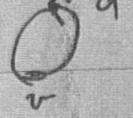
(c) S^1 

$$\Delta_1(S^1) = \mathbb{Z}\sigma_a$$
$$\Delta_0(S^1) = \mathbb{Z}\sigma_v$$

$$\partial_n = 0 \quad \forall n \in \mathbb{N}_0.$$

$$\Rightarrow H_n^{\Delta}(X) = \begin{cases} \mathbb{Z}, & n = 0,1 \\ 0, & n > 1. \end{cases}$$

(d) $T = S^1 \times S^1$.
Consider the Δ-complex structure from 108 (a).

$$\Delta_0(T) = \mathbb{Z}\sigma_v$$
$$\Delta_1(T) = \mathbb{Z}\sigma_a \oplus \mathbb{Z}\sigma_b \oplus \mathbb{Z}\sigma_c$$
$$\Delta_2(T) = \mathbb{Z}\sigma_U \oplus \mathbb{Z}\sigma_V$$

$$\partial_1 = 0 \Rightarrow H_0^{\Delta}(T) = \Delta_0(T) \cong \mathbb{Z}$$

$$\partial_2(n_U\sigma_U + n_V\sigma_V) = n_U(\sigma_b - \sigma_c + \sigma_a) + n_V(\sigma_a - \sigma_c + \sigma_b)$$

So $H_2^{\Delta}(T) = \ker(\partial_2) = \mathbb{Z}(\sigma_U - \sigma_V)$
$\simeq \mathbb{Z}$

$$H_1^{\Delta}(T) = \langle \sigma_a, \sigma_b, \sigma_c \rangle \big/ \langle \sigma_a - \sigma_c + \sigma_b \rangle$$

$$\simeq \mathbb{Z}\sigma_c + \mathbb{Z}\sigma_b$$

$$\simeq \mathbb{Z} \oplus \mathbb{Z}.$$

So $H_n^{\Delta}(T) \simeq \begin{cases} \mathbb{Z}, & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z}, & n = 1 \\ 0, & n > 2. \end{cases}$

(e) $H_n^{\Delta}(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}/2\mathbb{Z}, & n = 1 \\ 0, & n > 1. \end{cases}$

(exercise)

V 3. Singular homology

Def 114: (Singular homology)
Let X be a top. space.

(a) A singular n-simplex in X is a cont. map $\sigma : \Delta^n \to X$. ($n \in \mathbb{N}_0$)

(b) $\cdot\; C_n(X) := \bigoplus_{\sigma : \Delta^n \to X} \mathbb{Z}\sigma$ $\quad (n \in \mathbb{N}_0)$

set of singular n-chains

- ($n \in \mathbb{N}$)
  The boundary map
  $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$
  is defined via linear extension from

$$\partial_n(\sigma) := \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

- $\partial_0 : C_0(X) \longrightarrow 0 \quad \partial_0 := 0$

The complex (note $\partial\partial = 0$)

$$\cdots \to C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{0} 0$$

is called <u>singular chain complex for X</u>.

We define for $n \in \mathbb{N}_0$

$B_n(X) := \mathrm{im}(\partial_{n+1})$ set of <u>singular n-boundaries</u>

$Z_n(X) := \ker(\partial_n)$ set of <u>singular cycles</u>

$H_n(X) := Z_n(X) / B_n(X)$ n-th <u>singular homology group</u>.

<u>Rk 115</u>: Advantage: The definition does not need a choice of a Δ-complex structure

Disadvantage: $H_n(X)$ is hard to

compute directly from the definition.

<u>Prop 116</u>: Let X_α, $\alpha \in A$, be the path-connected components of X. Then for all $n \in \mathbb{N}_0$

$$H_n(X) \cong \bigoplus_{\alpha} H_n(X_\alpha).$$

<u>Proof</u>: Given an n-simplex $\sigma: \Delta^n \to X$, as σ is continuous, there exists a unique $\alpha \in A$ such that $\mathrm{im}(\sigma) \subseteq X_\alpha$.

□

<u>Prop 117</u>: Suppose $X \neq \emptyset$ is path-connected. Then $H_0(X) \cong \mathbb{Z}$.

<u>Proof</u>: Define $\epsilon: C_0(X) \longrightarrow \mathbb{Z}$

$$\epsilon\left(\sum_{i=1}^{\ell} n_i \sigma_i\right) := \sum_{i=1}^{\ell} n_i$$

(Here σ_i, $i = 1, \dots, \ell$, are pairwise different singular 0-simplices.)

To show: $im(\partial_1) = ker(\epsilon)$.

"$\subseteq$": σ a sing. 1-simplex

$$\epsilon(\partial_1(\sigma)) = \epsilon(\sigma|_{[v_1]} - \sigma|_{[v_0]})$$

$$= 0.$$

"$\supseteq$" By induction on $l \geq 1$.

Let $c = \sum_{i=1}^{l} n_i \sigma_i \in ker(\epsilon)$.

<u>$l=1$</u>: $0 = \epsilon(c) = \epsilon(n_1\sigma_1) = n_1$

$\Rightarrow$ $c = 0\sigma_1 = 0 \in im(\partial_1)$.

<u>$l>1$</u>: Let σ be a path from Q to P where $im(\sigma_l) = \{P\}$ and $im(\sigma_{l-1}) = \{Q\}$.

Then $c' := c - n_l \partial_1(\sigma) = \sum_{i=1}^{l-2} n_i \sigma_i + (n_{l-1} + n_l)\sigma_{l-1}$

$c, \partial_1(\sigma) \in ker(\epsilon)$. $\Rightarrow$ $c' \in ker(\epsilon)$

$\underset{(IH)}{\Rightarrow}$ $c' \in im(\partial_1)$ $\square$

<u>Prop 118</u>: Let $X = \{x_0\}$. Then $H_n(X) = 0$ for all $n > 0$.

<u>Proof</u>: For each $n \in \mathbb{N}_0$ there is a unique singular n-simplex σ_n.

We have $C_n(X) = \mathbb{Z}\sigma_n$.

Now for $n>0$:

$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_n|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$

$$= \left(\sum_{i=0}^{n} (-1)^i\right)\sigma_{n-1}$$

$$= \begin{cases} 0, & 2 \nmid n \\ \sigma_{n-1}, & 2 \mid n \end{cases}.$$

Thus the singular chain complex for X is

$$\xrightarrow{0} C_4(X) \xrightarrow{\sim} C_3(X) \xrightarrow{0} C_2(X) \xrightarrow{\sim} C_1(X) \xrightarrow{0} C_0(X) \longrightarrow 0$$

Thus $H_n(X) = 0 \ \forall n > 0$ $\square$

Def 119: The homology groups for the complex

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

are called reduced homology groups.

$$\tilde{H}_n(X) = \begin{cases} H_n(X), & n>0 \\ \ker \epsilon / \operatorname{im} \partial_1, & n=0 \end{cases}$$

Note: As $H_0(X)$ is a free $\mathbb{Z}$-module of rank $= \#$(path-connected components of X), we have that $\tilde{H}_0(X)$ is free $\mathbb{Z}$-module of $rk = rk(H_0(X)) - 1$.

(Pf: $\mathbb{Z}$ is projective
$\Rightarrow \ker(\epsilon) \oplus \mathbb{Z} \simeq C_0(X)$.
$\Rightarrow \tilde{H}_0(X) \oplus \mathbb{Z} \simeq H_0(X)$ $\square$)

Example 120: $\tilde{H}_0(\{x_0\}) = 0$.

V 4 Homotopy invariance.

Def 121: A continuous map $f: X \to Y$ induces maps $f_\#: C_n(X) \to C_n(Y)$ $(n \in \mathbb{N}_0)$

$$f_\#\left(\sum_i n_i \sigma_i\right) := \sum_i n_i (f \circ \sigma_i)$$

and they satisfy

$$\partial_{n+1} \circ f_{\#, n+1} = f_{\# n} \circ \partial_{n+1}.$$

because ∂_{n+1} is defined by restrictions on the domain Δ^{n+1}.

Thus $f_\#(B_n(X)) \subseteq B_n(Y)$
and $f_\#(Z_n(X)) \subseteq Z_n(Y)$.

Thus f induces group homomorphisms

$$f_*: H_n(X) \longrightarrow H_n(Y) \quad (n \in \mathbb{N}_0)$$

<u>Theorem 122</u>: Suppose $f, g : X \to Y$ are homotopic. Then $\forall_{n \in \mathbb{N}_0}$: $f_* = g_* : H_n(X) \longrightarrow H_n(Y)$.

A first consequence we get for contractible spaces.

<u>Def 123</u>: A top. space X is called <u>contractible</u>, if X is homotopy equivalent to a point.
(See Def. 30 on Page 26.)

<u>Example 124</u>: (a) $\mathbb{R}$ is contractible:

$$\mathbb{R} \underset{g}{\overset{f}{\rightleftarrows}} \{P\} \qquad f(P) = 0$$
$$g(x) = P \ \forall_{x \in \mathbb{R}}$$

$g \circ f = id_{\{P\}} \sim id_{\{P\}}$

$f \circ g = c_0 \underset{\uparrow}{\simeq} id_{\mathbb{R}}$
To show;

We use $F : \mathbb{R} \times I \longrightarrow \mathbb{R}$
$F(x, t) := tx.$
Then $F_0 = f \circ g = c_0$
$F_1 = id_{\mathbb{R}}.$

(b) $S^1 \not\simeq \{P\}$.
<u>Proof</u>: Assume $S^1 \simeq \{P\}$.
$\Rightarrow \exists \ \{P\} \underset{g}{\overset{f}{\rightleftarrows}} S^1$
such that $f \circ g \sim id_{S^1}$ and $g \circ f \sim id_{\{P\}}$.

W.l.o.g. $f(P) = 1 \in \mathbb{C}$.
Take a homotopy $F : S^1 \times I \to S^1$ from $f \circ g = c_1$ to id_{S^1}.

Put $G(z, t) := F(1, t)^{-1} F(z, t)$
$G : S^1 \times I \longrightarrow S^1$
give a homotopy $\tilde{G} : I \times I \longrightarrow S^1$ between loops ω_1 to c_1.
$\Rightarrow [\omega_1] = [c_1] \in \pi_1(S^1, 1)$ $\lightning$.

(c) Show that there is a contractible space X s.t. X does not deformation retract to a point. (exercise.)

See textbook Ch 0. Page 18 Exercise 6.

Proof of Thm 122: We will construct a family of maps $P = (P_n)_{\mathbb{N}_0}$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[dl, "P_{n+1}"'] & C_n(X) \arrow[r] \arrow[dl, "P_n"'] & C_{n-1}(X) \arrow[r] \arrow[dl, "P_{n-1}"'] & \cdots \arrow[dl] \\
C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & \cdots &
\end{tikzcd}$$

s.t. $\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n = f_\# - g_\#$.

(P is a chain homotopy from $C_\bullet(X)$ to $C_\bullet(Y)$)

If we have this P then $f_\# - g_\#$ maps $Z_n(X)$ to $B_n(Y)$ and thus $f_* = g_*$ on $H_n(X)$.

$$\Big(c \in Z_n(X) \Rightarrow f_\#(c) - g_\#(c) = \partial_{n+1}(P_n(c)) + P_{n-1}(\underbrace{\partial_n c}_{=0})$$

$$= \partial_{n+1}(P_n(c)) \in B_n(Y). \Big)$$

The operator P_n is called prism operators. ($n \in \mathbb{N}_0$)

Take $F: X \times I \longrightarrow Y$ a homotopy from f to g.

Take $\sigma: \Delta^n \longrightarrow X$ a singular n-simplex.

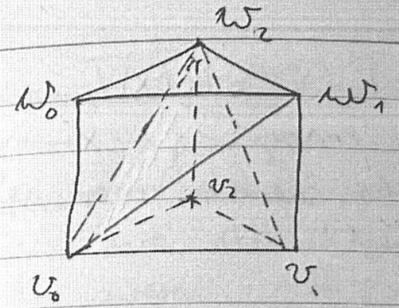
We get

$$\begin{array}{ccc} \Delta^n \times I & & \\ \sigma \times id \downarrow & \searrow^{\Sigma} & \\ X \times I & \xrightarrow{F} & Y \end{array}$$

$$\Sigma_0 = f_\#(\sigma) = f \circ \sigma$$

$$\Sigma_1 = g_\#(\sigma) = g \circ \sigma.$$

We subdevide $\Delta^n \times I$ into simplices of dimension $n+1$.



$\Delta^2 \times I$

We define $P_n: C_n(X) \longrightarrow C_{n+1}(Y)$: by

$$P_n(\sigma) := \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times id)\Big|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}$$

<u>Note:</u> The subdivision of $\Delta^n \times I$ induces a subdivision on its boundary which is equal to:

- the subdivision of $(\partial \Delta^n) \times I$ with
- $[v_0, v_1, v_2]$ and $[w_0, w_1, w_2]$.

This motivates: $\partial_{n+1}(P_n(\sigma)) + P_{n-1}(\partial_n(\sigma)) = f_\#(\sigma) - g_\#(\sigma)$.

(exercise)

□

Remark 125: As $\tilde{H}_0(X) \subseteq H_0(X)$, we have that homotopic maps $f, g : X \longrightarrow Y$ give rise to the same map $f_* = g_*$ on $\tilde{H}_0(X)$.

V 5. Excision

Idea: Given spaces $A \subseteq X$ we want a relation between $\tilde{H}_n(X)$, $\tilde{H}_n(A)$ and $\tilde{H}_n(X/A)$.

Theorem 126: Let X be a space and $\emptyset \neq A \subseteq X$ be closed s.t. $\exists\, U \subseteq X$ open with $A \subseteq U$ s.t. A is a deformation retract of U.
Then we have a long exact sequence

$$\cdots \longrightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \longrightarrow \tilde{H}_{n-1}(A) \longrightarrow \tilde{H}_{n-1}(X) \longrightarrow \tilde{H}_{n-1}(X/A) \longrightarrow \cdots \longrightarrow \tilde{H}_0(A) \longrightarrow \tilde{H}_0(X) \longrightarrow \tilde{H}_0(X/A) \rightarrow 0.$$

Thm 126 allows us to compute $\tilde{H}_n(S^m)$ $\forall n, m \in \mathbb{N}_0$.

<u>Corollary 127:</u> Let $n, m \in \mathbb{N}_0$.
Then
$$\tilde{H}_n(S^m) \simeq \begin{cases} \mathbb{Z}, & n = m \\ 0, & n \neq m. \end{cases}$$

<u>Proof:</u> <u>$m=0$</u>: Prop 118.
<u>$m>0$</u>: Take $(X, A) = (D^m, S^{m-1})$.
Then $X/A \simeq S^m$.

D^m contractible $\Rightarrow \tilde{H}_\ell(D^m) = 0 \; \forall_{\ell \in \mathbb{N}_0}$

Thm 126 (S^{m-1} is a deformation retract of $D^m \setminus \{\underline{0}\}$)

$$\Rightarrow \dots \to \tilde{H}_n(S^{m-1}) \to 0 \to \tilde{H}_n(S^m)$$
$$\hookrightarrow \tilde{H}_{n-1}(S^{m-1}) \to 0 \to \tilde{H}_{n-1}(S^m)$$
$$\hookrightarrow \dots$$
$$\hookrightarrow \tilde{H}_0(S^{m-1}) \to 0 \to \tilde{H}_0(S^m) \to 0$$

Thus $\tilde{H}_0(S^m) = 0$ and, for each $n \in \mathbb{N}$:

$$\tilde{H}_n(S^m) \simeq \tilde{H}_{n-1}(S^{m-1}) \simeq \begin{cases} \mathbb{Z}, & n = m \\ 0, & n \neq m \end{cases}$$

by induction hypothesis. □

<u>Corollary 128:</u> Let $n \in \mathbb{N}$.
Then ∂D^n is not a retract of D^n and every continuous map $f: D^n \to D^n$ has a fixed point.

<u>Proof:</u> The second assertion follows from the first by a standard proof.
Assume ∂D^n is a retract of D^n.
$r: D^n \to D^n$, s.t. $r \circ r = r$ and $\mathrm{im}(r) = \partial D^n$.
We get the identity $\mathrm{id}_{\tilde{H}_{n-1}(\partial D^n)}$:

$$\underset{\mathbb{Z}}{\underset{\shortparallel}{\tilde{H}_{n-1}(\partial D^n)}} \xrightarrow{\iota_*} \tilde{H}_{n-1}(D^n) \xrightarrow{r_*} \underset{\mathbb{Z}}{\underset{\shortparallel}{\tilde{H}_{n-1}(\partial D^n)}}$$

On the other hand $\tilde{H}_{n-1}(D^n) = 0$, and therefore $r_* = 0$.

$$\Rightarrow \quad id_{\tilde{H}_{n-1}(\partial D^n)} = (r \circ \iota)_* = r_* \circ \iota_* = 0$$

$$\lightning \quad \text{as} \quad \tilde{H}_{n-1}(\partial D^n) = \mathbb{Z} \neq 0.$$

$\square$

There is a canonical family of homology groups who fit in the long exact sequence in Thm 126 replacing $\tilde{H}_n(X/A)$.
Those are the homology groups of the quotient complex

$$C_\bullet(X) / C_\bullet(A).$$

<u>Def 129</u>: (relative homology groups)

Let A be a subspace of X.
Put $C_n(X,A) := C_n(X) / C_n(A)$.

The boundary maps factorize through the quotient

$$\begin{tikzcd}
C_n(X) \arrow[r, "\partial_n"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C_{n-1}(X) \arrow[d] \\
C_n(X,A) \arrow[r, "\partial_n"] & C_{n-1}(X,A)
\end{tikzcd}$$

because $\partial_n(C_n(A)) \subseteq C_{n-1}(A)$, $n \in \mathbb{N}$.
Thus we get a chain complex

$$\cdots \to C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A) \xrightarrow{\partial_{n-1}} \cdots \to C_1(X,A) \xrightarrow{\partial_1} C_0(X,A) \xrightarrow{\partial_0} 0$$

$$H_n(X,A) := \ker(\partial_n) / \operatorname{im}(\partial_{n+1})$$

is called <u>relative homology group</u>.

Elements of
$Z_n(X,A) = \ker(\partial_n)$ are called
relative n-cycles
$B_n(X,A) := \operatorname{im}(\partial_{n+1})$ are called
relative n-boundaries.

Remark 130: (a) Let $n \in \mathbb{N}$.
We have

$$B_n(X,A) \subseteq Z_n(X,A) \subseteq C_n(X,A) = C_n(X)/C_n(A)$$

(a1) $\bar{\alpha} \in Z_n(X,A) \iff \alpha \in C_n(X)$ s.t.
$\partial_n(\alpha) \in C_{n-1}(A)$.

(a2) $\bar{\beta} \in B_{n-1}(X,A) \iff \exists\, \alpha \in C_n(X)$
$\beta - \partial_n \alpha \in C_{n-1}(A)$

(b). $H_n(X,\emptyset) = H_n(X)$. $\forall n \in \mathbb{N}_0$!

because $C_n(\emptyset) = 0 \ \forall n \in \mathbb{N}_0$.

Thm 131 (homological algebra Thm 25)

Suppose A is a sub-space of X. Then we have a long exact sequence

$$\cdots \to H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{q_*} H_n(X,A)$$
$$\xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{\iota_*} H_{n-1}(X) \xrightarrow{q_*} H_{n-1}(X,A)$$
$$\xrightarrow{\partial_{n-1}} \cdots$$
$$\xrightarrow{\partial_1} H_0(A) \xrightarrow{\iota_*} H_0(X) \xrightarrow{q_*} H_0(X,A) \to 0$$

Here q_n is the quotient map

$$q_n : C_n(X) \longrightarrow C_n(X)/C_n(A).$$

Proof: It follows from
HA (Thm 25), because
$$0 \to C_\cdot(A) \xrightarrow{\iota_\#} C_\cdot(X) \xrightarrow{q} C_\cdot(X)/C_\cdot(A) \to 0$$

is an exact sequence of chain complexes, i.e.
$\iota_\#$ and q are chain maps and for each $n \in \mathbb{N}_0$ the sequence

$$0 \to C_n(A) \longrightarrow C_n(X) \to C_n(X)/C_n(A) \to 0$$

is exact. $\square$

We recall the construction of connecting morphisms δ_n, $n \in \mathbb{N}$.

<u>Remark 132:</u> Let $0 \to A_\bullet \xrightarrow{\alpha} B_\bullet \xrightarrow{\beta} C_\bullet \to 0$ be an exact sequence of chain complexes.
We construct

$$\delta_n : H_n(C) \longrightarrow H_{n-1}(A).$$

$$\begin{tikzcd}
0 \arrow[r] & \ker(d_{n,A}) \arrow[r] \arrow[d] & \ker(d_{n,B}) \arrow[r] \arrow[d] & \ker(d_{n,C}) \arrow[d] \arrow[dddll, bend left, "\tilde{\delta}_n"] & \\
0 \arrow[r] & A_n \arrow[r, "\alpha_n"] \arrow[d, "d_n"] & B_n \arrow[r, "\beta_n"] \arrow[d, "d_n"] & C_n \arrow[r] \arrow[d, "d_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d] & B_{n-1} \arrow[r, "\beta_{n-1}"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker}(d_{n,A}) \arrow[r] & \operatorname{coker}(d_{n,B}) \arrow[r] & \operatorname{coker}(d_{n,C}) \arrow[r] & 0
\end{tikzcd}$$

Take $[c] \in H_n(C) \Rightarrow c \in \ker(d_{n,C})$
Take $b \in B_n$ s.t. $\beta_n(b) = c$

$\beta_{n-1}(d_{n,B}(b)) = d_{n,C}(c) = 0$
$\Rightarrow \exists\, a \in A_{n-1} : \alpha_{n-1}(a) = d_{n,B}(b)$.
We put

$$\delta_n([c]) := [a] \in H_{n-1}(A).$$

<u>Remark 133:</u> If $A \neq \emptyset$ then there is a long exact sequence

$$\hookrightarrow \tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X,A) \hookrightarrow$$
$$\hookrightarrow \cdots \to \tilde{H}_0(A) \to \tilde{H}_0(X) \to H_0(X,A) \to 0$$

(Use the augmented chain complexes.)