<u>Definition 55:</u> A closed subset $A \subseteq X$ of a cell complex X is called subcomplex if A is the union of cells of X.

<u>Example 56:</u> (a) $k \leq n$:

$\mathbb{R}P^k \subseteq \mathbb{R}P^n$ is a subcomplex.
$\parallel$
$e^0 \cup e^1 \cup \ldots \cup e^k$

(b) We have
$S^0 \subsetneq S^1 \subsetneq S^2 \subsetneq \ldots \subsetneq S^n$
in $\mathbb{R}^{n+1}$.
S^k is not a subcomplex of S^n $(k < n)$ w.r.t. the cell structure of 54(a)
W.r.t. this cell structure S^n has only e^0 and S^n as subcomplexes if $n > 0$.

We can choose another cell structure

so that S^k is a subcomplex.
By induction:

$\underline{n=0}$: S^0 has two zero cells.
$\underline{n>0}$: $S^n \cong \left(S^{n-1} \sqcup (D^n_1 \sqcup D^n_{-1})\right) / \varphi_1^{(n)}, \varphi_{-1}^{(n)}$

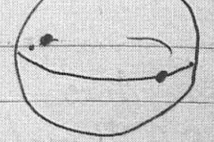
$\varphi_1^{(n)}, \varphi_{-1}^{(n)} : S^{n-1} \xrightarrow{id} S^{n-1}$.

(We glue here n-discs along their boundary by the identity.)

On S^{n-1} we consider the cell-structure coming from the (IH).

Then by construction w.r.t. this cell structure S^k is a subcomplex of S^n $\forall\, 0 \leq k \leq n$.

Example S^2:



$$S^2 = e_1^0 \sqcup e_{-1}^0 \sqcup e_1^1 \sqcup e_{-1}^1 \sqcup e_1^2 \sqcup e_{-1}^2$$

Terminology 57:

$$\varphi_\alpha^{(k)} : S^{k-1} \longrightarrow X^{k-1}$$

is called the attaching map (for D_α^k) of e_α^n

Its extension

$$\Phi_\alpha^{(n)} : D_\alpha^n \longrightarrow X$$

is called the characteristic map of e_α^n.

We now come to the application of van Kampen's theorem in case of attaching 2-cells.

Consider the following setting:
Let X be path-connected, $x_0 \in X$.
We attach 2-cells e_α^2 to X via $\varphi_\alpha : S^1 \longrightarrow X$ and obtain Y.
Take $s_0 \in S^1$ and $\gamma_\alpha : x_0 \rightsquigarrow \varphi_\alpha(s_0)$.
We get a loop at x_0

$$\gamma_\alpha \varphi_\alpha \overline{\gamma}_\alpha =: \varphi_{\alpha, x_0}.$$

Let $N := \langle\langle \{ [\varphi_{\alpha, x_0}]_X \mid \alpha \in A \} \rangle\rangle$

Proposition 58 The inclusion $i = \iota_X^Y : X \longrightarrow Y$ induces an isomorphism

$$\pi_1(X, x_0)/N \xrightarrow[\cong]{\overline{i_*}} \pi_1(Y, x_0)$$

Proof: (Note that $N \subseteq \ker i_*$ as $\varphi_{\alpha, x_0} \simeq c_{x_0}$ passing through e^2_α)

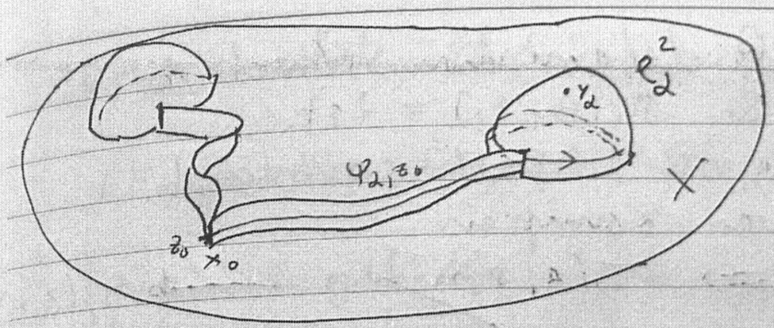
Steps:

1) We enlarge Y slightly to a space Z which deformation retracts to Y
2) We compute $\pi_1(Z, z_0)$ for some z_0 near x_0
3) We move back to the base point x_0.

Step 1: We glue a strip $S_\alpha = I \times I$ on γ_α along $I \times \{0\}$, for each α, such that

- $\{1\} \times (I \setminus \{0\})$ is attached to an arc on e^2_α (arc to $\varphi_\alpha(s_0) = \gamma_\alpha(1)$)

- All S_α are identified on $\{0\} \times I$.



Step 2: $z_0 := (0, \frac{1}{2})$.

Take $y_\alpha \in e^2_\alpha$.

Let ψ_{α, z_0} be the loop constructed as follows.

- first go from z_0 to e^2_α via S_α on $I \times \{\frac{1}{2}\}$.
- then go 1-time around y_α in e^2_α

- then go back to z_0 on $I \times \{\frac{1}{2}\}$.

Put $A := Z \setminus \{y_\alpha \mid \alpha \in A\}$

$B := Z - X$.

B deformation retracts to z_0.
So $\pi_1(B, z_0) = \{1\}$.
$A \cap B$ is path-connected.
van Kampen
$\Rightarrow$ $\pi_1(A, z_0) \Big/ \langle\langle \iota_* \pi_1(A \cap B, z_0) \rangle\rangle \xrightarrow{\sim} \pi_1(Z, z_0)$

($\iota : A \cap B \hookrightarrow A$ inclusion)

Another application of van Kampen shows

$$\pi_1(A \cap B, z_0) \simeq \mathop{*}_{\alpha \in A} \langle [\psi_{\alpha, z_0}]_{A \cap B} \rangle$$
$$\simeq \mathop{*}_{\alpha \in A} \mathbb{Z}$$

Step 3: Note that

$[x_0, z_0]\, \psi_{\alpha, z_0}\, [z_0, x_0] \simeq \varphi_{\alpha, x_0}$ in A.

Thus we get the following diagram

$$\begin{tikzcd}
\pi_1(Y, x_0) \arrow[r, "\sim"] & \pi_1(Z, x_0) \arrow[r, "\beta_{[x_0, z_0]}", "\sim"'] & \pi_1(Z, z_0) \\
\pi_1(X, x_0)/\wr \arrow[u, "\bar{i}_*"] \arrow[r, "\sim"] \arrow[ur, phantom, "\circlearrowleft"] & \pi_1(A, x_0) \Big/ \langle\langle \{[\varphi_{\alpha, x_0}]_A \mid \alpha\} \rangle\rangle \arrow[u] \arrow[r, "\overline{\beta_{[x_0, z_0]}}", "\sim"'] \arrow[ur, phantom, "\circlearrowleft"] & \pi_1(A, z_0) \Big/ \langle\langle \{[\psi_{\alpha, z_0}]_A \mid \alpha\} \rangle\rangle \arrow[u, "\wr"']
\end{tikzcd}$$

Thus $\bar{i}_*$ is an isomorphism. $\square$

<u>Example 59:</u> $(g \geq 1)$.

$$\pi_1(M_g) \simeq \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \dots [a_g, b_g] \rangle$$

Here $X = S^1_{a_1} \vee S^1_{b_1} \vee \dots \vee S^1_{b_g}$

and we attach one 2-cell.

<u>Corollary 60:</u> Let G be a group. Then there is a 2-dimensional cell complex (path connected) such that
$$\pi_1(X) \simeq G.$$

<u>Proof:</u> $G = \langle (x_\lambda)_\Lambda \mid r_\alpha, \alpha \in A \rangle$

Glue on $\bigvee_{\lambda \in \Lambda} S^1_\lambda$

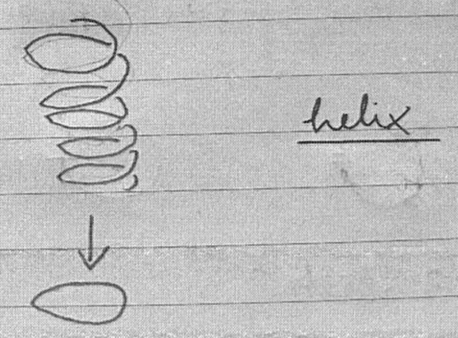
2-discs according to r_α.
Prop 58 gives $\pi_1(X) \simeq G$. □

IV Covering spaces

IV 1. Definition and first examples

In this section we generalize the exponential map

$$\mathbb{R} \longrightarrow S^1 \qquad x \longmapsto e^{2\pi i x}.$$



It was very helpful to compute $\pi_1(S^1)$.

<u>Def 61:</u> (covering space)
A <u>covering space</u> of a space X is a continuous map $p: \tilde{X} \to X$ such that $\exists\, (U_\alpha)_{\alpha \in A}$ an open cover of X such that:

for each $\alpha \in A$:

- $p^{-1}(U_\alpha)$ is a disjoint union of open sets V in $\tilde{X}$ such that $p|_V : V \longrightarrow U_\alpha$ is a homeomorphism.

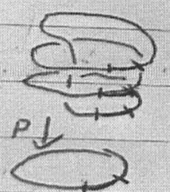
Empty disjoint union is allowed.

Example 62:

(a) $p : S^1 \longrightarrow S^1$ $\quad p(z) := z^n$. $\quad (n > 0)$

$\underline{n=3}$

3 sheets.

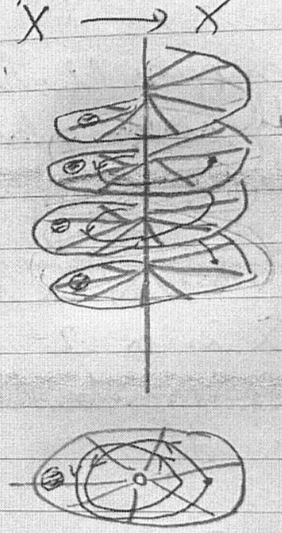


(b) $X = \mathbb{R}^2 \setminus \{0\}$

$$\tilde{X} = \{(s\cos(2\pi t), s\sin(2\pi t), t) \mid s \in]0,\infty[,\ t \in \mathbb{R}\} \subseteq \mathbb{R}^3$$

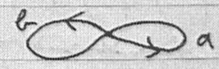
helicoid.

$p : \tilde{X} \longrightarrow X \qquad p(x,y,z) := (x,y)$.

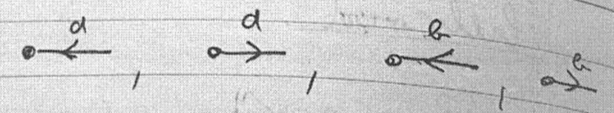


The analogue for $D^2 - \{(0,0)\}$.

(c) $X = S^1 \vee S^1$

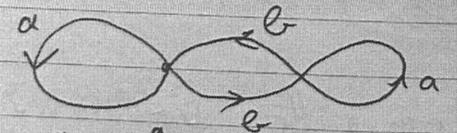


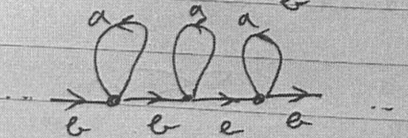
A graph $\tilde{X}$ (i.e. a 1-dim cell complex) is called 2-oriented if at every vertex there is exactly one of each of the following classes of edges:

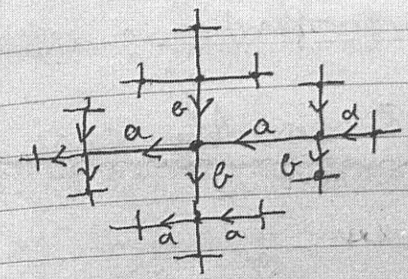


(Here we suppose that $\tilde{X}$ is labelled with a and b and oriented)

$\tilde{X} \xrightarrow{p} X = S^1 \vee S^1$ is a covering space iff $\tilde{X}$ is a 2-oriented graph and p respects the type of the edges.

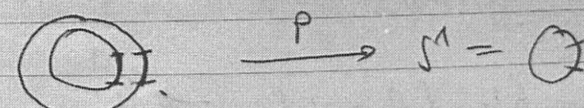






infinite tree of valence 4.

(d) (non-connected covering space)



$S^1 \sqcup S^1$

IV Lifting properties

See Lemma 13 for the case $\mathbb{R} \longrightarrow S^1$.

Def 63: Given $p : \tilde{X} \longrightarrow X$ and $f : Y \longrightarrow X$, a *lift* of f is a map $\tilde{f} : Y \longrightarrow \tilde{X}$ such that $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Prop 64: (homotopy lifting property)

Let $p : \tilde{X} \longrightarrow X$ be a covering space and $f_t : Y \longrightarrow X$ be a homotopy and $\tilde{f}_0$ a lift of f_0.

Then $\exists$ homotopy $\tilde{f}_t : Y \longrightarrow \tilde{X}$ of $\tilde{f}_0$ which lifts f_t.

Proof: See the proof of Lem 13. $\square$

Remark 65: (two special cases of Prop. 64)

(a) path lifting property (see the red curve in Example 62 (b))
Case $|Y| = 1$.

(b) path homotopy lifting property.

$Y = I$. (exercise)

Prop 64 has consequences for p_*.

Corollary 66: Suppose $p(\tilde{x}_0) = x_0$.

(i) $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ is injective.

(ii) Let $[f]_X \in \pi_1(X, x_0)$.

We denote by $\tilde{f}_{\tilde{x}_0}$ the lift of f starting at $\tilde{x}_0$.

Then are equivalent:

1° $[f]_X$ in (p_*)

2° $\tilde{f}_{\tilde{x}_0}$ is a loop in $\tilde{X}$.

Proof: (i) $p_*([\tilde{g}]_{\tilde{X}}) = [c_{x_0}]_X$

Put $g := p \circ \tilde{g}$.

$[g]_X = [c_{x_0}]_X \Rightarrow g \simeq c_{x_0}$

$\overset{64}{\Rightarrow} \tilde{g} \simeq c_{\tilde{x}_0} \Rightarrow [\tilde{g}]_{\tilde{X}} = [c_{\tilde{x}_0}]_{\tilde{X}}$

(ii) $2^\circ \Rightarrow 1^\circ$ ✓

$1^\circ \Rightarrow 2^\circ$ $[f]_X = p_*([\tilde{g}]_{\tilde{X}})$

$\Rightarrow g := p \circ \tilde{g} \in \Omega(X, x_0)$ homotopic to f.

$\overset{64}{\Rightarrow} \tilde{g} = \tilde{g}_{\tilde{x}_0} \simeq \tilde{f}_{\tilde{x}_0}$

This a homotopy of path.

Thus $\tilde{f}_{\tilde{x}_0}(1) = \tilde{g}(1) = \tilde{x}_0$.

Thus $\tilde{f}_{x_0}$ is a loop. □

Def 67: Let $p : \tilde{X} \to X$ be a covering space and suppose X is connected.
Then $|p^{-1}(x)|$ is independent of x.
It is called the number of sheets of p.

Rk 68: Let $X, \tilde{X}$ path-connected and $p(\tilde{x}_0) = x_0$. Put $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.
Then

$$p^{-1}(x_0) \xrightarrow{\sim} H \backslash \pi_1(X, x_0)$$

$$\tilde{x} \longmapsto H[p \circ \tilde{\gamma}]_X$$

$$(\tilde{x}_0 \overset{\tilde{\gamma}}{\rightsquigarrow} \tilde{x})$$

$$\tilde{f}_{\tilde{x}_0}(1) \longleftarrow H[f]_X$$

In 62(a) we get $\mathbb{Z}/n\mathbb{Z}$.

There is a criterion for lifting general maps (not just homotopies)

__Prop 69:__ (lifting criterion)
Let $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a (based) covering space and suppose Y is path-connected and locally path-connected.

Let $f: Y \longrightarrow X$ with $f(y_0) = x_0$ be continuous s.t.

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

__Then__ f has a lift to $\tilde{X}$.

__Proof:__ (Recall: "__loc. path-connected__"
$\forall y \in Y\ \forall U \subseteq Y$ open wt $y \in U$
$\exists V \subseteq U$ open wt. $y \in V$: V path-connected.)

__Step 1:__ We define $\tilde{f}$.

Take $y \in Y$ and $\gamma: y_0 \rightsquigarrow y$.
Lift $f\gamma$ to $\tilde{X}$ starting at $\tilde{x}_0$.

Put $\tilde{f}(y) := \widetilde{f\gamma}(1)$.

__Well-defined:__ $\gamma: y_0 \rightsquigarrow y$
$\gamma': y_0 \rightsquigarrow y$

$$\Rightarrow [(f\gamma)(\overline{f\gamma'})]_X = f_*([\gamma\bar{\gamma}']_Y)$$

$$\in p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

$\overset{66}{\Longrightarrow}$ $(f\gamma)(\overline{f\gamma'})$ lifts to a loop in $\tilde{X}$ at $\tilde{x}_0$.

In particular $\widetilde{f\gamma'}(1) = \widetilde{f\gamma}(1)$.

__Step 2:__ $\tilde{f}$ is continuous.

Take $y \in Y$ and an open nbhd U

of $f(y)$ s.t. $\exists\, \tilde{U} \subseteq \tilde{X}$ open nbhd of $\tilde{f}(y)$ s.t. $\tilde{U} \xrightarrow[p]{\sim} U$ (homeom.)

Take $V \subseteq f^{-1}(U)$ path-connected open nbhd of y.

$$\begin{array}{ccc} & & \tilde{U} \\ & & \downarrow p \\ V & \xrightarrow{f} & U \end{array}$$

To show $\tilde{f}(V) \subseteq \tilde{U}$.

Take $\gamma : y_0 \rightsquigarrow y$ in Y
and $\eta : y \rightsquigarrow y'$ in V.

Then $\tilde{f}(y') = \widetilde{f\circ(\gamma\eta)}(1)$

$$= (\widetilde{f\circ\gamma})_{\tilde{x}_0}(\widetilde{f\circ\eta})_{\tilde{f}(y)}(1)$$

We have
$\tilde{f}(y) \in \tilde{U} \xrightarrow[\sim]{p} U \supseteq$ path $f\eta$

So the unique lift is in $\tilde{U}$.
So $\tilde{f}(y') \in \tilde{U}$. $\square$

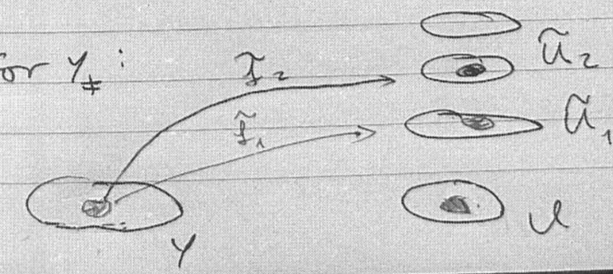
<u>Prop 70</u> (Uniqueness)
Let $p : \tilde{X} \to X$ be a covering space and Y be connected.
Let $\tilde{f}_1, \tilde{f}_2$ be lifts of a continuous map $f : Y \to X$.
Suppose $\exists\, y \in Y$: $\tilde{f}_1(y) = \tilde{f}_2(y)$

Then $\tilde{f}_1 = \tilde{f}_2$

<u>Proof:</u> $Y_= := \{y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\}$
$Y_{\neq} := Y \setminus Y_=$.

$Y_=$ and $Y_{\neq}$ are open.
Y connected and $Y_= \neq \emptyset \Rightarrow Y = Y_=$

For $Y_{\neq}$:



$\square$