We will give some applications of Thm 12.

Thm 15: (FTA)
Let $p \in \mathbb{C}[X] \setminus \mathbb{C}$. Then p has a root.

Proof: $(f_r)_{r \geq 0}$, $f_r(s) := \dfrac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$

is a homotopy of loops at 1 to the constant loop if p has no root.

W.l.o.g. p has the form

$p(z) = z^n + a_1 z^{n-1} + \dots + a_n$,
i.e. p is monic.

Take $r_0 > \max\{1, |a_1| + \dots + |a_n|\}$

Then for $0 \leq t \leq 1$ the polynomial
$p_t(z) = z^n + t(a_1 z^{n-1} + \dots + a_n)$

has no root on $\{z \in \mathbb{C} \mid |z| = r_0\}$,
as for $|z| = r_0$:
$|p_t(z)| \geq r_0^n - t(|a_1| + \dots + |a_n|) r_0^{n-1}$

$\geq (r_0 - |a_1| - \dots - |a_n|) r_0^{n-1} > 0$

$$g_t(s) := \frac{p_t(r_0 e^{2\pi i s})/p_t(r_0)}{|p_t(r_0 e^{2\pi i s})/p_t(r_0)|}$$

gives a homotopy from ω_n to f_{r_0}

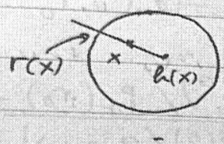
Thus $[\omega_n] = [f_{r_0}] = [f_0] = [\omega_0]$

Injectivity in Thm 12 $\Rightarrow$ $n = 0$
$\Rightarrow$ $p \in \mathbb{C}$ ↯ $\square$

Brouwer's fixed point theorem for $D^2 := \{x \in \mathbb{R}^2 \mid |x|_2 \leq 1\}$

Thm 16: Let $f: D^2 \longrightarrow D^2$ be continuous.
Then f has a fixed point, i.e.
$\exists x \in D^2: f(x) = x$.

__Proof:__ Assume that h has no fixed point. Define the map $r: D^2 \to S^1$, via intersection with S^1 of the ray $h(x) + \mathbb{R}^{\geq 0}(x - h(x))$.



Then r is continuous and $r|_{S^1} = id$.

Fix $x_0 \in S^1$ and take $[f]_{S^1} \in \pi_1(S^1, x_0)$.
Then $[f]_{D^2} = [c]_{D^2}$,

for example by a linear homotopy $(g_t)_t$, see Example 2.

$\Longrightarrow$ $(r \circ g_t)_{0 \leq t \leq 1}$ is a homotopy from

$f = r \circ f$ to $c = r \circ c$ in S^1.

$\Longrightarrow$ $[f]_{S^1} = [c]_{S^1}$.

$\Longrightarrow$ $\pi_1(S^1, x_0) = 0$ ↯ to Thm 12 □
$[f]_{S^1}$ arbitrary

__Thm 17:__ (Borsuk–Ulam) for dimension 2)
Let $f: S^2 \to \mathbb{R}^2$ be continuous.
Then $\exists\, x \in S^2$ s.t. $f(-x) = f(x)$.

__Rem 18:__ By Thm 17 there is a point x on earth such that x and $-x$ have the same temperature and the same barometric pressure.

__Proof__ (of Thm 17): Assume that there is no such x. Put

$$g(x) := \frac{(f(x) - f(-x))}{|f(x) - f(-x)|} \in S^1, \quad x \in S^2.$$

and

$$\eta(s) = (\cos(s 2\pi), \sin(2\pi s), 0), \; s \in I.$$

and $h := g \circ \eta : I \longrightarrow S^1$

Then $h(s + \frac{1}{2}) = -h(s)$ for all $s \in [0, \frac{1}{2}]$. Take a lift $\tilde{h}$ of h.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[ur, "\tilde{h}"] \arrow[r, "h"'] & S^1 \end{tikzcd}$$

Look at the graph of p.

$$\overset{(*)}{\Longrightarrow} \underset{\substack{n \in \mathbb{Z} \\ \text{odd}}}{\exists} : \tilde{h}(s+\tfrac{1}{2}) = \tilde{h}(s) + \tfrac{n}{2} \text{ for all } s \in [0,\tfrac{1}{2}]$$

$$\Longrightarrow \quad n + \tilde{h}(0) = \tfrac{n}{2} + \tilde{h}(\tfrac{1}{2}) = \tilde{h}(1).$$

W.l.o.g. we can assume that $h(0) = (1,0)$.
Otherwise compose an axial rotation with f.
Then we can choose $\tilde{h}$ s.t. $\tilde{h}(0) = 0$.

Then $\tilde{h} \simeq \tilde{\omega}_n$ in $\mathbb{R}$
$\Rightarrow [h]_{S^1} = [\omega_n]_{S^1}$ is non-trivial as $2 \nmid n$.

On the other hand: $\eta \simeq c_{(1,0)}$ in S^2
$\Rightarrow h = g \circ \eta \simeq g \circ c_{(1,0)} = c_{(1,0)}$ in S^1

$\Rightarrow [h]_{S^1}$ is trivial $\lightning$ □

<u>Def 19:</u> A pair (X, x_0) consisting of a top. space and a point $x_0 \in X$ is called a <u>based space</u>. A morphism $f: (X, x_0) \longrightarrow (Y, y_0)$ of based spaces is a map $f: X \to Y$, continuous, s.t. $f(x_0) = y_0$.

$\mathrm{Top}_{\text{based}} :=$ Category of based topological spaces.

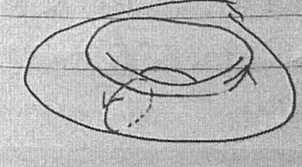
<u>Prop 20:</u> $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$

<u>Proof:</u> $[f]_{X \times Y} \longmapsto ([pr_X \circ f], [pr_Y \circ f])$

$[f_X * f_Y] \longleftarrow ([f_X], [f_Y])$

□

<u>Example 21:</u> $T^2 := S^1 \times S^1$ 2-dim. torus.
$\pi_1(T^2) \simeq \pi_1(S^1) \times \pi_1(S^1) \simeq \mathbb{Z} \times \mathbb{Z}$.



<u>Prop 22</u> A morphism $\alpha : (X, x_0) \longrightarrow (Y, y_0)$ induces a group homomorphism

$\alpha_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$

defined via

$\alpha_*([f]) := [\alpha \circ f]$.

Further we have

(a) $(\alpha \circ \beta)_* = \alpha_* \circ \beta_*$

(b) $(id_X)_* = id_{\pi_1(X, x_0)}$. (identity map of the set $\pi_1(X, x_0)$)

<u>Proof:</u> easy. □

<u>Cor 23:</u> Given $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ continuous such that $\beta \circ \alpha$ is a homeomorphism. Take $x_0 \in X$.

Then $\alpha_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, \alpha(x_0))$ is injective and

$\beta_* : \pi_1(Y, \alpha(x_0)) \longrightarrow \pi_1(Z, \beta(\alpha(x_0)))$ is surjective.

<u>Example 24:</u> $n \geq 1$. Take $x \in \mathbb{R}^n$.

Then $\mathbb{R}^n \setminus \{x\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$

Thus $\pi_1(\mathbb{R}^n \setminus \{x\}) \simeq \pi_1(S^{n-1}) \times \pi_1(\mathbb{R})$
$\simeq \pi_1(S^{n-1})$.

<u>Prop 25:</u> Let $n \geq 2$. Then $\pi_1(S^n) = 0$.

<u>Proof:</u> $[f]_{S^n} \in \pi_1(S^n, x_0)$

<u>Case 1:</u> $\exists\, x \in S^n$ s.t. $x \notin im(f)$.

$S^n \setminus \{x\}$ homeomorphic to $\mathbb{R}^n$ and $\pi_1(\mathbb{R}^n) = 0$.

So f is homotop to c_{x_0} in $S^n \setminus \{x\}$.

So $[f]_{S^n} = [c_{x_0}]_{S^n}$

<u>Case 2:</u> $im(f) = S^n$. We show that $\exists\, g$ a non-surjective loop at x_0 homotop to f. (Then apply Case 1)

Take $x \neq x_0$ and an open ball $B \subseteq S^n$ around x s.t. $x_0 \notin \overline{B}$. (B is an intersection of an open ball in $\mathbb{R}^{n+1}$ with S^n.)

$f^{-1}(B)$ is a disjoint union of open intervals in $I = [0,1]$.
$f^{-1}(x)$ is compact, so there are only finitely many those intervals $]a_1, b_1[, \dots,]a_m, b_m[$ intersecting $f^{-1}(x)$.
$f(a_i), f(b_i) \in \partial B$ for $i = 1, \dots, m$. So $f|_{[a_i, b_i]}$ is homotop to a path $g : [a_i, b_i] \longrightarrow \partial B$. (Why?)

Thus f is homotop to a loop g in S^n with $\mathrm{im}(g) \not\ni x$.

$\square$

<u>Cor 26:</u> $\mathbb{R}^2$ is not homoemorphic to $\mathbb{R}^n$ for $n \neq 2$.

<u>Proof:</u> $n=1$: $\mathbb{R} \setminus \{pt\}$ disconnected
$\mathbb{R}^2 \setminus \{pt\}$ connected.

$n > 2$: $\pi_1(\mathbb{R}^n \setminus \{pt\}) \overset{24}{\simeq} \pi_1(S^{n-1}) \overset{25}{=} 0$.
$\pi_1(\mathbb{R}^2 \setminus \{pt\}) \overset{24}{\simeq} \pi_1(S^1) \overset{12}{\simeq} \mathbb{Z}$.

$\square$

For a subset $A \subseteq X$ and $x_0 \in A$ it is reasonable to study
$$\iota_* : \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$$
for the inclusion $\iota : A \longrightarrow X$
$(\iota(x) = x,\ x \in A)$.

<u>Theorem 27'</u> (a) If A is a retract of X then ι_* is injective
(b) If A is a deformation retract of X then ι_* bijective.

We recall the notions:

<u>Def 28:</u> (a) A continuous map $r : X \to X$ is called <u>retraction of X onto A</u> if $\mathrm{im}(r) = A$ and $r|_A = id_A$

(b) A homotopy $F: X \times I \longrightarrow X$ is called a <u>deformation retraction of X onto A</u> if
- $F(-,0) = id_X$
- $F(a,t) = a \quad \forall a \in A \ \forall t \in I$
- $im(F(-,1)) = A$.

(i.e. F is a homotopy from id_X to a retraction onto A relative to A)

<u>Example 29:</u> $r: \mathbb{R}^2 \setminus \{0\} =: X \longrightarrow X$
$r(x) := \frac{x}{|x|_2}$

Then r is a retraction from X onto S^1.
(Indeed: $im(r) = S^1$ and $r(x) = x$ for all $x \in S^1$.)

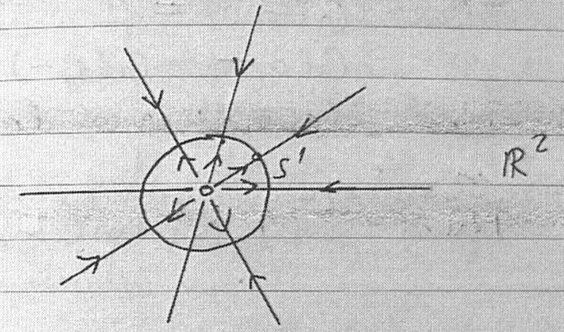
<u>Claim:</u> S^1 is a deformation retract of X.

<u>Proof:</u> $F: X \times I \longrightarrow X$,

$F(x,t) := (1-t)\,x + t\,\frac{x}{|x|}$.

Then $F(-,0) = id_X$, $F(-,1) = r$ and $F(-,t)\big|_{S^1} = id\big|_{S^1}$.

□



Then Thm 27 implies that
$\iota_*: \Pi_1(S^1) \longrightarrow \Pi_1(\mathbb{R}^2 \setminus \{0\})$
is a group isomorphism.

<u>Proof of Thm 27:</u> (a) $r: X \to X$ a retraction of X onto A
Put $\varrho: X \to A$, $\varrho(x) := r(x)$.
$\Rightarrow \varrho \circ \iota = id_A \overset{23}{\Rightarrow} \iota_*$ is injective.

(b) $F : X \times I \longrightarrow X$ a deformation retraction of ~~X~~ onto r.
Take $[f]_X \in \pi_1(X, x_0)$. $(x_0 \in A)$.

Put $G(s,t) := F(f(s), t)$.
Then $G(-, 0) = f$
$G(0, -) = G(1, -) = x_0$
and $G(-, 1)$ is a loop in A.
Thus $[f]_X \in \mathrm{im}(\iota_*)$ □

<u>Def 30</u> $\alpha : X \longrightarrow Y$ continuous is called a homotopy equivalence if $\exists\ \beta : Y \longrightarrow X$ cont. s.t.
$\alpha \circ \beta \simeq id_Y$ and $\beta \circ \alpha \simeq id_X$

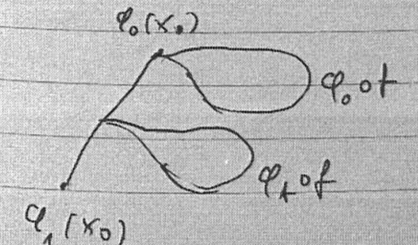
In that case we say that X is <u>homotopy equivalent</u> to Y (or we say: X and Y have the <u>same homotopy type</u>)

<u>Prop 31</u>: Let $\alpha : X \longrightarrow Y$ be a homotopy equivalence and $x_0 \in X$.
Then $\alpha_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, \alpha(x_0))$ is bijective.

<u>Lem 32</u>: Let $\Phi : X \times I \longrightarrow Y$ be a homotopy from φ_0 to φ_1.
Put $h := \Phi(x_0, -)$, i.e. h is a path from $\varphi_0(x_0)$ to $\varphi_1(x_0)$.
Then the following diagram commutes.

$$\begin{tikzcd}
 & \pi_1(Y, \varphi_0(x_0)) \arrow[d, "\beta_h"] & {[\varphi]} \arrow[d] \\
\pi_1(X, x_0) \arrow[ur, "\varphi_{0*}"] \arrow[r, "\varphi_{1*}"'] & \pi_1(Y, \varphi_1(x_0)) & {[\bar{h}\varphi h]}
\end{tikzcd}$$

<u>Proof</u>: Take $[f]_X \in \pi_1(X, x_0)$.
To show $\bar{h}(\varphi_0 \circ f)h \simeq \varphi_1 \circ f$.



□

<u>Proof of Prop. 31:</u> Note that in lemma 32 β_h is bijective.

Take $\beta : Y \to X$ s.t. $\alpha \circ \beta \simeq id_Y$ and $\beta \circ \alpha \simeq id_X$.

By Lem 32

$$(\beta \circ \alpha)_* : \pi_1(X, x_0) \to \pi_1(X, \beta(\alpha(x_0)))$$

and $(\alpha \circ \beta)_* : \pi_1(Y, \alpha(x_0)) \to \pi_1(Y, \alpha(\beta(\alpha(x_0))))$

are isomorphisms

$\Longrightarrow$ α_* and β_* are injective

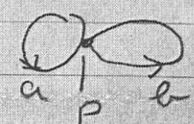
$\underset{\beta_* \circ \alpha_* \text{ bijective}}{\Longrightarrow}$ α_* is bijective.

$\square$

II Van Kampen's Theorem

II 1. Free product of groups

<u>Example 33:</u> $X = S^1 \vee S^1$



two loops at P.

A loop in X at P is homotopic to some word of the form

$$\underbrace{a^{i_1} b^{i_2} a^{i_3} \cdots}_{\text{finitely many}}$$

(It might start with b.)
In fact this can be interpreted as an element of $\pi_1(X)$.
We want to define the group of such words without geometry.

<u>Def 34:</u> (free product of groups)

Let $(G_\lambda)_{\lambda\in\Lambda}$ be a family of groups, such that $\forall \lambda,\mu\in\Lambda, \lambda\neq\mu : G_\lambda \cap G_\mu = \{1\}$.

Put $A := \bigcup_{\lambda\in\Lambda} G_\lambda$.

$= \{1\} \,\dot\cup\, \dot{\bigcup_{\lambda\in\Lambda}} (G_\lambda \setminus \{1\})$.

(disjoint union)

(a) A word in A is a tuple $\underline{g} = (g_1,\dots,g_m)$ with $m\geq 0$ and $g_i \in A$, for all i.

(b) $\underline{g} \overset{i}{\sim} \underline{h}$ (incident)

if $\left(\exists j\in\{1,\dots,m-1\}: \underline{h} = (g_1,\dots,g_j g_{j+1},\dots,g_m)\right)$

or $\left(\underline{g} = (1) \text{ and } \underline{h} = ()\right)$

or vice versa.

(c) $\underline{g} \sim \underline{h}$ (equivalent)

$\underset{\text{Def}}{\Longleftrightarrow} \exists\, l\geq 0\ \exists\, \underline{c}^{(0)},\dots,\underline{c}^{(l)}$ words in A:

$\underline{g} = \underline{c}^{(0)} \overset{i}{\sim} \underline{c}^{(1)} \overset{i}{\sim} \underline{c}^{(2)} \overset{i}{\sim} \dots \overset{i}{\sim} \underline{c}^{(l)} = \underline{h}$.

(d) $\mathop{*}_{\lambda\in\Lambda} G_\lambda := \{ [\underline{g}]_\sim \mid \underline{g} \text{ word in } A \}$

"free product of the family $(G_\lambda)_{\lambda\in\Lambda}$."

<u>Example 35:</u> $G = \langle a\rangle \simeq \mathbb{Z}$

$H = \langle b\rangle \simeq \mathbb{Z}$.

$A = \{ a^j, b^k \mid j,k\in\mathbb{Z}\setminus\{0\}\} \cup \{1\}$

Consider the word $\underline{g} = (a,1,b,b,a,a^{-2})$

$\underline{g} \overset{i}{\sim} (a,b,b,a,a^{-2}) \overset{i}{\sim} (a,b^2,a,a^{-2})$

$\overset{i}{\sim} (a,b^2,a^{-1})$

<u>Prop. 36:</u> $\underset{\Lambda}{*} G_\lambda$ is a group with group structure

$$[\underline{g}]_\sim [\underline{h}]_\sim := [\underline{g} \cup \underline{h}]_\sim$$

where $\underline{g} \cup \underline{h} := (g_1, \dots, g_m, h_1, \dots, h_n)$,

for $\underline{g} = (g_1, \dots, g_m), \underline{h} = (h_1, \dots, h_n) \in \mathcal{A}$.

<u>Proof:</u> well-def. ✓
associativity ✓
unit element $[()]_\sim$ empty word
inverse: $[\underline{g}]_\sim^{-1} = [(g_m^{-1}, \dots, g_1^{-1})]_\sim$

□

<u>Def. 37:</u> A word $\underline{g}$ in $\mathcal{A}$ is called <u>reduced</u> if $\nexists$ word $\underline{h} = (h_1, \dots, h_n)$ in $\mathcal{A}$: $n < m$ and $\underline{g} \sim \underline{h}$.

<u>Lemma 38:</u> Let $\underline{g}$ be a word $= (g_1, \dots, g_m)$ in $\mathcal{A}$.
Suppose (1) $\forall_i : g_i \neq 1$ and
(2) $\forall_{i=1,\dots,m-1} : \nexists_{\lambda \in \Lambda}\ g_i, g_{i+1} \in G_\lambda$.

Then $\underline{g}$ is reduced.

<u>Proof:</u> exercise. □

<u>Prop. 39:</u> (a) $G_\lambda \xrightarrow{\iota_\lambda} \underset{\Lambda}{*} G_\lambda$

$$g \longmapsto [(g)]$$

is injective.

(b) Let $(G_\lambda \xrightarrow{\varphi_\lambda} H)_\Lambda$ be a family of group homomorphisms.
Then $\exists!$ group homomorphism

$$\underset{\Lambda}{*} G_\lambda \xrightarrow{\varphi} H \quad \text{such that}$$

$\forall_{\lambda \in \Lambda}$:

$$\begin{tikzcd} G_\lambda \arrow[r, hook] \arrow[dr, "\varphi_\lambda"'] & \underset{\Lambda}{*} G_\lambda \arrow[d, "\varphi"] \\ & H \end{tikzcd}$$

Proof: (a) ι_λ is a group homomorph.
$\iota_\lambda(g) = [()] \Rightarrow (g) \sim ()$

$\Rightarrow (g)$ is not reduced.

$\underset{38}{\Rightarrow} \quad g = 1.$

(b) We are given $(G_\lambda \xrightarrow{\varphi_\lambda} H)_\Lambda$

and put $\underset{\Lambda}{*} G_\lambda \xrightarrow{\varphi} H$
via
$$\varphi([\underline{g}]) := \varphi_{\lambda_1}(g_1) \cdots \varphi_{\lambda_m}(g_m)$$

if $g_i \in G_{\lambda_i}, \ i = 1, \dots, m.$

φ is a well-defined group homom. and unique w.r.t. to the diagrams as

- $\bigcup_\Lambda \iota_\lambda(G_\lambda)$ generates $\underset{\Lambda}{*} G_\lambda$ and
- $\varphi \circ \iota_\lambda = \varphi_\lambda.$

$\square$

Example 40

(a) $\mathbb{Z} * \mathbb{Z} = \{ a^{i_1} b^{i_2} a^{i_3} \cdots a^{i_{2k-1}},$
$a^{i_1} b^{i_2} \cdots a^{i_{2k-1}} b^{i_{2k}},$
$b^{i_1} a^{i_2} \cdots b^{i_{2k-1}},$
$b^{i_1} \cdots a^{i_{2k}} \mid i_1, \dots, i_{2k} \in \mathbb{Z} \setminus \{0\}$
and $k \geq 0 \}$

example: $b a^2 b^{-1} a b.$

(b) free group on m-letters

Write $\mathbb{Z} = \mathbb{Z} x = \langle x \rangle$
(free $\mathbb{Z}$ module generated by x.)

$$\mathbb{Z} * \dots * \mathbb{Z} \cong \langle x_1 \rangle * \dots * \langle x_m \rangle =: F(x_1, \dots, x_m)$$

Exercise 41: The abelianization of $F(x_1, \dots, x_m)$ is isomorphic to $\underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{m}.$

<u>Hint:</u> Prop 39 (b) gives a group homo.

$$F(\underline{x}) \longrightarrow \mathbb{Z}^m$$
$$x_i \longmapsto e_i = (0, \dots, 1, \dots, 0)$$

Compute the kernel. $\square$

<u>Def 42:</u> (presentation of a group)

$G = \langle x_1, \dots, x_m \mid w \rangle$
is the group generated by $x_1, \dots, x_m$ with relation w
(w word in $x_i^{\pm 1}$)

$$:= F(\underline{x}) \big/ \underbrace{\langle\langle w \rangle\rangle}$$

smallest normal subgroup containing w

This can be done with more words.

$R \subseteq F(\underline{x})$
$$\langle \underline{x} \mid R \rangle := F(\underline{x}) \big/ \langle\langle R \rangle\rangle$$

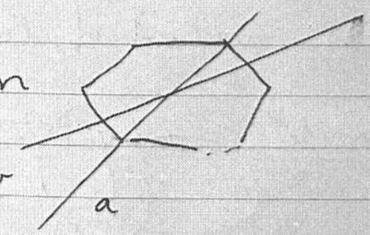
<u>Example 43:</u>

$$\langle a \mid a^m \rangle \cong \mathbb{Z}/m\mathbb{Z} = \{ [z]_m \mid z \in \mathbb{Z} \}$$
$$= \{ [0]_m, \dots, [m-1]_m \}$$

$\langle a, b \mid a^2, b^2, (ab)^m \rangle$
D_{2m} dihedral group with $2m$ elements.

<u>model:</u> $m \geq 3$

regular m-gon



a and b are represented by reflections.

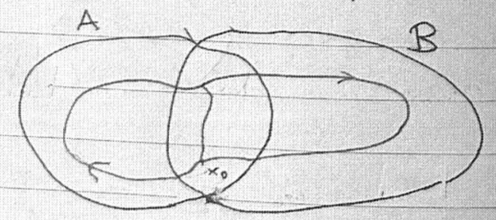
II 2. The van Kampen Theorem

We want to compute $\pi_1(X, x_0)$ using $\pi_1(A_\lambda, x_0)$ for an open cover $X = \bigcup_\lambda A_\lambda$ of X.

By Prop 39(b) we have a map

$$\Phi : \mathop{*}_{\Lambda} \pi_1(A_\lambda, x_0) \longrightarrow \pi_1(X, x_0)$$

van Kampen says that Φ is very often surjective.
(The condition is that $\forall \lambda \neq \mu$: $A_\lambda \cap A_\mu$ is path connected, so that we can express a loop in X as a product of loops in A_λ s.)



Φ is in general not injective.
consider $X = A \cup B$, $x_0 \in A \cap B$, A and B open.
We have in

$$\begin{array}{ccc} & \pi_1(A, x_0) \xrightarrow{\ \iota^A\ } \subset & \\ \pi_1(A \cap B, x_0) & & \pi_1(A, x_0) * \pi_1(B, x_0) \\ & \pi_1(B, x_0) \xrightarrow{\ \iota^B\ } \subset & \Big\downarrow \Phi \\ & & \pi_1(X, x_0) \end{array}$$

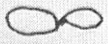
$$\Phi(\iota^A([f])) = \Phi(\iota^B([f])), \forall [f] \in \pi_1(A \cap B, x_0)$$

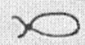
So $\iota^A([f])\, \iota^B([f])^{-1} \in \ker \Phi$.

van Kampen says, if $A \cap B$ is path connected, then
$N := \langle\langle \{ \iota^A([f]) \iota^B([f])^{-1} \mid [f]_{A \cap B} \} \rangle\rangle$
is equal to $\ker \Phi$, i.e.

$$\pi_1(A, x_0) * \pi_1(B, x_0) / N \xrightarrow{\sim} \pi_1(X, x_0).$$

Example 44: (a) $S^1 \vee S^1$ (wedge sum)

$X =$ 

$A =$  $B =$ 

(Then $A \cap B = x$ deformation retracts to a point)

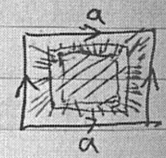
$\pi_1(A \cap B, x_0) = 0 \Rightarrow N = \{1\}$

van Kampen $\Rightarrow \pi_1(A) * \pi_1(B) \xrightarrow{\sim} \pi_1(X)$

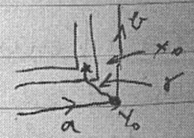
$\cong$

$\mathbb{Z} * \mathbb{Z}$.

(b) torus $S^1 \times S^1$



$\pi_1(B, x_0) = 0$

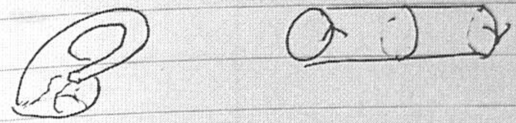


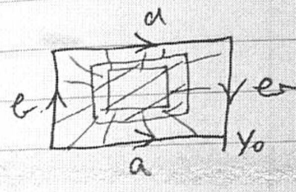
$A \cap B$ deformation retracts to a circle which is homotop to $\gamma b a^{-1} b^{-1} a \gamma^{-1}$ in A.

So $\pi_1(X) \cong \pi_1(A, x_0) / \langle\langle \gamma b a^{-1} b^{-1} a \gamma^{-1} \rangle\rangle$

$\cong \pi_1(A, y_0) / \langle\langle b a^{-1} b^{-1} a \rangle\rangle \cong \mathbb{Z} * \mathbb{Z} / \langle\langle b a^{-1} b^{-1} a \rangle\rangle = \mathbb{Z} \oplus \mathbb{Z}$

(c) Kleinian bottle.





A, B

$\pi_1(X, x_0) \cong \mathbb{Z} * \mathbb{Z} / \langle\langle b^{-1} a^{-1} b^{-1} a \rangle\rangle$

$= \langle a, b \mid b^{-1} a = a b \rangle$

(d) $X = \mathbb{R}P^2$ 

$\pi_1(X, x_0) \underset{\uparrow}{\cong} \mathbb{Z} / \langle a^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

$\mathbb{Z} = \langle a \rangle$.