IV 3. Classification of covering spaces

Let X be path-connected and loc. path-connected
Under an extra condition
we will observe that

(1) isomorphism classes of connected covering spaces are 1-1 to the set of conjugacy classes of subgroups of $\pi_1(X, x_0)$

(2) isomorphism classes of connected based covering spaces are 1-1 to set of subgroups of $\pi_1(X, x_0)$

The extra condition we need is "semilocally simply-connected".

<u>Example 71:</u> $\mathbb{R}P^2 =: X$.

$$\pi_1(X, x_0) \simeq \langle a \mid a^2 \rangle \simeq \mathbb{Z}/2\mathbb{Z}.$$

So we have only two isomorphism classes of
- covering spaces
- based covering spaces.

- $\mathbb{R}P^2 \xrightarrow{id} \mathbb{R}P^2$ $\quad \pi_1(\mathbb{R}P^2, \tilde{x}_0) \cong \mathbb{Z}/2\mathbb{Z}$.
- $S^2 \longrightarrow \mathbb{R}P^2$ $\quad \pi_1(S, \tilde{x}_0) = \{1\}$

<u>Def 72:</u> A space X is called <u>semilocally simply-connected</u> if $\forall x \in X$ $\exists U$ open nbhd of x: the map $\pi_1(U,x) \longrightarrow \pi_1(X,x)$ is trivial. Write (slsc)

<u>Remark 73:</u> (a) A simply connected space is semilocally simply-connected.
(b) The condition that $\pi_1(U,x) \longrightarrow \pi_1(X,x)$ is trivial means that for every loop $f \in \Omega(U,x)$

there exists a homotopy in X to the constant loop c_x.

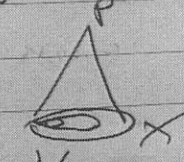
<u>Example 74:</u> $X = \bigcup_{n \in \mathbb{N}} C_n$ (see Example 50)
$C_n \subseteq \mathbb{R}^2$ circle with center $(0, \frac{1}{n})$ and radius $\frac{1}{n}$.

X is not slsc, but the cone on X is slsc.

<u>Proof:</u> (i) $\pi_1(X, 0) \xrightarrow{g} \prod_{n \in \mathbb{N}} \pi_1(C_n, 0)$

So for every open nbhd of 0 the map
$$\pi_1(U, 0) \xrightarrow{i_*} \pi_1(X, 0)$$
is non-trivial, because $g \circ i_*$ is non-trivial.

(ii) The cone on X deformation retracts to a point
$\Rightarrow \pi_1(CX, p) = \{1\}$



$\Rightarrow \pi_1(CX, x) = \{1\}, \ \forall x \in CX.$ $\square$

Proposition 75: (Existence of a simply-connected covering space)

Suppose that X is
(pls) $\begin{cases} \cdot \text{ path-connected} \\ \cdot \text{ locally path-connected} \\ \cdot \text{ semilocally simply-connected} \end{cases}$ (pl)

Then X has a simply-connected covering space.

Proof: · $\tilde{X} := \{ [\gamma]_X \mid \gamma \text{ path in } X \text{ starting at } x_0 \}$

$$\tilde{X} \xrightarrow{p} X \qquad p([\gamma]) := \gamma(1).$$

(homotopy means hoe homotopy of path $\Rightarrow p$ is well-defined)

X path-connected $\Rightarrow p$ is surjective.

· We need a topology on $\tilde{X}$.

$\mathcal{G} := \{ U \subseteq X \mid U \text{ path-conn. and slsc.} \}$

$\mathcal{G}$ is a basis of the top. of X.

Take $U \in \mathcal{G}$ and $[\gamma] \in \tilde{X}$ with $\gamma(1) \in U$.
Put
$$U_{[\gamma]} := \{ [\gamma\eta] \mid \eta \text{ path in } U \text{ with } \gamma(1) = \eta(0) \}$$

Claim: $\tilde{\mathcal{G}} := \{ U_{[\gamma]} \mid U \in \mathcal{G}, [\gamma] \in \tilde{X} \text{ s.t. } \gamma(1) \in U \}$ is a basis of a topology on $\tilde{X}$.

Proof: $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$

$\Rightarrow U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$

Take $W \in \mathcal{G}$ s.t. $\gamma''(1) \in W \subseteq U \cap V$.
$\Rightarrow W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']}$ □(claim)

- **Claim**: $\tilde{X} \xrightarrow{p} X$ is a covering space.

Proof: $U \in \mathcal{G} \Rightarrow p^{-1}(U) = \bigcup_{\substack{\gamma(1)\in U \\ [\gamma]\in\tilde{X}}} U_{[\gamma]}$

("$\supseteq$" ✓ "$\subseteq$" $[\gamma] \in p^{-1}(U) \Rightarrow \gamma(1) = p([\gamma]) \in U$)
$\Rightarrow [\gamma] \in U_{[\gamma]}$)

Thus p is continuous.
Further we have $\forall U_{[\gamma]}$: $p(U_{[\gamma]}) = U$ is open.
Thus p is an open map.
Thus $\forall U_{[\gamma]}$:

$$p|_{U_{[\gamma]}} : U_{[\gamma]} \longrightarrow U \text{ is a}$$

homeomorphism.
We need to show that
$\{U_{[\gamma]} \mid [\gamma] \in \tilde{X}, \gamma(1) \in U\}$
gives a partition of $p^{-1}(U)$.

Pf: $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']} \Rightarrow U_{[\gamma]} = U_{[\gamma'']} = U_{[\gamma']}$ □

□(Claim)

- $\tilde{X}$ is simply connected.

(1) path-connected: $[\gamma] \in \tilde{X}$.

$$\gamma_t(s) := \begin{cases} \gamma(s), & 0 \le s \le t \\ \gamma(t), & t \le s \le 1. \end{cases}$$

$t \longmapsto [\gamma_t]$ is a path in $\tilde{X}$, because it lifts γ and p is a covering space, so that $t \longmapsto [\gamma_t]$ is continuous.

This path connects $[c_{x_0}] =: \tilde{x}_0$ with $[\gamma] = [\gamma_1]$.

(2) $\pi_1(\tilde{X}, \tilde{x}_0) = \{1\}$.
p is a covering space
$\overset{66(i)}{\Rightarrow}$ p_* is injective.
We show that p_* is trivial.
Take $[\gamma] \in \operatorname{im}(p_*)$
Then the lift $t \longmapsto [\gamma_t]$ is a loop by 66(ii).
$\Rightarrow [c_{x_0}] = [\gamma_0] = [\gamma_1] = [\gamma]$

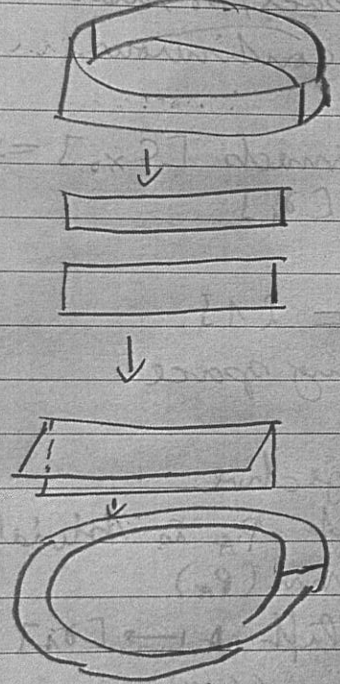
□

<u>Example 76</u>: $X_n := S^1 \times I \big/ \sim$, $n \in \mathbb{N}$

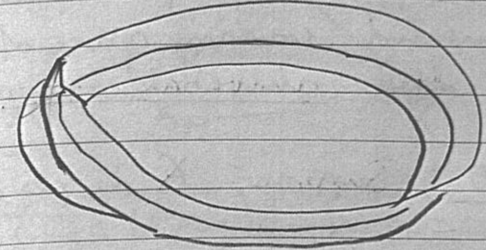
The $\sim$ is given by $(z,1) \sim (z e^{\frac{2\pi i}{n}}, 1)$

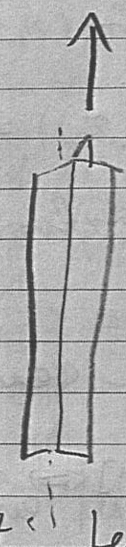
X_2 : Möbius band.

How to see this?



X_3:





This is the universal cover of X_3

<u>Homework 77</u>: Let $X_{n,m} := X_n \sqcup X_m \big/ S^1 \times \{0\}$.

Read about the construction of the universal cover of $X_{n,m}$.

Def 78: Suppose X is pl. A simply connected covering space of X is called <u>universal cover of X</u>.

Prop. 79: Suppose X is pls and $x_0 \in X$. Let $H \leq \pi_1(X, x_0)$. Then there is a based covering space $(X_H, x_H) \xrightarrow{p} (X, x_0)$ such that

$$p_*(\pi_1(X_H, x_H)) = H.$$

Proof: We use X_H from the proof of Prop. 75.

$$X_H := \tilde{X}/\sim_H \quad \text{(quotient topology)}$$

$\sim_H$: $[\gamma] \sim_H [\gamma'] \overset{\text{def.}}{\iff} \gamma(1) = \gamma'(1)$ and $[\gamma\bar{\gamma}'] \in H$.

Claim: $X_H \xrightarrow{p_H} X$, $p_H(H[\gamma]) = \gamma(1)$ is a covering space.

Proof:

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "q"'] \arrow[dr, "p_{[1]}"] & \\ X_H \arrow[rr, "p_H"] & & X \end{tikzcd}$$

$$q^{-1}(q(U_{[\gamma]})) = \bigcup_{[\gamma'] \in H[\gamma]} U_{[\gamma']}$$

Thus q is open. (Continuous by definition of quotient topology)

We restrict to $U_{[\gamma]}$

$$\begin{tikzcd} & U_{[\gamma]} \arrow[dl, "q"'] \arrow[dr, "p_{[1]}"] & \\ q(U_{[\gamma]}) \arrow[rr, "p_H"] & & U \end{tikzcd} \qquad (*)$$

In $(*)$ $p_{[1]}|_{U_{[\gamma]}}$ is bijective

Thus $q|_{U_{[\gamma]}}$ and $p_H|_{q(U_{[\gamma]})}$ are bijective. $\Rightarrow q|_{U_{[\gamma]}}$ is a homeo.

$\Rightarrow$ $p_H|$ is a homeo.

$p_{[1]}|$ homeo.

$$p_H^{-1}(\mathcal{U}) = \bigcup_{\gamma(1)\in\mathcal{U}} q(\mathcal{U}_{[\gamma]})$$

We have to show that we obtain a partition by this union

$$q(\mathcal{U}_{[\gamma]}) \cap q(\mathcal{U}_{[\gamma']}) \neq \emptyset$$

$\Rightarrow \exists\, \eta, \eta'$ path in $\mathcal{U}$ from $\gamma(1), \gamma'(1)$

$\exists [f] \in H: \quad [f\gamma\eta] = [\gamma'\eta']$

$\Rightarrow \mathcal{U}_{[f\gamma]} = \mathcal{U}_{[f\gamma\eta]} = \mathcal{U}_{[\gamma'\eta']} = \mathcal{U}_{[\gamma']}$

$\Rightarrow q(\mathcal{U}_{[\gamma]}) = q(\mathcal{U}_{[f\gamma]}) = q(\mathcal{U}_{[\gamma']})$

This proves that $X_H \xrightarrow{p_H} X$ is a covering space.

Take $x_H := H[c_{x_0}] \in X_H$.

Then $p(x_H) = x_0$

<u>Claim</u>: $p_*(\pi_1(X_H, x_H)) = H$.

<u>Proof</u>: Take $[\gamma] \in \pi_1(X, x_0)$.

$[\gamma] \in \operatorname{im} p_{H*}$

$\overset{66(iii)}{\Leftrightarrow}$ lift of γ to X_H (starting at x_H) is a loop.

$\Leftrightarrow$ The endpoints of the path $t \mapsto [\gamma_t]$ in $\tilde{X}$ are H-equivalent.

$\Leftrightarrow H[c_{x_0}] \ni [\gamma_1] = [\gamma]$

$\Leftrightarrow [\gamma] \in H.$ $\square$(Claim)

$\square$

<u>Def 80</u>: (a) An isomorphism of covering space $\tilde{X} \xrightarrow{p} X$ and $\tilde{X}' \xrightarrow{p'} X$ is a homeom $f: \tilde{X} \to \tilde{X}'$ s.t. $p' \circ f = p$.

(b) For the based setting we require $f(\tilde{x}_0) = \tilde{x}_0'$.

Theorem 81: We have the following bijections if X is pls.
(Fix $x_0 \in X$).

(i) $\{ [\tilde{X} \to X]_{\cong} \mid \tilde{X} \to X$ covering space $\}$

$$\longrightarrow \{ \{ gHg^{-1} \mid g \in \pi_1(X,x_0) \} \mid H \leq \pi_1(X,x_0) \}$$

defined via

$$[\tilde{X} \xrightarrow{p} X]_{\cong} \longmapsto \{ p_*(\pi_1(\tilde{X},\tilde{x}_0)) \mid \tilde{x}_0 \in p^{-1}(x_0) \}$$

(ii) $\{ [(\tilde{X},\tilde{x}_0) \longrightarrow (X,x_0)]_{\cong} \mid (\tilde{X},\tilde{x}_0) \to (X,x_0)$ based covering space $\}$

$$\longrightarrow \{ H \mid H \leq \pi_1(X,x_0) \}$$

$$[(\tilde{X},\tilde{x}_0) \xrightarrow{p} (X,x_0)]_{\cong} \longmapsto p_*(\pi_1(\tilde{X},\tilde{x}_0)).$$

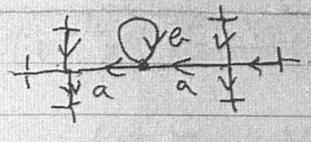
Proof: exercise □

Example 82: $X =$  $= S^1 \vee S^1$

$$\pi_1(X,x_0) \simeq \langle a, b \mid \rangle = F(a,b)$$

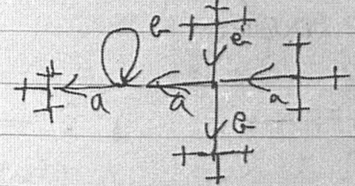
$$\simeq \mathbb{Z} * \mathbb{Z}.$$

(i) $H = \langle b \rangle \leq \pi_1(X,x_0)$:

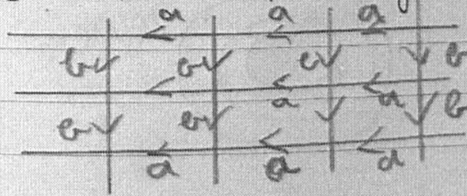


$= X_H$

(ii) $H = \langle aba^{-1} \rangle \leq \pi_1(X,x_0)$:



(iii) $H = [\pi_1(X,x_0), \pi_1(X,x_0)]$
(the commutatorsubgroup.)



$= X_H$

Remark 83: Let $\tilde{X} \xrightarrow{p} (X, x_0)$ be a covering space and put $F := p^{-1}(x_0)$.
Then we get a right action of $\pi_1(X, x_0)$ on F

$$\rho_r : F \times \pi_1(X, x_0) \longrightarrow F$$

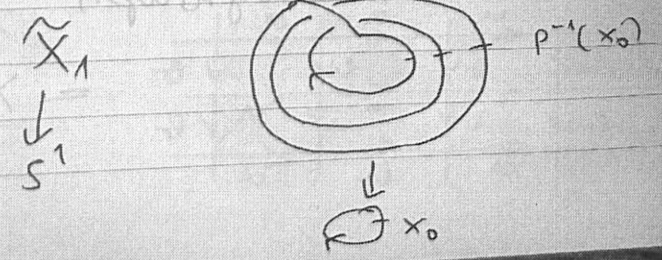
$$(\tilde{x}, [\gamma]) \longmapsto \tilde{x}.[\gamma] := \tilde{\gamma}_{\tilde{x}}(1)$$

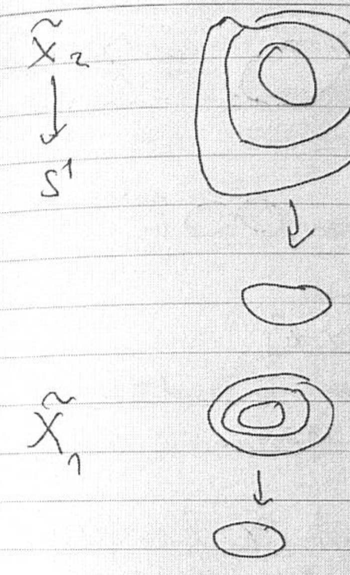
(Also write $\tilde{x}.\gamma$)

and a left action.

$$\rho : \pi_1(X, x_0) \times F \longrightarrow F \qquad \gamma.\tilde{x} := \tilde{x}.\gamma$$

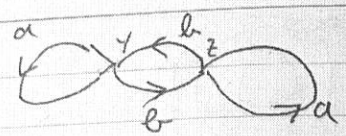
Example 84: (i) Consider the 3-sheeted covering spaces of S^1





The clockwise generator of $\pi_1(S^1, x_0)$ gives a cyclic permutation on $p^{-1}(x_0)$.

(Exercise: What happens for $\tilde{X}_2$ and $\tilde{X}_1$?)

(ii)  $= \tilde{X}$

$\downarrow$

$S^1 \vee S^1 =$ 

$p^{-1}(x_0) = \{y, z\}$
b permutes y and z
a fixes them.

<u>Theorem 85</u>: Let X be pls, $x_0 \in X$ and ρ be a left action of $\pi_1(X, x_0)$ on a set F.

(1) Then there is a covering space $X_\rho \xrightarrow{p_\rho} X$ whose left-action on $p_\rho^{-1}(x_0)$ is isomorphic to ρ.

(2) Let $\tilde{X} \xrightarrow{p} X$ be a covering space whose left action on $p^{-1}(x_0)$ is isomorphic to ρ. Then $\tilde{X} \xrightarrow{p} X$ is isom. to p_ρ.

<u>Proof</u>: (1) X has a universal cover $\tilde{X}_0 \longrightarrow X$. We take the one constructed in the proof of Prop. 75.

$$\tilde{X}_0 = \{ [\gamma] \mid \gamma \text{ path in } X \text{ with start at } x_0 \}$$

$\pi_1(X_0, x_0)$ acts on $\tilde{X}_0 \times F$ from the left:

$$[f] \circ ([\gamma], z) := ([f\gamma], \rho([f]).z)$$

(Geometrically $([\gamma], z)$ and $([f\gamma], \rho([f]).z)$ should represent the same point of X_ρ)

$$X_\rho := \pi_1(X, x_0) \backslash \tilde{X}_0$$

$$X_\rho \xrightarrow{p_\rho} X \qquad p_\rho({}_{\pi_1}([\gamma], z)) := \gamma(1).$$

Standard proof $\Rightarrow$ $X_\rho \longrightarrow X$ is a covering space.

(2) exercise. $\square$