Def 86: (Deck transformations) Let $p: \tilde{X} \to X$ be a covering space. An isomorphism of p is called a *deck transformation* (or *covering transformation*).

Example 87:

(a) $\mathbb{R} \xrightarrow{p} S^1$

The deck transformations are
$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x + m \end{aligned}, \quad m \in \mathbb{Z}.$$

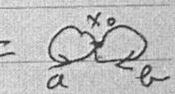
(They are all by uniqueness as $\mathbb{R}$ is connected, see 70.)

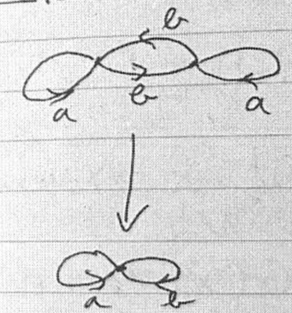
(b) $$\begin{aligned} S^1 &\longrightarrow S^1 \\ z &\longmapsto z^n \end{aligned} \qquad (n \in \mathbb{N})$$

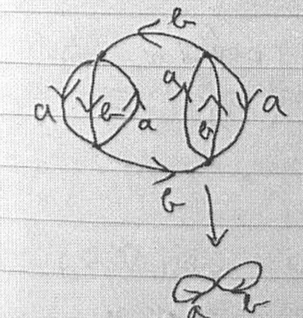
The deck transformations are the rotations by
$$0, \frac{2\pi}{n}, 2\frac{2\pi}{n}, \dots, (n-1)\frac{2\pi}{n}.$$

Def 88: A covering space $p: \tilde{X} \to X$ is called *normal* if

$\forall_{x \in X} \forall \tilde{x}, \tilde{x}' \in p^{-1}(x) \; \exists \; \tau: \tilde{X} \to \tilde{X}$ deck trafo: $\tau(\tilde{x}) = \tilde{x}'$.

Example 89: $X = S^1 \vee S^1 =$ 

(i)  is normal.

(ii)  is not normal.

<u>Prop. 90:</u> Let $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$ be a covering space, $\tilde{X}$ path-connected and X path-conn. and loc. path-conn.

Let $H := p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

(i) p is normal $\Leftrightarrow H \trianglelefteq \pi_1(X, x_0)$

(ii) $G(\tilde{X})$ = group of deck transf.

$$\simeq N(H)/H.$$

In particular if $\tilde{X} \to X$ is a universal cover then $H = \{1\}$ and $G(\tilde{X}) \simeq \pi_1(X, x_0)$.

<u>Proof:</u> (i) Let $[f] \in \pi_1(X_0, x_0)$ and $\tilde{x}_1 \in p^{-1}(x_0)$ s.t. $\tilde{f}_{\tilde{x}_0}(1) = \tilde{x}_1$.

$$[f] \in N(H) \Leftrightarrow p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

$\underset{\substack{\uparrow \\ \text{lifting}}}{\Leftrightarrow}$ $\exists$ deck transformation Σ: $\Sigma(\tilde{x}_0) = \tilde{x}_1$

criterion 69, as $\tilde{X}$ path-conn. & loc. path-conn.

(ii) We define a group homomorphism. Fix $\tilde{x}_0 \in \tilde{X}$.

$$N(H) \longrightarrow G(\tilde{X}).$$
$$[f] \longmapsto \Sigma_f$$

$$\Sigma_f : \tilde{X} \longrightarrow \tilde{X}$$
$$\tilde{\gamma}(1) \longmapsto \widetilde{f\gamma}(1) \ (\text{start } \tilde{x}_0)$$

(γ path in X, start x_0)

well-defined: Take γ, γ' (start x_0)

s.t. $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$

$\Rightarrow$ $\widetilde{\gamma\bar{\gamma}'} \in \Omega(\tilde{X}_0, \tilde{x}_0)$

$\overset{66(ii)}{\Rightarrow}$ $\gamma\bar{\gamma}' \in H$

$\underset{\substack{\uparrow \\ [f] \in N(H)}}{\Longrightarrow}$ $\widetilde{\gamma\bar{\gamma}'}_{\tilde{f}(1)} \in \Omega(\tilde{X}_0, \tilde{f}(1))$.

$\Longrightarrow$ $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$.

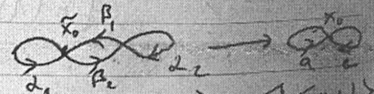
· group homomorphism: exercise

· kernel $= H$: exercise $\square$

<u>Remark 91:</u> Let X be pls, $x_0 \in X$ and $\tilde{X} \xrightarrow{p} X$ be a connected covering space.
(Exercise: Then $\tilde{X}$ is pls)
Put $F := p^{-1}(x_0)$ and $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$
We have

$$\begin{tikzcd}
\pi_1(X_0, x_0) \arrow[rr] & & \mathrm{Bij}(F) \\
\cup\!| & G & \\
N(H) \arrow[uurr] \arrow[d] & \sigma & \sigma \\
N(H)/H \arrow[uuurr] & \simeq & G(\tilde{X}) \arrow[uuu]
\end{tikzcd}$$

If $\tilde{X}$ is a universal cover, i.e. $\tilde{X}$ is simply-connected, then $\pi_1(X_0, x_0)$ act freely on $\mathrm{Bij}(F)$ via deck transformations.

<u>Example:</u> (see 89 (i)) 

Then $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \langle a, b^2, b a^{-1} b^{-1} \rangle_{\pi_1(X, x_0)} \trianglelefteq \langle a, b \rangle$
and $G(\tilde{X}) \simeq N(H)/H \simeq F(a,b)/H \simeq \langle b \mid b^2 \rangle$
$\simeq \mathbb{Z}/2\mathbb{Z}$.

<u>IV 6 Covering space actions</u>

The action of $G(\tilde{X})$ on a covering space was an example of a group action on a space.

<u>Def 92:</u> Let G be a group and Y be a top. space. An <u>action of G on Y</u> is a group homomorphism $\varsigma : G \to \mathrm{Homeo}(Y)$

("G acts on Y via homeomorphisms")

<u>Example 93:</u> (a) $G = S^1$, $Y = \mathbb{C}$
$\varsigma(z)(y) := z \cdot y$ (rotation by θ, where $z = e^{2\pi i \theta}$)

(b) $G = \mathbb{Z}$, $Y = \mathbb{R}$
$\varsigma(z)(y) := y + z$ (translation by z)

(c)  $= Y$

$G := \mathbb{Z}/3\mathbb{Z}$

$\rho([a]_3) :=$ rotation by $\frac{2\pi}{3}a$

(d) $\tilde{X} \xrightarrow{p} X$ a normal covering space, X and $\tilde{X}$ pl.

Then $G(\tilde{X})$ acts on $\tilde{X}$.
We get the following diagram:

$$\begin{tikzcd}
 & \tilde{X} \arrow[dl] \arrow[dr, "p"] \arrow[d, phantom, "\alpha"] & \\
G(\tilde{X}) \backslash \tilde{X} \arrow[rr, "\varphi"] & & X
\end{tikzcd}$$

<u>Exercise</u>: Prove that φ is a homeomorphism.

<u>Def 94</u> An action of G on Y is called a <u>covering space action</u> if
$\forall y \in Y \ \exists$ open nbhd U of y:

$\forall g \in G \setminus \{1\}: \quad g(U) \cap U = \emptyset$.

<u>Example 95</u>: The actions in 93(b),(c),(d) are covering space actions.
The action in 93(a) is not.

<u>Remark 96</u>: The condition in Def 94 is equivalent to
$\forall g_1, g_2 \in G, \ g_1 \neq g_2: \quad g_1(U) \cap g_2(U) = \emptyset$.

In fact 93(d) is the whole picture for covering space actions, if Y is pl.