Theorem 45: Let $X = \bigcup_{\lambda \in \Lambda} A_\lambda$ such that

- each A_λ is path-connected
- $\forall_{\lambda \in \Lambda}: x_0 \in A_\lambda$
- $\forall_{\lambda, \mu \in \Lambda}$ with $\lambda \neq \mu$: $A_\lambda \cap A_\mu$ is path-connected.

Then the map

$$\Phi : \mathop{*}_{\Lambda} \pi_1(A_\lambda, x_0) \longrightarrow \pi_1(X, x_0)$$

is surjective.

If in addition $\forall_{\lambda, \mu, \gamma \in \Lambda}$ $A_\lambda \cap A_\mu \cap A_\gamma$ is path connected then Φ has kernel

$$N := \langle\langle \iota^{A_\lambda}_{A_\lambda \cap A_\mu}([f]) \, \iota^{A_\mu}_{A_\lambda \cap A_\mu}([f])^{-1} \mid \lambda, \mu \in \Lambda, [f] \in \pi_1(A_\lambda \cap A_\mu, x_0) \rangle\rangle$$

Example 46: (wedge sum)

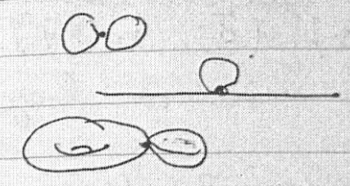
$(X_\lambda)_{\lambda \in \Lambda}$ topological spaces

Fix $x_\lambda \in X_\lambda$.

wedge sum of $(X_\lambda)_\Lambda$:

$$\bigvee_{\lambda \in \Lambda} X_\lambda := \coprod_{\lambda \in \Lambda} X_\lambda \Big/ x_\lambda \sim x_\mu \ \forall_{\lambda, \mu}$$

$S^1 \vee S^1$:
$\mathbb{R} \vee S^1$
$T^2 \vee S^2$



Suppose X_λ is path-connected $\forall_{\lambda \in \Lambda}$ and each x_λ is a deformation retract of an open neighbourhood U_λ.
Then

$$\pi_1\Big(\bigvee_{\lambda \in \Lambda} X_\lambda\Big) \cong \mathop{*}_{\lambda \in \Lambda} \pi_1(X_\lambda, x_\lambda)$$

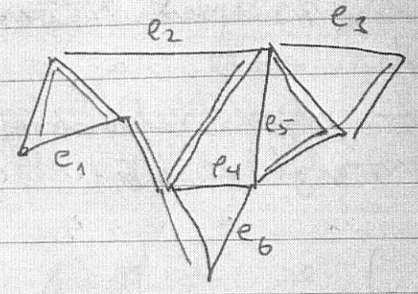
Proof: $A_\lambda := X_\lambda \vee \Big(\bigvee_{\mu \neq \lambda} U_\mu\Big)$

- $\forall_\lambda$: A_λ is path connected and open.
- $\forall_{\lambda \neq \mu}$: $A_\lambda \cap A_\mu = \bigvee_{\alpha \in \Lambda} U_\alpha$ is path-connected, and deformation retracts to a point
- $\forall_{\lambda, \mu, \alpha}$ p.w. different: $A_\lambda \cap A_\mu \cap A_\alpha = \bigvee_{\beta \in \Lambda} U_\beta$ is path-connected

Thus $\ker(\Phi) = \{1\}$ and
Φ is bijective by van Kampen. $\square$

<u>Example 47:</u> (connected graph.

Γ



Consider a spanning tree. (blue)
(tree = connected graph without circles)

$e_1, \dots, e_6$ edges which are not in the tree T.

On every edge e_i we consider two half intervals

$U := T \cup$ half intervals is open in Γ!

$A_i := U \cup e_i$ it deformation retracts to a circle (along the tree)

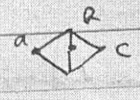
$\forall_{|\{i,j,k\}| \geq 2}: A_i \cap A_j \cap A_k = U$ is path-connected.

and $\forall_{i \neq j}: A_i \cap A_j = U$ def. retracts to a point.

$$\Rightarrow \quad \pi_1(\Gamma, x_0) \simeq \mathop{\ast}_{i=1}^{6} \pi_1(A_i, x_0) \simeq \mathop{\ast}_{i=1}^{6} \mathbb{Z} \simeq F(x_1, \dots, x_6).$$

<u>Example 48:</u> (a) $X = S^1$

$A = \subset \quad B = \supset$
Take $x_0 \in A \cap B$.
$A \cap B$ is not path-connected.
$$\underbrace{\pi_1(A, x_0)}_{\{1\}} \ast \underbrace{\pi_1(B, x_0)}_{\{1\}} \longrightarrow \underbrace{\pi_1(S^1, x_0)}_{\mathbb{Z}}$$
is not surjective.

(b) $X =$ 

$A_a := X \setminus \{a\}$,
$A_b := X \setminus \{b\}$
$A_c := X \setminus \{c\}$

$A_a \cap A_b$, $A_a \cap A_c$, $A_b \cap A_c$
are path-connected and $\sim$ point.
$A_a \cap A_b \cap A_c$ is not
path connected.

$\mathbb{Z} * \mathbb{Z} * \mathbb{Z} \cong \pi_1(A_a, x_0) * \pi_1(A_b, x_0) * \pi_1(A_c, x_0)$

$\longrightarrow\!\!\!\!\!> \pi_1(X, x_0)$ is not injective.
$\qquad\qquad\cong$
$\qquad\quad \mathbb{Z} * \mathbb{Z}$

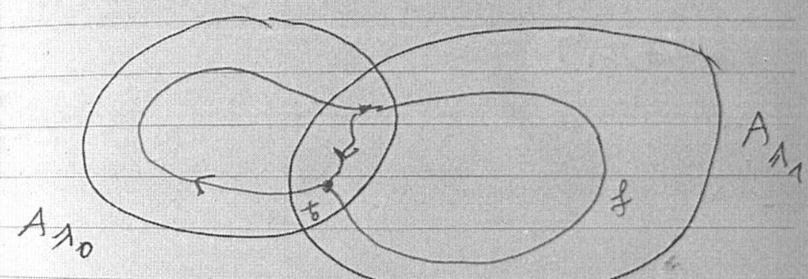
($\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ and $\mathbb{Z} * \mathbb{Z}$ have
non-isomorphic abelianizations.)

But $N = \langle\langle\, \iota^{A}_{A \cap B}([\lambda])\, \iota^{B}_{A \cap B}([\lambda])^{-1} \mid A, B \in \{A_a, A_b, A_c\},\ [\lambda] \in \pi_1(A \cap B, x_0) \,\rangle\rangle = \{1\}$.

Proof (Theorem 45): Write

$$\Omega(X, x_0) := \{ f : I \to X \mid f \text{ a loop at } x_0 \}$$

Surjectivity of Φ:



Let $f \in \Omega(X, x_0)$.
Then $\exists\ 0 = s_0 < s_1 < \dots < s_a = 1$
such that
$\forall_{i \in \{0, \dots, a-1\}}\ \exists_{\lambda_i \in \Lambda} : f([s_i, s_{i+1}]) \subseteq A_{\lambda_i}$.

For $0 < i < a$ choose a path α_i in $A_{\lambda_i} \cap A_{\lambda_{i-1}}$ from $f(s_i)$ to x_0, and put $\alpha_0 := \alpha_a := c_{x_0}$.

Put $f_i := \bar{\alpha}_i \, f|_{[s_i, s_{i+1}]} \, \alpha_{i+1}$

for $0 \le i < a$.

Then $\Phi([f_0]_{\lambda_0} [f_1]_{\lambda_1} \dots [f_{k-1}]_{\lambda_{k-1}})$

$= [f]_X$

(exercise)

$\underline{\ker \Phi = N}$: We only need to show $\ker \Phi \subseteq N$.

The proof needs several steps.
In step 1 we compare for $f \in \Omega(X, x_0)$ two different cuttings of I.
In Step 2 we compare decompositions for homotopic loops.

__Step 1__: Let $f \in \Omega(X, x_0)$. Suppose we have

$0 = s_0 < s_1 < s_2 < \dots < s_a = 1$

$0 = s_0' < s_1' < s_2' < \dots < s_b' = 1$

and sets A_{λ_i}, $A_{\lambda_j'}$ as above (surjectivity part).
Suppose further that for all $0 < i < a$, $0 < j < b$ we have

$s_i \neq s_j'$.

We take $0 < i < a$ a path α_i in $A_{\lambda_{i-1}} \cap A_{\lambda_i} \cap A_{\lambda_j'}$ if

$s_j' < s_i < s_{j+1}'$

from $f(s_i)$ to x_0

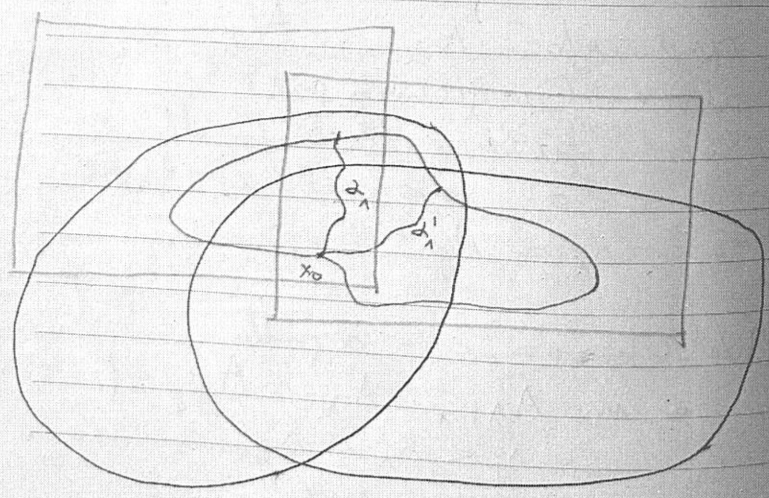
and for $0 < j < b$
α'_j in $A_{\lambda'_{j-1}} \cap A_{\lambda'_j} \cap A_{\lambda_i}$

from $f(s'_j)$ to x_0 if
$s_i < s'_j < s_{i+1}$.

Then for the corresponding words we have

$[f_0]_{\lambda_0} [f_1]_{\lambda_1} \dots [f_a]_{\lambda_a}$

$\equiv_N [f'_0]_{\lambda'_0} [f'_1]_{\lambda'_1} \dots [f'_b]_{\lambda_b}$.



Pf: (Step 1) Suppose $s_{a-1} < s'_{b-1} < 1$

$[f_{a-1}]_{\lambda_{a-1}} = [\bar{\alpha}_{a-1} f|_{[s_{a-1},1]}]_{\lambda_{a-1}}$

$= [\bar{\alpha}_{a-1} f|_{[s_{a-1}, s'_{b-1}]} \alpha'_{b-1}]_{\lambda_{a-1}}$

$[\bar{\alpha}'_{b-1} f|_{[s'_{b-1},1]}]_{\lambda_{a-1}}$

$\equiv_N [\bar{\alpha}_{a-1} f|_{[s_{a-1}, s'_{b-1}]} \alpha'_{b-1}]_{\lambda_{a-1}}$

$[f_{b-1}]_{\lambda_{b-1}}$

Now apply induction on $a+b$
The base case is $a+b=2$.

$0 = s_0 < s_1 = 1$
$0 = s'_0 < s'_1 = 1$

$[f]_{\lambda_0} \equiv_N [f]_{\lambda'_0}$ by definition of N.

$\square$

Step 2: Suppose

$\Phi([f_0]_{\lambda_0} \cdots [f_{a-1}]_{\lambda_{a-1}})$

$= \Phi([g_0]_{\mu_0} \cdots [g_{b-1}]_{\mu_{b-1}})$

To show

$[f_0]_{\lambda_0} \cdots [f_{a-1}]_{\lambda_{a-1}} \equiv_N [g_0]_{\mu_0} \cdots [g_{b-1}]_{\mu_{b-1}}$

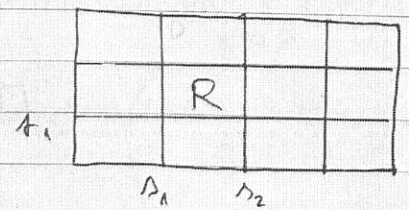
Pf: We have

$f := f_0 \cdots f_{a-1} \overset{F}{\simeq} g := g_0 \cdots g_{b-1}$

for some homotopy F.

We find $0 = s_0 < s_1 < \dots < s_k = 1$

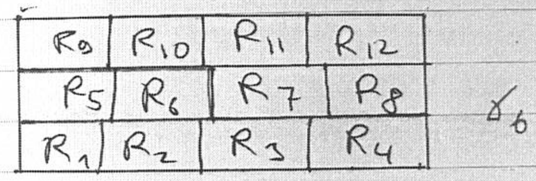
$0 = t_0 < t_1 < \dots < t_\ell = 1.$



such that for each rectangle R we find $\lambda_R \in \Lambda$ such that

$F(R) \subseteq A_{\lambda_R}.$

We move the vertical edges a bit such that no vertex is attached to 4 edges.



We count the rectangles as above indicated.

A path γ from left to right defines a loop $F \circ \gamma$.
Let γ_i be the path separating

$R_1, \dots, R_i$ from $R_{i+1}, \dots, R_{(k-1)(\ell-1)}$.

The decomposition of $[F \circ \gamma_i]$ given by edges on γ_i is equivalent (mod N) to the one of $[F \circ \gamma_{i+1}]$, by homotopy across R_{i+1}.

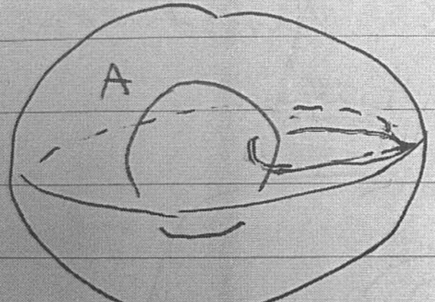
$\square$

Example 49 : (linking of circles)

(a) $A = S^1 \times \{0\} \subseteq \mathbb{R}^3$

$X := \mathbb{R}^3 \setminus A$. Want $\pi_1(X)$.

X deformation retracts to $S^2 \vee S^1$



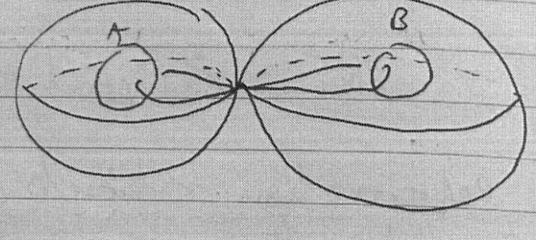
$$\Rightarrow \quad \pi_1(X) \simeq \pi_1(S^2 \vee S^1) \simeq \pi_1(S^2) * \pi_1(S^1) \simeq \pi_1(S^1) \simeq \mathbb{Z}.$$

$\pi_1(S^2) = \{1\}$

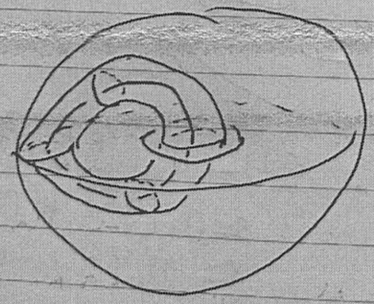
(b) A and B two unlinked circles in $\mathbb{R}^3$

$Y := \mathbb{R}^3 \setminus (A \cup B)$.

Y deformation retracts to $S^2 \vee S^1 \vee S^2 \vee S^1$.
So $\pi_1(Y) \simeq \mathbb{Z} * \mathbb{Z}$.



(c) A and B linked circles
$Z = \mathbb{R}^3 \setminus (A \cup B)$



Z deformation retracts onto $S^2 \vee (S^1 \times S^1)$
$\Rightarrow \pi_1(Z) \simeq \mathbb{Z} \times \mathbb{Z}$

<u>Example 50</u>: (The shrinking wedge of circles)

$C_n \subseteq \mathbb{R}^2$ the circle with radius $\frac{1}{n}$ and center $(\frac{1}{n}, 0)$. $(n \in \mathbb{N})$

$$X := \bigcup_{n=1}^{\infty} C_n \text{ with subspace topology.}$$

(This is not homeomorphic to $\bigvee_{n \in \mathbb{N}} S^1$)

We will see that $\pi_1(X)$ is not countable.
Let $x_0 := (0,0)$.
Let $r_n : X \to C_n$ be the retraction which send C_m $(m \neq n)$ to x_0 and fixes C_n pointwise.

$$\varphi : \pi_1(X, x_0) \to P := \prod_{n \in \mathbb{N}} \pi_1(C_n, x_0)$$

$$\varphi([f]_X) := ([r_n \circ f]_{C_n})_{\mathbb{N}}.$$

is surjective.

Pf: Given $([w_{k_n}]_{C_n})_{\mathbb{N}} \in P$ take the loop f which on $[\frac{1}{n+1}, \frac{1}{n}]$ winds k_n-times around C_n, for all $n \in \mathbb{N}$.

Then

$$\varrho([f]_x) = ([w_{k_n}]_{C_n})_{\mathbb{N}}$$

□

P is uncountable. (Why?)

is

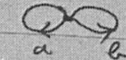
$$\prod_{\mathbb{N}} \mathbb{Z}$$

Thus $\pi_1(X, x_0)$ is uncountable.

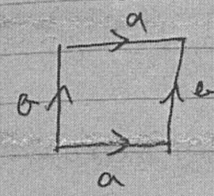
Rk 51: We are going to apply van Kampen to cell-complexes later.

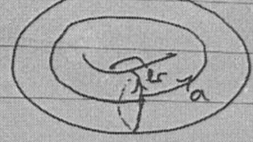
In the next chapter we give an introduction to cell-complexes.

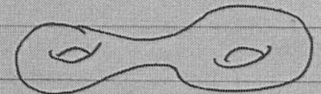
III Cell complexes

Example 52: (a) We attach on $S^1 \vee S^1 =$ 

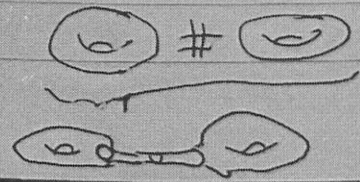
a 2-disc in the following way:

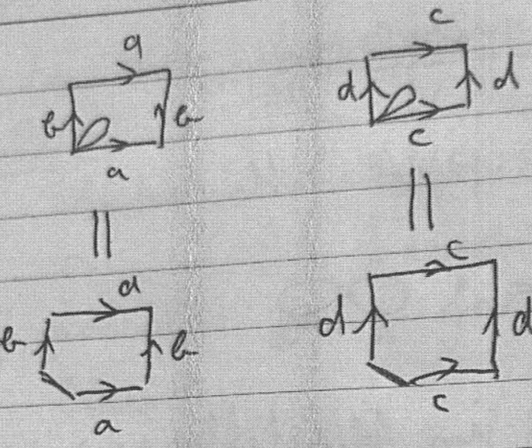


We get a torus 

(b) How to obtain  $=: X$

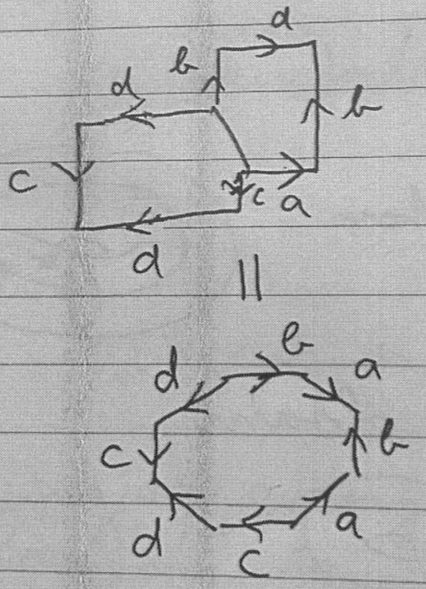
via attaching 2-cells on a graph?

$X =$  (connected sum)



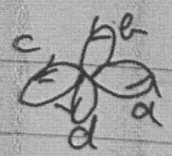
We cut out open discs.

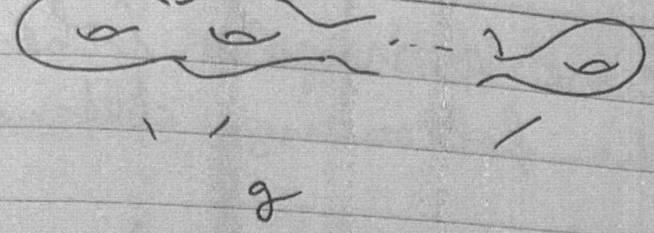
Glue the blue segments together



One gets X by glueing a disc on

$$S^1 \vee S^1 \vee S^1 \vee S^1$$

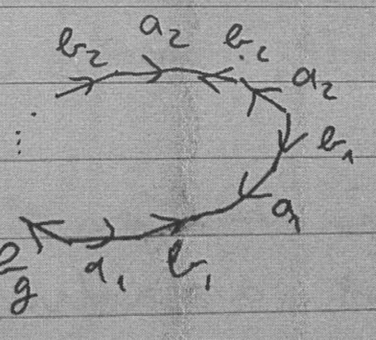


(c) M_g = 

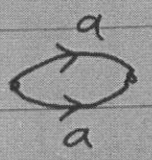
The closed orientable surface of genus g. ($g \in \mathbb{N}_0$)

$M_0 = S^2$

M_g is constructed by gluing



for $g \geq 1$.

M_0 : 

Those examples are motivating the following definition.

<u>Definition 53:</u> A space X constructed in the following way is called cell complex (or CW complex)

(1) Start with a discrete set X^0
The elements of X^0 are called <u>0-cells</u>. X^0 is called <u>0-skeleton</u>.

(2) By induction:
"We are attaching n-cells (open n-disc) via continuous maps $\varphi_\alpha : S^{n-1} \longrightarrow X^{n-1}, \alpha \in A$"

$$X^n = X^{n-1} \sqcup \Big(\bigsqcup_{\alpha \in A} D^n_\alpha \Big) \Big/ \underset{\underset{S^1}{\cap}}{x} \sim \varphi_\alpha(x), \alpha \in A$$

with quotient topology.

We call $e^n_\alpha = \operatorname{int}(D^n_\alpha) \subseteq X^n, \alpha \in A$ the <u>n-cells</u>.

(3) If the induction stops we put $X = X^n$.
Otherwise put
$$X := \bigcup_{\mathbb{N}} X^n$$

with the <u>weak topology</u>:

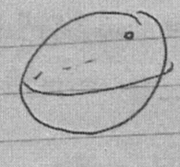
$A \subseteq X$ is <u>closed</u> iff
$\forall_{n \in \mathbb{N}}$ $A \cap X^n$ is closed.

$$\dim X := \inf \{ n \in \mathbb{N}_0 \mid X = X^n \}$$

<u>Example 54:</u> (a) A one dimensional cell complex is called a graph.

(b) S^n $(n \geq 1)$ has the structure of a cell complex

$$X^0 = \{pt\} = X^1 = \dots = X^{n-1}$$
$$S^n = X^n = \{pt\} \sqcup D^n / S^{n-1} \sim pt.$$



(The gluing map here is $\varphi : S^{n-1} \longrightarrow \{pt\}$)

(c) $\mathbb{R}P^n$ $(n \in \mathbb{N}_0)$
We will see
$\mathbb{R}P^n = e^0 \cup e^1 \cup e^2 \cup \cdots \cup e^n$
(one cell for each dimension $\leq n$)

$$\mathbb{R}P^0 = \{pt\} = e$$

$$\mathbb{R}P^1 = \mathbb{R}P^0 \sqcup D^1 \big/ _{pt \sim \text{vertexes}}$$

$(D^1 = \bullet\!\!-\!\!\bullet)$

$= \bigcirc$

$$\mathbb{R}P^2 = \mathbb{R}P^1 \sqcup D^2 \big/ _{\varphi^{(2)}}$$

$$\varphi^{(2)} : S^1 \longrightarrow \mathbb{R}P^1$$

$$\varphi^{(2)}(e^{2\pi i \theta}) := e^{4\pi i \theta}$$

(winding twice)

identify antipodal points

$$\mathbb{R}P^3 = \mathbb{R}P^2 \sqcup D^3 \big/ _{\varphi^{(3)}}$$

$$\varphi^{(3)} : S^2 = S^{2,+} \sqcup S^{2,-} \sqcup S^1 \longrightarrow \mathbb{R}P^2$$

with
$\varphi^{(3)} : S^{2,+} \sqcup S^{2,-} \longrightarrow (D^2)^\circ = e^3$
antipodal retraction

$$\varphi^{(3)}\big|_{S^1} = \overline{\varphi^{(2)}} : S^1 \longrightarrow \mathbb{R}P^2$$

$$\mathbb{R}P^n = \mathbb{R}P^{n-1} \sqcup D^n \big/ _{\varphi^{(n)}}$$