^0}$$

$n>0$: $\Lambda := \bigcup_{i=0}^{n-1} [v_0, \dots, \hat{v}_i, \dots, v_n] \subsetneq \partial\Delta^n$.

(e.g. $n=2$:



Δ^2 $\quad \partial\Delta^2$ $\quad \Lambda$)

We have

$$H_n(\Delta^n, \partial\Delta^n) \xrightarrow[\sim]{\partial} H_{n-1}(\partial\Delta^n, \Lambda) \xleftarrow[\sim]{i_*} H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$$

- The 1st isom. comes from the long exact sequence for $(\Delta^n, \partial\Delta^n, \Lambda)$ using that $H_k(\Delta^n, \Lambda) \simeq \tilde{H}_k(\Delta^n/\Lambda) \simeq \tilde{H}_k(D^n) = 0$.

- The second follows from

$$H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \simeq \tilde{H}_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}) \simeq \tilde{H}_{n-1}(\partial\Delta^n/\Lambda) \simeq H_{n-1}(\partial\Delta^n, \Lambda).$$

$\Delta^{n-1}/\partial\Delta^{n-1}$ homeom. to $\partial\Delta^n/\Lambda$

Let $i_n \in H_n(\Delta^n, \partial\Delta^n)$ be represented by

id_{Δ^n}. Then $\partial i_n \in H_{n-1}(\partial\Delta^n, \Lambda)$ is represented by $\pm id_{\Delta^{n-1}}$.
Thus
$$\partial i_n = \pm \iota_*(i_{n-1})$$

So, by induction, i_n generates $H_n(\Delta^n, \partial\Delta^n)$.

Corollary 147: Let X be a CW complex and A, B be subcomplexes s.t. $A \cap B$ is also a non-empty subcomplex and $A \cup B = X$.
Then $H_n(B, A \cap B) \xrightarrow{\sim} H_n(X, A)$ for all $n \in \mathbb{N}_0$.

Proof: (X, A) and $(B, A \cap B)$ are good pairs and
$$X/A \cong B/A \cap B.$$
Apply Prop. 142. $\square$