Textbook: Allen Hatcher, "Algebraic Topology"

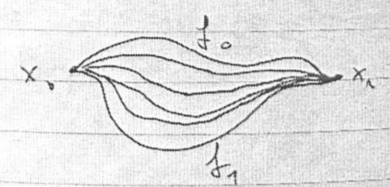


# I The fundamental group

<u>Def 1:</u> (a) Let $X$ be a top. space. A path in $X$ is a continuous map $f: I = [0,1] \longrightarrow X$

(b) Two paths $f_0, f_1 : I \longrightarrow X$ are called <u>homotopic</u> if $\exists$ $F: I \times I \longrightarrow X$ cont s.t. $F(-,0) = f_0$ and $F(-,1) = f_1$

(The family $f_t := F(-,t)$, $t \in I$, is called a <u>homotopy</u>.)




and $F(0,-)$ and $F(1,-)$ are constant.

We write $f_0 \simeq f_1$

**Example 2:** In $\mathbb{R}^n$. Let $x_0, x_1 \in \mathbb{R}^n$ and $f$ be a path from $x_0$ to $x_1$.

Then $f$ is homotopic to the segment.

$[x_0, x_1] = \{ s x_1 + (1-s) x_0 \mid s \in [0,1] \}$

more precisely

$g : I \longrightarrow \mathbb{R}^n \quad g(s) = s x_1 + (1-s) x_0$.

Proof: Take $F : I \times I \longrightarrow \mathbb{R}^n$

$F(s,t) := t\, g(s) + (1-t) f(s)$. □

**Remark 3:** Let $x_0, x_1 \in X$.

Then homotopy is an equivalence relation on the set of path from $x_0$ to $x_1$.

Proof: reflexive: $f \simeq f$ via $F(s,t) = f(s)$

symmetric: $f \simeq g$ via $F$ then $g \simeq f$ via



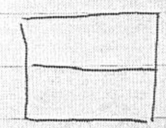
$G(s,t) := F(s, 1-t)$.

transitive: $f \overset{F}{\simeq} g \overset{G}{\simeq} h$

Then $f \simeq h$ via

$$H(s,t) := \begin{cases} F(s, 2t), & 0 \le t \le \frac{1}{2} \\ G(s, 2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

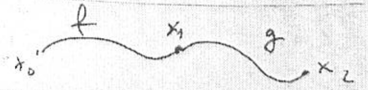
$H$ is continuous, as $H$ is continuous on $I \times [0, \frac{1}{2}]$ and $I \times [\frac{1}{2}, 1]$



□

**Def 4:** Given two paths $f, g : I \longrightarrow X$ s.t. $f(1) = g(0)$ we define the composition $f \cdot g$:

$$(f \cdot g)(s) := \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$



<u>Lem 5:</u> $f_0 \simeq f_1$ from $x_0$ to $x_1$

and $g_0 \simeq g_1$ from $x_1$ to $x_2$

Then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ from $x_0$ to $x_2$.

<u>Proof:</u> $f_0 \overset{F}{\simeq} f_1$, $g_0 \overset{G}{\simeq} g_1$

Take $H(s,t) := \begin{cases} F(2s,t), & 0 \le s \le \frac{1}{2} \\ G(2s-1,t), & \frac{1}{2} \le s \le 1. \end{cases}$

$\square$

<u>Def 6:</u> (a) Let $x_0 \in X$. A loop at $x_0$ is a path $f$ s.t. $f(0) = f(1) = x_0$.

(b) $\pi_1(X, x_0) := \{ [f] \mid f \text{ a loop at } x_0 \text{ in } X \}$

is called the <u>fundamental group</u> of <u>X at the base point</u> $x_0$.



Here $[f]$ is the homotopy class of $f$.

<u>Prop. 7:</u> $\pi_1(X, x_0)$ is a group w.r.t. the product $[f][g] := [f g]$.

<u>Lem 8:</u> (reparametrization)

Let $\varphi : I \longrightarrow I$ be continuous with $\varphi(0) = 0$ and $\varphi(1) = 1$, and let $f : I \longrightarrow X$ be a path. Then $f \simeq f \circ \varphi$.

<u>Proof:</u> Put $\varphi_t(s) := (1-t)s + t\varphi(s)$

$(f \circ \varphi_t)_t$ is a homotopy from $f$ to $f \circ \varphi$.

$\square$

<u>Proof of Prop 7:</u> Let $c$ be the constant path $c(s) := x_0$, $s \in I$.

$c \cdot f \simeq f \simeq f \cdot c$ by lemma 8.

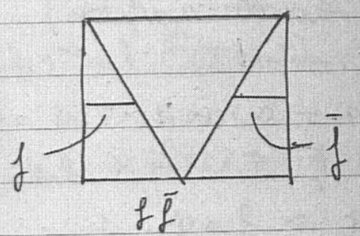
- $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$ by Lemma 8.

- Given $f$ we put

$$\bar{f}(s) = f(1-s).$$

<u>Claim</u>: $f \cdot \bar{f} \simeq \bar{f} \cdot f \simeq c$.

<u>Proof</u>:




$$F(s,t) := \begin{cases} f(2s), & 0 \le s \le \frac{1-t}{2} \\ f(1-2s), & \frac{1+t}{2} \le s \le 1 \\ f(1-t), & \frac{1-t}{2} \le s \le \frac{1+t}{2} \end{cases}$$

□



<u>Example 9</u>: $\pi_1(\mathbb{R}^n, x_0) = 0$.

<u>Proof</u>: Take a linear homotopy. □

<u>Prop 10</u>: Let $x_0, x_1 \in X$ s.t. $\exists$ path $h$ from $x_0$ to $x_1$.
Then

$$\beta_h : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$
$$[f] \longmapsto [h f \bar{h}]$$

is an isomorphism.

($\beta_h$ "change of base point map")

<u>Def 11</u>: (simply connected)
A topological space $X$ is called simply connected if $X$ is path-connected and $\pi_1(X) = 0$.

<u>Thm 12</u>: $\pi_1(S^1) \simeq \mathbb{Z}$

**Lemma 13:** (homotopy extension property)

Consider $p : \mathbb{R} \longrightarrow S^1$ $\quad p(u) = e^{2\pi i u}$

Suppose we are given two cont. maps

$$F : Y \times I \longrightarrow S^1$$
$$\tilde{F} : Y \times \{0\} \longrightarrow \mathbb{R}$$

s.t. $F\big|_{Y\times\{0\}} = p \circ \tilde{F}$.

Then $\exists!\ \tilde{F} : Y \times I \longrightarrow \mathbb{R}$ s.t. $p \circ \tilde{F} = F$ extending $\tilde{F}|_{Y\times\{0\}}$.

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{F}"] \arrow[d, phantom, "\cap"] & \mathbb{R} \arrow[d, "p"] \\ Y \times I \arrow[r, "F"] \arrow[ur, dashed, "\exists!\tilde{F}"] & S^1 \end{tikzcd}$$



**Proof:** **Uniqueness:**

We reduce to the case of $|Y| = 1$.
Say $\tilde{F}, \tilde{G}$ are two lifts of $F$ with $\tilde{F}|_{Y\times\{0\}} = \tilde{G}|_{Y\times\{0\}}$.

Then for each $y \in Y$
$$\tilde{F}|_{\{y\}\times I},\ \tilde{G}|_{\{y\}\times I} \text{ lift } F|_{\{y\}\times I}$$

with $\tilde{F}(y,0) = \tilde{G}(y,0)$.
From the case $|Y| = 1$ we get
$$\tilde{F}|_{\{y\}\times I} = \tilde{G}|_{\{y\}\times I}.$$

So we need to prove the case $|Y| = 1$.

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d, phantom, "\cap"] & \mathbb{R} \arrow[d, "p"] & r_0 \\ I \arrow[r, "F"] & S^1 & x_0 \end{tikzcd}$$

Let $\tilde{F}, \tilde{G}$ be two lifts of $F$ with $\tilde{F}(0) = \tilde{G}(0) = r_0$.
Take $\delta$ maximal s.t. $\tilde{F}|_{[0,\delta]} = \tilde{G}|_{[0,\delta]}$.

Assume $\delta < 1$.

W.l.o.g. $\delta = 0$.

Take $U \subseteq S^1$ open s.t. $x_0 \in U$

(*) and $p^{-1}(U) = \dot{\bigcup_i} U_i$

s.t. $p|_{U_i} : U_i \xrightarrow{\sim} U$ homeomorphism.

$\exists i_0 : r_0 \in U_{i_0}$ and $\exists \varepsilon > 0 : F([0,\varepsilon]) \subseteq U$.

$\tilde{F}([0,\varepsilon])$ and $\tilde{G}([0,\varepsilon])$ are connected and $r_0 \in U_{i_0}$

So $\tilde{F}([0,\varepsilon]) \cup \tilde{G}([0,\varepsilon]) \subseteq U_{i_0}$

$p|_{U_{i_0}}$ is bijective $\Rightarrow$ $\tilde{F}|_{[0,\varepsilon]} = \tilde{G}|_{[0,\varepsilon]}$ $\lightning$

<u>Existence:</u> It is enough to show

$\forall_{y_0 \in Y}$ $\exists$ open nbhd $N$ of $y_0$:

$\exists$ lift $\tilde{F}$ of $F$ on $N \times I$.

(By the uniqueness part we can then glue them together.)



We cover $S^1$ by open sets $U^{(\alpha)}$, $\alpha \in A$, which satisfy (*) (replacing $U$ by $U^{(\alpha)}$). In fact two sets are enough.

As $I$ is compact and $F$ is continuous $\exists$ open nbhd $N$ of $y_0$ s.t.

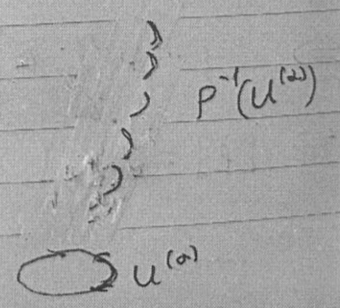
$\exists\ 0 = t_0 < t_1 < \dots < t_m = 1$:

$\forall_{j \in \{0,\dots,m-1\}} \exists_{\alpha_j} : F(N \times [t_j, t_{j+1}]) \subseteq U^{(\alpha_j)}$

Now we inductively lift $F$ using that

$$p|_{U_i^{(\alpha_j)}} : U_i^{(\alpha_j)} \xrightarrow{\sim} U^{(\alpha_j)}.$$

$\square$



Proof of Thm 12:

We show that

$$z \in \mathbb{Z} \longmapsto [\omega_z] \in \pi_1(S^1, 1)$$

with $\omega_z(s) := e^{2\pi i z s}$

is bijective.

Surjective: $[f] \in \pi_1(S^1, 1)$

Lemma 13 $\Rightarrow \exists \tilde{f} : I \longrightarrow \mathbb{R}$ : $\tilde{f}(0) = 0$ and $p \circ \tilde{f} = f$.

$\Rightarrow \tilde{f}(1) \in \mathbb{Z}$

$\Rightarrow \tilde{f} \simeq \tilde{\omega}_z$ in $\mathbb{R}$

$\Rightarrow f = p \circ \tilde{f} \simeq p \circ \tilde{\omega}_z = \omega_z$ in $S^1$

$\Rightarrow [f] = [\omega_z]$.



Injectivity: Suppose $[\omega_n] = [\omega_m]$

$\Rightarrow \omega_m \simeq \omega_n$

$\Rightarrow \tilde{\omega}_m \simeq \tilde{\omega}_n$ , by Lemma 13 (fill the details)

$\Rightarrow m = n$

$\square$

Exercise 14: Show that the map is a group homomorphism.