<u>Remark 91:</u> Let X be plc, $x_0 \in X$ and $\tilde{X} \xrightarrow{p} X$ be a connected covering space.
(Exercise: Then $\tilde{X}$ is plc)
Put $F := p^{-1}(x_0)$ and $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$
We have

$$\begin{tikzcd}
\pi_1(X_0, x_0) \arrow[rr] & & \mathrm{Bij}(F) \\
N(H) \arrow[u, phantom, "\cup" rotate=90] \arrow[urr, "G"] \arrow[d] & & \\
N(H)/H \arrow[uurr, "\sigma"] \arrow[r, "\simeq"] & G(\tilde{X}) \arrow[uur, "\sigma"'] &
\end{tikzcd}$$

If $\tilde{X}$ is a universal cover, i.e. $\tilde{X}$ is simply-connected, then $\pi_1(X_0, x_0)$ act freely on $\mathrm{Bij}(F)$ via deck transformations.

<u>Example:</u> (see 89(x)) 
Then $H := p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \langle a, b^2, ba^{-1}b^{-1} \rangle \underset{\pi_1(X, x_0)}{\trianglelefteq} \langle a, b \rangle$

and $G(\tilde{X}) \simeq N(H)/H \simeq F(a,b)/H \simeq \langle b \mid b^2 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$.

IV 6 Covering space actions

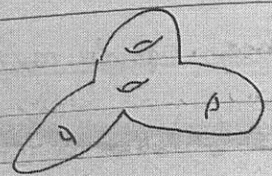
The action of $G(\tilde{X})$ on a covering space was an example of a group action on a space.

<u>Def 92:</u> Let G be a group and Y be a top. space. An <u>action of G on Y</u> is a group homomorphism $\varsigma : G \to \mathrm{Homeo}(Y)$

("G acts on Y via homeomorphisms")

<u>Example 93:</u> (a) $G = S^1$, $Y = \mathbb{C}$
$\varsigma(z)(y) := z \cdot y$ (rotation by θ, where $z = e^{i\theta}$)

(b) $G = \mathbb{Z}$, $Y = \mathbb{R}$
$\varsigma(z)(y) := y + z$ (translation by z)

(c)  $= Y$

$G := \mathbb{Z}/3\mathbb{Z}$

$\rho([a]_3) :=$ rotation by $\frac{2\pi}{3} \cdot a$.

(d) $\tilde{X} \xrightarrow{p} X$ a normal covering space, X and $\tilde{X}$ pl.

Then $G(\tilde{X})$ acts on $\tilde{X}$.
We get the following diagram:

$$\begin{tikzcd} & \tilde{X} \arrow[dl] \arrow[dr, "p"] \arrow[d, phantom, "\alpha"] & \\ G(\tilde{X}) \backslash \tilde{X} \arrow[rr, "\varphi"] & & X \end{tikzcd}$$

Exercise: Prove that φ is a homeomorphism.

Def 94: An action of G on Y is called a covering space action if $\forall y \in Y\ \exists$ open nbhd U of y:

$$\forall g \in G \setminus \{1\}: \quad g(U) \cap U = \emptyset.$$

Example 95: The actions in 93(b), (c), (d) are covering space actions. The action in 93(a) is not.

Remark 96: The condition in Def 94 is equivalent to

$$\forall g_1, g_2 \in G,\ g_1 \neq g_2: \quad g_1(U) \cap g_2(U) = \emptyset.$$

In fact 93(d) is the whole picture for covering space actions, if Y is pl.

<u>Theorem 97:</u> (covering space as a quotient of a csa)
Suppose $\rho: G \to \mathrm{Homeo}(Y)$ is a covering space action. Then

(a) $Y \xrightarrow{p} Y/G$ is a normal covering space

(b) $\mathrm{im}(\rho)$ is the group of deck transformations of p if Y is path-connected

(c) Take $y_0 \in Y$. Then

$$G \cong \pi_1\left(Y/G, p(y_0)\right) / \pi_1(Y, y_0)$$

if Y is pl.

<u>Proof:</u> (a1) p is a covering space:

p is the quotient map for a group action via homeom.
$\Rightarrow$ p is continuous and <u>open</u>.

The csa property now implies that p is a covering space.

(a2) p is normal:

Let $y_0 \in Y$. Take a nbhd U of y_0 with the csa property.

Then $p^{-1}(p(U)) = \dot{\bigcup}_{g \in G} g(U)$.

Take $y_1 \in Y$ s.t. $p(y_0) = p(y_1)$.

$\underset{\text{Def of } p}{\overset{\Rightarrow}{\uparrow}}$ $\exists g \in G: \; gy_0 = y_1$

$\rho(g) \in \mathrm{Homeo}(Y)$ is a deck transformation of p.
Thus p is normal.

(b) Let $d \in \mathrm{Homeo}(Y)$ be a deck transformation of p. Take $y_0 \in Y$. $y_1 := d(y_0)$
Then $p(y_1) = p(d(y_0)) = p(y_0)$

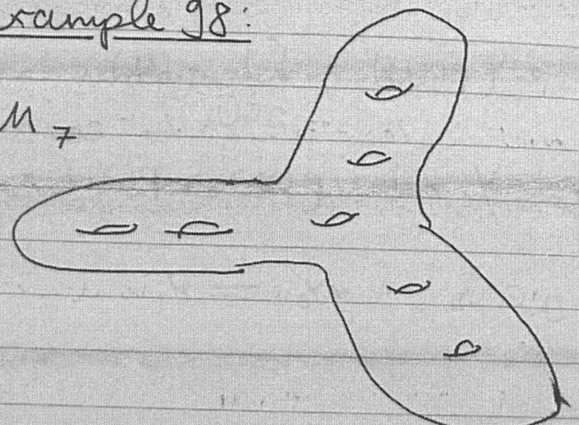
$\Rightarrow \exists\, g \in G : g y_0 = y_1$

$\underset{\substack{\uparrow \\ Y \text{ path-} \\ \text{connected}}}{\Longrightarrow} \quad d = \rho(g).$

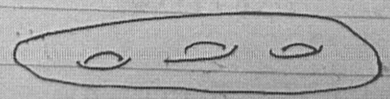
(c) follows from Prop. 90 (ii) $\square$

<u>Example 98:</u>

M_7



is part of a covering space of

M_3 

because we have a csa of $\mathbb{Z}/3\mathbb{Z}$ on M_7.

Further we have $\dfrac{\pi_1(M_3, x_0)}{p_*(\pi_1(M_7, \tilde{x}_0))}$

$\simeq \mathbb{Z}/3\mathbb{Z}$.

<u>Example 99:</u> (Symmetry groups as fundamental groups)

<u>Note</u>: Let Y be simply-connected and locally path-connected. Then $\pi_1\left(Y/G, y_0\right) \simeq G$, if

$\rho : G \longrightarrow \mathrm{Homeo}(Y)$ is an csa.

(a) $G := \{\pm 1\} \xrightarrow{\rho} \mathrm{Homeo}(S^2)$

$\rho(1)(\underline{x}) := \underline{x}$

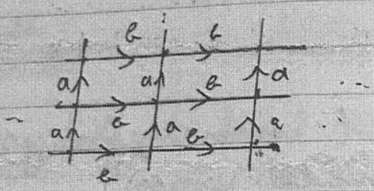
$\rho(-1)(\underline{x}) := -\underline{x}$

ρ is a csa.

S^2 is simply-conn. and loc. path-conn.

$\overset{97(c)}{\Longrightarrow} \quad \pi_1(\mathbb{R}P^2, x_0) \simeq G \simeq \mathbb{Z}/2\mathbb{Z}.$

(b) $Y = \mathbb{R}^2$ with the following graph (vertexes $= \mathbb{Z}\times\mathbb{Z}$)



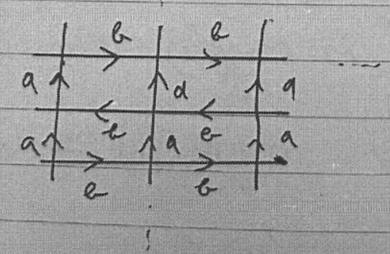
$G = \{ A_{n,m} \mid n, m \in \mathbb{Z} \} \simeq \mathbb{Z}\times\mathbb{Z}$.

$A_{n,m}(x,y) := (x+n, y+m)$

$\mathbb{R}^2$ simply-conn. and loc. path-connected

$\overset{}{\underset{\substack{\uparrow \\ G \curvearrowright \mathbb{R}^2 \\ \text{is csa}}}{\Longrightarrow}} \quad \pi_1(S^1\times S^1) \simeq \pi_1({}_G\backslash\mathbb{R}^2) \simeq G \underset{\mathbb{Z}\times\mathbb{Z}.}{\overset{}{\shortparallel}}$

(c) $Y = \mathbb{R}^2$



Let $r_z : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad (z \in \mathbb{R})$

be the orthogonal reflection on the line $\mathbb{R}(0,1) + (z,0)$

$$G := \{ A_{n,2m} \mid n, m \in \mathbb{Z} \}$$
$$\dot\cup \{ r_0 \circ A_{n,2m+1} \mid n,m \in \mathbb{Z} \}$$
$$\dot\cup \{ r_{\frac{1}{2}} \circ A_{n,2m+1} \mid n,m \in \mathbb{Z} \}$$
$$= \{ A_{n,2m} \mid n,m \in \mathbb{Z} \}$$
$$\cup \{ r_z \circ A_{n,2m+1} \mid n,m \in \mathbb{Z},\ z \in \tfrac{1}{2}\mathbb{Z} \}$$

$G \curvearrowright \mathbb{R}^2$ is a csa. and $\mathbb{R}^2$ is simply-conn. and loc. path-conn.

$\Longrightarrow \quad \pi_1(K) \simeq \pi_1({}_G\backslash\mathbb{R}^2) \simeq G.$

Example 100: (Cayley complex)

$G = \langle g_\alpha \mid r_\beta \rangle$. The Cayley complex is the universal cover of X_G, see Corollary 60 for X_G.

$\tilde{X}_G$ (Cayley complex) 2-dim cell complex:

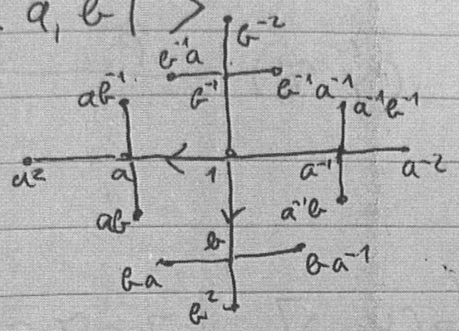
- A vertex for each group element $g \in G$
- For each $g \in G$ and α an oriented edge $(g, g g_\alpha)$
- For each $g \in G$ and β attach a 2-cell to the oriented circle given by $r_\beta = g_{\alpha_1}^{\varepsilon_1} g_{\alpha_2}^{\varepsilon_2} \cdots g_{\alpha_l}^{\varepsilon_l}$

($\varepsilon_i \in \{\pm 1\}$.)

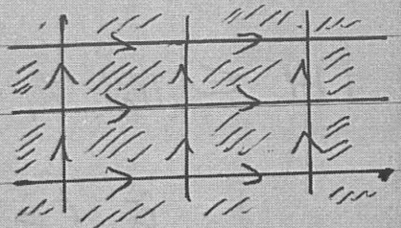
$g, g g_{\alpha_1}^{\varepsilon_1}, g g_{\alpha_1}^{\varepsilon_1} g_{\alpha_2}^{\varepsilon_2}, \ldots, g g_{\alpha_1}^{\varepsilon_1} \cdots g_{\alpha_{l-1}}^{\varepsilon_{l-1}}, g r_\beta = g.$

(Exercise: $\tilde{X}_G$ is simply-connected and $\tilde{X}_G \to X_G$ a universal cover)

(a) $G = \langle a, b \mid \ \rangle$



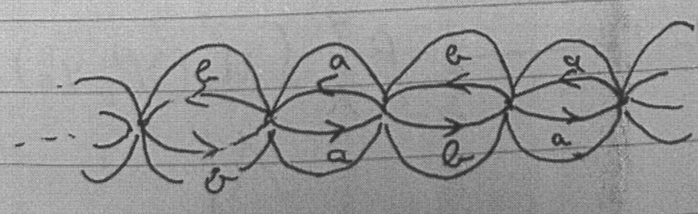
(b) $G = \langle a, b \mid aba^{-1}b^{-1} \rangle \simeq \mathbb{Z} \times \mathbb{Z}$



$\tilde{X}_G$

$X_G \simeq S^1 \times S^1$

(c) $G = \langle a, b \mid a^2, b^2 \rangle$



$\tilde{X}_G$

$X_G \simeq \mathbb{R}P^2 \vee \mathbb{R}P^2$.

<u>Overview 10.1</u> Suppose (Y, y_0) is pls. We only consider connected spaces in this remark.

We fix $(\tilde{Y}, \tilde{y}_0) \xrightarrow{\tilde{p}} (Y, y_0)$ a c.s.

Then

$$\{[(Y', y_0') \xrightarrow{p^{Y'}} (Y, y_0)]_{\cong} \mid \exists\, (\tilde{Y}, \tilde{y}_0) \xrightarrow{\tilde{p}_{Y'}} (Y', y_0')$$

$$\text{s.t. } p^{Y'} \circ p^{\tilde{Y}}_{Y'} = \tilde{p} \}$$

<u>1–1</u> $\{ H \mid \pi_1(Y, y_0) \supseteq H \supseteq \tilde{p}_*(\pi_1(\tilde{Y}, \tilde{y}_0)) \}$

(a)

$$\begin{array}{ccc}
(\tilde{Y}, \tilde{y}_0) & & \tilde{p}_*(\pi_1(\tilde{Y}, \tilde{y}_0)) \\
\downarrow{\scriptstyle \tilde{p}_{Y'}} & & \wedge| \\
(Y', y_0') & \longmapsto & p^{Y'}_*(\pi_1(Y', y_0')) \\
\downarrow{\scriptstyle p^{Y'}} & & \wedge| \\
(Y, y_0) & & \pi_1(Y, y_0)
\end{array}$$

(b)

$$\begin{array}{cccc}
 & (\tilde{Y}, \tilde{y}_0) & & \tilde{H} \\
N(\tilde{H})/\tilde{H} & \downarrow{\scriptstyle p^{H'}} & & \wedge| \\
 & (Y_{H'}, y_{H'}) & \longleftarrow\!\shortmid & H' \\
N(H')/H' & \downarrow{\scriptstyle p_{H'}} & & \wedge| \\
 & (Y, y_0) & & \pi_1(Y, y_0)
\end{array}$$

(c) Analogy with Galois theory

$F^{sep} \supseteq \tilde{E} \supseteq E' \supseteq F$ field extensions.

$$\begin{array}{ccc}
 & \tilde{E} & \mathcal{G}(F^{sep} \mid \tilde{E}) \\
\mathrm{Aut}(\tilde{E} \mid E') & | & \wedge| \\
 & E' & \mathcal{G}(F^{sep} \mid E') \\
\mathrm{Aut}(E \mid F) & | & \wedge| \\
 & F & \mathcal{G}(F^{sep} \mid F).
\end{array}$$

V Homology

Fundamental groups are not enough to distinguish between spaces, e.g. $\pi_1(S^m) = 0 \; \forall m \geq 2$.

Higher homotopy groups are difficult to compute and behave unintuitively.
For example $\pi_n(S^m) \neq 0$ for ∞-many (n,m) with $n > m$.

Homology groups are much easier to compute and still provide a lot of information about the space.

To give an idea about the singular homology groups we first define simplicial homology groups.

V 1. Δ-complexes

Def 102: (n-simplex)
An n-simplex in $\mathbb{R}^m$ is a set of the form

$$[v_0, v_1, \dots, v_n] := \left\{ \sum_{i=0}^{n} t_i v_i \;\middle|\; t_i \geq 0, \sum_{i=0}^{n} t_i = 1 \right\}$$

where $v_0, v_1, \dots, v_n$ are elements of $\mathbb{R}^m$ such that $v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$ are linearly independent.

Example 103: (a) 0-simplex $\overset{\bullet}{v_0}$

(b) 1-simplex 

(c) 2-simplex 

(d) standard n-simplex

$$\Delta^n := \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; (0 \leq t_i \;\forall i) \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}$$

$$= [e_0, e_1, e_2, \dots, e_n]$$

$$e_i := (0, \dots, 0, \overset{(i)}{1}, 0, \dots, 0) \in \mathbb{R}^{n+1}.$$

Terminology 104: The notation

$[v_0, v_1, \dots, v_n]$ indicates an ordering on the set of vertices: $v_0 < v_1 < v_2 < \dots < v_n$.

Thus n-simplex means n-simplex with ordering on the set of vertices.

Def 105: (a) The $(n-1)$-simplices
$[v_1, \dots, v_n]$, $[v_0, v_2, \dots, v_n]$,
$[v_0, \dots, \hat{v}_i, \dots, v_n]$, $i = 0, \dots, n$
are called faces of $[v_0, \dots, v_n]$.

(b) $\partial \Delta^n :=$ union of all faces.
<u>boundary of Δ^n</u>

$$\mathring{\Delta}^n := \Delta^n \setminus \partial \Delta^n \quad \underline{\text{interior of } \Delta^n}$$

Rk 106: For every n-simplex $[v_0, \dots, v_n]$ there is a canonical homeomorphism

$$\Delta^n \longrightarrow [v_0, \dots, v_n]$$
$$(t_0, t_1, \dots, t_n) \longmapsto \sum_{i=0}^{n} t_i v_i$$

This will be used a lot later.

Def 107 (Δ - complex)
A Δ - complex structure on a space X is a family of maps $\sigma_\alpha : \Delta^{d_\alpha} \longrightarrow X$, $\alpha \in A$, such that

(i) $\forall_\alpha$: $\mathring{\sigma}_\alpha := \sigma_\alpha|_{\mathring{\Delta}^{d_\alpha}}$ is injective

and $\dot{\bigcup}_{\alpha \in A} \operatorname{im}(\mathring{\sigma}_\alpha) = X$

(ii) $\forall_{\alpha \in A}$ $\forall_F$ face of Δ^{d_α} : $\exists_{\beta \in A}$

$$\sigma_\alpha|_F = \sigma_\beta$$

Here we used the map from Rk 106.

(iii) $\forall_{A \subseteq X}$: (A open in X

$\Leftrightarrow$ $\forall_{\alpha \in A}$: $\sigma_\alpha^{-1}(A)$ open in Δ^{d_α})

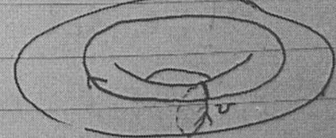
Examples 108:

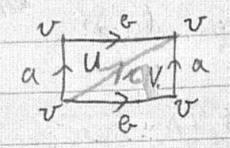
(a) Δ^n has a canonical Δ-complex structure
$$A = \{ S \mid \underset{\neq \emptyset}{S} \subseteq \{0, \dots, n\}\}$$
$$\sigma_S : \Delta^{|S|-1} \longrightarrow \Delta^n$$
$$\sigma_S(t_0, \dots, t_{|S|}) := \sum_{i=0}^{|S|} t_i e_{j_i}$$

where $S = \{ j_0 < j_1 < \dots < j_{|S|} \}$

$$|A| = 2^{n+1} - 1.$$

(b) T^2





T has a Δ-complex structure with $|\mathcal{A}| = 6$

$\mathcal{A} = \{a, b, U, V, c, v\}$

$\sigma_a : [e_0, e_1] \longrightarrow$ 

σ_b, σ_c similar

$\sigma_v : [e_0] \longrightarrow \{v\}$

$\sigma_U : [e_0, e_1, e_2] \longrightarrow \overline{U}$

$\sigma_V : [e_0, e_1, e_2] \longrightarrow \overline{V}.$

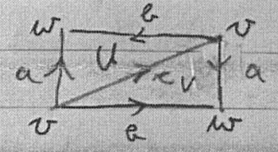
with $\sigma_U|_{[e_0,e_1]} = \sigma_a = \sigma_V|_{[e_1,e_2]}$

$$\sigma_U|_{[e_0,e_2]} = \sigma_c = \sigma_V|_{[e_0,e_2]}$$

$$\sigma_U|_{[e_1,e_2]} = \sigma_b = \sigma_V|_{[e_0,e_1]}$$

Note that with the above choice of σ_b, σ_a, we cannot have $\sigma_V|_{[e_0,e_1]} = \sigma_a$.

(c) $\mathbb{R}P^2$

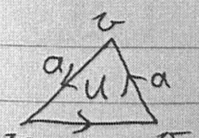


$$\sigma_U|_{[e_0,e_1]} = \sigma_c = \sigma_V|_{[e_0,e_1]}$$

$$\sigma_U|_{[e_0,e_2]} = \sigma_a = \sigma_V|_{[e_1,e_2]}$$

$$\sigma_U|_{[e_1,e_2]} = \sigma_b = \sigma_V|_{[e_0,e_2]}.$$

(d)



does not define a Δ-complex structure,

because if $\sigma_U|_{[e_0,e_1]} = \sigma_a$

then $\sigma_U|_{[e_0,e_2]} = \overline{\sigma_a}$

but $\mathrm{im}(\overline{\sigma_\alpha}) = \mathrm{im}(\sigma_\alpha)$ and $\overline{\sigma}_\alpha \neq \sigma_\alpha$.

(e) Exercise: Find a Δ-complex structure for the Kleinian bottle with $|\mathcal{A}| = 6$

V 2. Simplicial homology

We write $[v_0, v_1, \dots, v_n] = \Delta^n$ for the standard simplex.

<u>Def 109</u>: (simplicial chain complex)
Let X be a top space with a Δ-complex structure $(\sigma_\alpha)_{\alpha \in \mathcal{A}}$.

$n \in \mathbb{N}_0$: $\Delta_n(X) := \bigoplus_{\substack{\alpha \in \mathcal{A} \\ d_\alpha = n}} \mathbb{Z}\sigma_\alpha$

An element $\sum_{\alpha \in \mathcal{A}} n_\alpha \sigma_\alpha \in \Delta_n(X)$ is called an <u>n-chain.</u>

For each $n \in \mathbb{N}$ we define a <u>boundary homomorphism</u>

$$\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$$

$$\partial_n(\sigma) := \sum_{i=0}^{n} (-1)^i \sigma\big|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}.$$

and $\partial_0 : \Delta_0(X) \longrightarrow 0.$

$(\Delta_n(X), \partial_n)_{n\in\mathbb{N}_0}$ is called the simplicial chain complex for $(X, (\sigma_\alpha)_\alpha)$.

We write

$$\to \Delta_3(X) \xrightarrow{\partial_3} \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\partial_0} 0$$

<u>Lemma 110:</u> $\forall n \in \mathbb{N}: \partial_{n-1} \circ \partial_n = 0$

<u>Proof:</u>

$$\partial_{n-1}\circ\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \partial_{n-1}\left(\sigma\big|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\right)$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{i-1} (-1)^j(-1)^i \sigma\big|_{[v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]}$$

$$+ \sum_{i=0}^{n}\sum_{j=i+1}^{n} (-1)^{j-1}(-1)^i \sigma\big|_{[v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]}$$

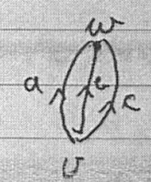
$= 0$ $\square$

<u>Def 111:</u> $n\geq 0$: $H_n^{\Delta}(X) = \ker\partial_n / \operatorname{im}\partial_{n+1}$

is called the nth simplicial homology group of X.

<u>Remark 112:</u> $H_n^{\Delta}(X)$ does not depend on the Δ-complex structure. We will prove this later.

<u>Examples 113:</u> (a)

$X =$ 

Δ-complex structure $\sigma_v, \sigma_w, \sigma_a, \sigma_b, \sigma_c$

$\Delta_0(X) = \mathbb{Z}\sigma_v \oplus \mathbb{Z}\sigma_w$

$\Delta_1(X) = \mathbb{Z}\sigma_a \oplus \mathbb{Z}\sigma_b \oplus \mathbb{Z}\sigma_c$

$n > 1$: $\Delta_n(X) = 0$. and
$H_n^{\Delta}(X) = 0$.

$$H_0^{\Delta}(X) = \ker(\partial_0) / \operatorname{im}(\partial_1)$$

$$\underset{\substack{\uparrow \\ \partial_0 = 0}}{=} \Delta_0(X) / \operatorname{im}(\partial_1)$$

$$= \Delta_0(X) / \langle \sigma_w - \sigma_v \rangle$$

$$\underset{\uparrow}{\cong} \mathbb{Z}\sigma_v \simeq \mathbb{Z}.$$

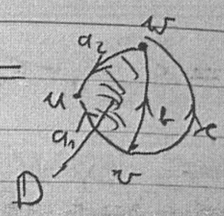
$\{\sigma_w - \sigma_v, \sigma_v\}$ is a basis of $\Delta_0(X)$

$$\Big(\partial_1(n_a \sigma_a + n_b \sigma_b + n_c \sigma_c)$$
$$= n_a(\sigma_w - \sigma_v) + n_b(\sigma_w - \sigma_v) + n_c(\sigma_w - \sigma_v)$$

$$= (n_a + n_b + n_c)(\sigma_w - \sigma_v). \Big)$$

$$H_1^{\Delta}(X) = \ker(\partial_1) / \operatorname{im}(\partial_2) \underset{\substack{\uparrow \\ \partial_2 = 0}}{\cong} \ker \partial_1$$

$$= \mathbb{Z}(\sigma_a - \sigma_b) + \mathbb{Z}(\sigma_a - \sigma_c)$$
$$\simeq \mathbb{Z} \oplus \mathbb{Z}.$$

(b) $Y =$ 

$$\Delta_2(Y) = \mathbb{Z} D$$
$$\Delta_1(Y) = \mathbb{Z}\sigma_{a_1} \oplus \mathbb{Z}\sigma_{a_2} \oplus \mathbb{Z}\sigma_b \oplus \mathbb{Z}\sigma_c$$
$$\Delta_0(Y) = \mathbb{Z}\sigma_u \oplus \mathbb{Z}\sigma_v \oplus \mathbb{Z}\sigma_w$$

$$\partial_2(D) = \sigma_{a_2} - \sigma_{a_1} + \sigma_b$$

$$\partial_1(n_{a_1}\sigma_{a_1} + n_{a_2}\sigma_{a_2} + n_b \sigma_b + n_c \sigma_c)$$

$$= n_{a_1}(\sigma_u - \sigma_w) + n_{a_2}(\sigma_u - \sigma_v)$$

$$+ (\sigma_w - \sigma_v) n_b + n_c(\sigma_w - \sigma_v)$$

$$= (n_{a_1} + n_{a_2})\sigma_u + (n_b + n_c - n_{a_1})\sigma_w$$

$$+ (-n_{a_2} - n_b - n_c)\sigma_v$$

$$\overset{!}{=} 0 \iff n_{a_1} = n_b + n_c = -n_{a_2}.$$

$$\Rightarrow \ker(\partial_1) = \langle \sigma_{a_1} - \sigma_{a_2} + \sigma_b, \sigma_{a_1} - \sigma_{a_2} + \sigma_c \rangle$$

$$\Rightarrow \quad H_1^{\Delta}(X) = \ker(\partial_1) / \operatorname{im}(\partial_2) \simeq \mathbb{Z}.$$

$$H_2^{\Delta}(X) = \ker(\partial_2) / \operatorname{im}(\partial_3)$$

$$\simeq \ker(\partial_2) \underset{\substack{\uparrow \\ \partial_2 \text{ injective}}}{\simeq} 0$$

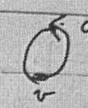
$$H_0^{\Delta}(X) = \Delta_0(X) / \langle \sigma_u - \sigma_w, \sigma_u - \sigma_v, \sigma_w - \sigma_v \rangle$$

$$\simeq \mathbb{Z}\sigma_v \simeq \mathbb{Z}.$$

$\{\sigma_u - \sigma_w, \sigma_u - \sigma_v, \sigma_v\}$

$\mathbb{Z}$ - basis of $\Delta_0(X)$

<u>$n \geq 2$:</u> $H_n^{\Delta}(X) = 0.$

(c) S^1 

$$\Delta_1(S^1) = \mathbb{Z}\sigma_a$$
$$\Delta_0(S^1) = \mathbb{Z}\sigma_v$$

$\partial_n = 0 \quad \forall_{n \in \mathbb{N}_0}.$

$$\Rightarrow H_n^{\Delta}(X) = \begin{cases} \mathbb{Z}, & n = 0,1 \\ 0, & n > 1. \end{cases}$$

(d) $T = S^1 \times S^1$

Consider the Δ - complex structure from 108 (a).

$$\Delta_0(T) = \mathbb{Z}\sigma_v$$
$$\Delta_1(T) = \mathbb{Z}\sigma_a \oplus \mathbb{Z}\sigma_b \oplus \mathbb{Z}\sigma_c$$
$$\Delta_2(T) = \mathbb{Z}\sigma_U \oplus \mathbb{Z}\sigma_V$$

$\partial_1 = 0 \Rightarrow H_0^{\Delta}(T) = \Delta_0(T) \simeq \mathbb{Z}$

$$\partial_2(n_U\sigma_U + n_V\sigma_V) = n_U(\sigma_b - \sigma_c + \sigma_a) + n_V(\sigma_a - \sigma_c + \sigma_b)$$