VI.2. Triangulated Categories

Let K be a category together with an automorphism $T : K \to K$. (T is a functor)

<u>Def 214</u>: A <u>triangle</u> on $(A, B, C) \in Obj(K)^3$ is a triple (u, v, w) of morphisms $u : A \to B$, $v : B \to C$, $w : C \to TA$

Write

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd}$$

A <u>morphism of</u> Δs is a triple (f, g, h)

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r, "v"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r, "w"] \arrow[d, "h"] \arrow[dr, phantom, "\circlearrowleft"] & TA \arrow[d, "Tf"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA' \end{tikzcd}$$

<u>Rk</u>: Exact triangles should be seen as a replacement of short exact sequences.

<u>Def 215</u>: (triangulated category, Verdier)
A <u>triangulated category</u> is an additive category K with an autom. $T : K \to K$ (<u>translation functor</u>) and a family of Δs (called <u>exact Δs</u>) subject to the following axioms

(TR1). $\forall$ morphisms $u : A \to B$ $\exists$ morph. v, w such that (u, v, w) is an exact Δ.

- $\forall_{A \in Obj(K)}$: $(id_A, 0, 0)$ is exact on $(A, A, 0)$.
- Every Δ isomorphic to an exact Δ is exact.

(TR2) (Rotation) If (u, v, w) is exact on (A, B, C) then $(v, w, -Tu)$ is exact on (B, C, TA) and $(-T^{-1}w, u, v)$ — " — $(T^{-1}C, A, B)$.

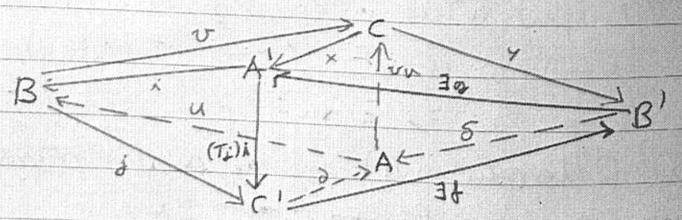
(TR3) (Morphisms) Given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] & TA \arrow[d, "Tf"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & TA' \end{tikzcd}$$

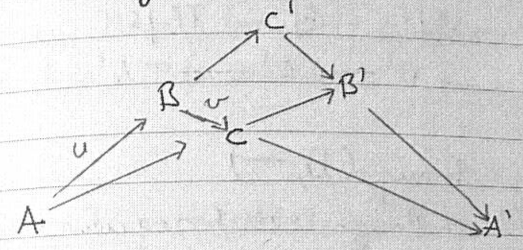
whose rows are exact Δ.
Then $\exists\, h : C \to C'$ such that (f, g, h) is a morphism of Δs.

(TR4) (The octahedral axiom)
Given $A, B, C, A', B', C' \in \mathrm{Obj}(K)$ and exact Δs
(u, j, ∂) on (A, B, C')
(v, x, i) on (B, C, A') and
(vu, y, δ) on (A, C, B')
there exists a fourth exact Δ
$(f, g, (Tj) i)$ on (C', B', A')
such that
$$\delta f = \partial, \quad g y = x \quad \text{and} \quad y v = f j.$$

<u>Visualization of (TR4):</u>



other way to visualize:



<u>Rk 216</u>: (a) Let (u, v, w) be an exact triangle. Then $vu = wv = (Tu)w = 0$
(b) (5-lemma)
If (f, g, h) is a morphism of exact triangles. Suppose f and g are isomorphisms. Then h is an isomorphism.

<u>Proof</u>: (a)
$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, dashed, "\exists h"] & TA \arrow[d, equal] \\ A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & TA \end{tikzcd}$$
(TR3) $\Rightarrow \exists\, h : 0 \to C$ (id_A, u, h) is a morphism of Δs.
<u>Note</u>: $h = 0$, so $vu = 0$.
The other zero identities we get by (TR2).

(b)
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"', "s"] & B \arrow[r] \arrow[d, "g"', "s"] & C \arrow[r] \arrow[d, "h"'] & TA \arrow[d, "Tf"', "s"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & TA' \end{tikzcd}$$

Apply $\mathrm{Hom}_K(D, -)$

We get a long exact sequence in Mod-$\mathbb{Z}$.

$$\hookrightarrow \mathrm{Hom}_K(D,A) \to \mathrm{Hom}_K(D,B) \to \mathrm{Hom}_K(D,C) \to$$
$$\hookrightarrow \mathrm{Hom}_K(D,TA) \to \mathrm{Hom}_K(D,TB) \to \mathrm{Hom}_K(D,TC) \to$$
$$\hookrightarrow$$

(Exactness: Exercise! Hint: Use (TR3) with (TR1).)

(f_*, g_*, h_*) defines a morphism between the long exact sequences. By the original 5-lemma, h_* is an isomorphism

$$\mathrm{Hom}_K(D,C) \xrightarrow{\sim} \mathrm{Hom}_K(D,C').$$

Take $D = C'$ $\Rightarrow$ h has a right inverse. Use $\mathrm{Hom}_K(-, D)$ to show that h has a left inverse □

Prop 217: $K(\mathcal{A})$ is a triangulated category.

Proof: [Weibel, Prop. 10.2.4.] □

Cor. 218: $K^b(\mathcal{A})$, $K^-(\mathcal{A})$, $K^+(\mathcal{A})$ are triangulated categories

Proof: $\mathcal{C} \in \{Ch^b(\mathcal{A}), Ch^-(\mathcal{A}), Ch^+(\mathcal{A})\}$
$\mathcal{C}$ full subcategory of $Ch(\mathcal{A})$
$\mathcal{C}$ is additive and closed under T and taking mapping cones
□

They are examples of triangulated subcategories of $K(\mathcal{A})$, see Def 219(b).

Def 219: (a) An additive functor $F: K' \longrightarrow K$ between Δ categories is called a morphism of Δ —"— if

- $FT = TF$ and
- F sends exact Δs to exact Δs.

(b) Let $(K, \text{Exact}\Delta s)$ be a triangulated category.
A full sub-category K' of K is called triangulated subcategory if $(K', K' \cap \text{Exact}\Delta s)$ is a triangulated category with translation functor $T|_{K'}$.

Def 220: Let K be a triangulated category and $\mathcal{A}$ be an abelian category. An additive functor $H: K \longrightarrow \mathcal{A}$ is called (covariant) cohomological functor if $\forall\ (u, v, w)$ exact triangle:

$$\hookrightarrow H(T^i A) \xrightarrow{H(T^i u)} H(T^i B) \longrightarrow H(T^i C) \hookrightarrow$$

$$\hookrightarrow H(T^{i+1} A) \longrightarrow \cdots \text{ is exact.}$$

We write $H^i(A)$ for $H(T^i A)$.

Example 221: (a) $H^0 : K(\mathcal{A}) \longrightarrow \mathcal{A}$.

(b) Take $D \in Obj(K)$.
$H(A) := Hom_K(D, A)$.

(c) Take $D \in Obj(K)$
$H(A) := Hom_K(A, D)$
is a contravariant cohomological functor.

VI 3. Localization and the calculus of fractions

We want to define the derived category of $\mathcal{A}$:

$$D(\mathcal{A}) := Q^{-1}K(\mathcal{A}).$$

where Q is the class of quasi-isomorphisms in $K(\mathcal{A})$.

Def 222: Let $\mathcal{C}$ be a category and S a class of morphisms in $\mathcal{C}$.
A category $S^{-1}\mathcal{C}$ together with a functor $q: \mathcal{C} \to S^{-1}\mathcal{C}$ is called a *localization of $\mathcal{C}$ w.r.t. S* if

1. $\forall s \in S$: $q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$
2. $\forall F: \mathcal{C} \to \mathcal{D}$ functor s.t. for all $s \in S$ $F(s)$ is an isom. in $\mathcal{D}$
   $\exists!\ G: S^{-1}\mathcal{C} \to \mathcal{D}$: $G \circ q = F$.

Example 223: (i) $S \subseteq Ch(\mathcal{A})$ class of all homotopy equivalences.

Prop 210 $\Rightarrow$ $K(\mathcal{A}) = S^{-1}Ch(\mathcal{A})$.

(ii) $\tilde{Q} :=$ class of all quasi-isomorphisms in $Ch(\mathcal{A})$

$$\Rightarrow \tilde{Q}^{-1}Ch(\mathcal{A}) = \tilde{Q}^{-1}(S^{-1}Ch(\mathcal{A})) = Q^{-1}K(\mathcal{A}) = D(\mathcal{A})$$

Problem: We don't know if $\tilde{Q}\,Ch(\mathcal{A})$ exists.

To prove existence for the cases
$\mathcal{A} =$ mod-R and
$\mathcal{A} =$ Sheaves(X)
one proves that $Q^{-1}K(\mathcal{A})$ exists in giving the morphisms in $Q^{-1}K(\mathcal{A})$ explicitely using Q and $K(\mathcal{A})$.

(iii) If $\mathcal{C}$ is small then $S^{-1}\mathcal{C}$ exists.
Too restrictive.
(iv) We will see that $S^{-1}\mathcal{C}$ exists if S is a locally small multiplicative system.

<u>Def 224:</u> Let $\mathcal{C}$ be a category. $S \subseteq \mathcal{C}$ is called <u>multiplicative system</u> if (1) $\cdot\ \forall X \in Obj(\mathcal{C}) : id_X \in S$

$\cdot\ \forall s : A \to B \in S\ \forall \underset{\in S}{t} : B \to C : ts \in S$

(2) (Ore condition)
$\cdot\ \forall t : Z \to Y$ in S $\forall g : X \to Y$ in $\mathcal{C}$
$\exists f : U \to Z$ in $\mathcal{C}$ $\exists s : U \to X$ in S : $tf = gs$

$$\begin{tikzcd} U \arrow[r, "\exists f"] \arrow[d, "\exists s"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[d, "t"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

(Write $t^{-1}g = fs^{-1}$ for some f, s)

$\cdot\ \forall f, s\ \exists g, t : fs^{-1} = t^{-1}g$.

(3) (cancellation) Let $f, g : X \to Y$ in $\mathcal{C}$.
T.a.e.:
1° $\exists s \in S : fs = gs$
2° $\exists t \in S : tf = tg$

<u>Example 225:</u> Let $(R, +, \cdot)$ be a commutative unital ring.
$\mathcal{R}$: $Obj(\mathcal{R}) := \{\cdot\}$
$End_{\mathcal{R}}(\cdot) := (R, +)$

$r_1 \circ r_2 := r_1 r_2$

Let $S \subseteq R$ be multiplicatively closed and $1_R \in S$.
Then S is a multiplicative system in $\mathcal{R}$ and $S^{-1}\mathcal{R}$ is the category with

$Obj(S^{-1}\mathcal{R}) = \{\cdot\}$
$End_{S^{-1}\mathcal{R}}(\cdot) = \{\frac{r}{s} \mid r \in R, s \in S\} = S^{-1}R$

(Note: $\frac{r}{s} = [(r,s)]_\sim$, $(r,s) \sim (r',s')$ if

$\exists\, t \in S: \quad t(rs' - r's) = 0)$

Let $\mathcal{C}$ be a category and $S \subseteq \mathcal{C}$ be a multiplicative system.
We now construct $S^{-1}\mathcal{C}$.

<u>Def 226:</u> A left fraction from X to Y is a diagram $X \xleftarrow{s} X_1 \xrightarrow{f} Y$ with $s \in S$.
Two left fractions
$$X \xleftarrow{s} X_1 \xrightarrow{f} Y$$
and $X \xleftarrow{t} X_2 \xrightarrow{g} Y$
are equivalent if $\exists$ left fraction
$X \leftarrow X_3 \longrightarrow Y$ such that $\exists \begin{smallmatrix} X_3 \to X_1 \\ X_3 \to X_2 \end{smallmatrix}$
such that

$$\begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & X_3 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \\ & X_2 \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes.
$\mathrm{Hom}_S(X,Y) := \mathrm{Hom}_{S^{-1}\mathcal{C}}(X,Y) := \{\text{left fractions } X \text{ to } Y\}/\sim$

<u>Problem 227:</u> In general $\mathrm{Hom}_S(X,Y)$ is not a set.

<u>Def 228:</u> S is called <u>locally small</u> (on the left) if
$\forall X \in \mathrm{Obj}(\mathcal{C})\ \exists$ set $S_X \subseteq S\ \forall X_1 \xrightarrow{f} X$ in S
$\exists\, X_2 \xrightarrow{g} X_1$ in $\mathcal{C}$: $f \circ g \in S_X$.

<u>Lemma 229</u> ([Weibel, 10.3.6])
If S is loc. small then $\mathrm{Hom}_S(X,Y)$ is a set $\forall X, Y \in \mathrm{Obj}(\mathcal{C})$.

We compose left fractions as follows.

$$\mathrm{Hom}_S(X,Y) \times \mathrm{Hom}_S(Y,Z) \longrightarrow \mathrm{Hom}_S(X,Z)$$
$$X \xleftarrow{t} X' \xrightarrow{g} Y \qquad Y \xleftarrow{s} Y' \xrightarrow{f} Z \qquad \underbrace{X \xleftarrow{r} X'' \xrightarrow{h} Z}_{\text{we want}}$$

<u>Idea:</u>

$$\begin{tikzcd} X'' \arrow[r, "i"] \arrow[d, "u"'] \arrow[dr, phantom, "\text{Ore}"] & Y' \arrow[r, "f"] \arrow[d, "s"] & Z \\ X' \arrow[r, "g"] \arrow[d, phantom] & Y & \end{tikzcd}$$
$X \xleftarrow{t} X'$

Take $r = tu$ and $h = fi$.

<u>Calculus:</u> $f s^{-1} g t^{-1} = f i u^{-1} t^{-1} = (fi)(tu)^{-1}$.

Theorem 230: (Gabriel–Zisman Thm)
Suppose the S is loc. small
Then $S^{-1}\mathcal{C}$ constructed above exists, i.e. $\forall x, y \in \mathcal{C}$: $\mathrm{Hom}_S(X,Y)$ is a set, and it is the localization of $\mathcal{C}$ w.r.t. S.
with $q : \mathcal{C} \longrightarrow S^{-1}\mathcal{C}$
given by $q(X \xrightarrow{f} Y) := X \xleftarrow{id_X} X \xrightarrow{f} Y$.

Proof. [Weibel, 10.3.7] . □

Def 231: A full subcategory $\mathcal{B}$ of $\mathcal{C}$ is called localizing subcategory of $\mathcal{C}$ w.r.t. S if
$\mathcal{B} \cap S$ is a multiplicative system too and

$$(\mathcal{B} \cap S)^{-1}\mathcal{B} \longrightarrow S^{-1}\mathcal{C}$$

is fully faithfull.

One still writes $S^{-1}\mathcal{B}$ for $(\mathcal{B} \cap S)^{-1}\mathcal{B}$.

Examples 232: $K^b(\mathcal{A}), K^+(\mathcal{A}), K^-(\mathcal{A})$ in $K(\mathcal{A})$

VI 4. The Derived Category

$$D(\mathcal{A}) = Q^{-1} K(\mathcal{A})$$

The goal is the following theorem

Theorem 233:
(a) $D(\mathcal{A}), D^b(\mathcal{A}), D^+(\mathcal{A}), D^-(\mathcal{A})$ are triangulated categories when they exist.
(b) They exist when
$\mathcal{A}$ = mod-R
$\mathcal{A}$ = Presheaves(X)
$\mathcal{A}$ = Sheaves(X),
because in those cases
Q is locally small from the left.
(c) Suppose $\mathcal{A}$ has enough injectives then $D^+(\mathcal{A})$ exists
and $D^+(\mathcal{A}) \underset{eq}{\cong} K^+(\mathcal{I})$
($\mathcal{I}$ full subcategory of all injectives of $\mathcal{A}$)

(d) Suppose $\mathcal{A}$ has enough projectives. Then $D^-(\mathcal{A})$ exists and

$$D^-(\mathcal{A}) \simeq K^-(\mathcal{P})$$

<u>Def 234:</u> Let K be a triang. cat and $H : K \longrightarrow \mathcal{A}$ a cohomological functor.

$S := \{\, s$ morph in $K \mid \forall_i \in \mathbb{Z}$: $H^i(s)$ is an isomorphism in $\mathcal{A} \,\}$

S is called the system arising from H.

<u>Example 235:</u> Q is the system arising from H^0.

<u>Prop 236:</u> Suppose S arises from H. Then

(a) S is a mult. system

(b) $S^{-1}K$ is a Δ-cat and $K \longrightarrow S^{-1}K$ is a morph. of Δ-cat.

<u>Proof:</u> (a) (See Def 224).

<u>(1):</u> ✓

<u>(2):</u>

$$\begin{tikzcd}
W \arrow[r, "s"] \arrow[d, "\exists f"'] & X \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, equal] & TW \arrow[d, "Tf"] \\
Z \arrow[r, "t"] & Y \arrow[r] & C \arrow[r] & TZ
\end{tikzcd}$$

(TR3)

Consider long exact seq.

$(H^i(t)$ iso $\forall_{i \in \mathbb{Z}}) \Rightarrow H^i(C) = 0\ \forall_i \in \mathbb{Z}$

$\Rightarrow H^i(s)$ iso $\forall_{i \in \mathbb{Z}}$.

<u>(3):</u> $sf = sg$, $s : Y \to Y'$ in S.

Embed s in an exact Δ: (u, s, δ) on (Z, Y, Y').

$\mathrm{Hom}_K(X, Z) \xrightarrow{u} \mathrm{Hom}_K(X, Y) \xrightarrow{s} \mathrm{Hom}_K(X, Y')$ is exact.

$\Rightarrow \exists\, h \in \mathrm{Hom}_K(X, Z) : uh = f - g$.

Take exact Δ (t, h, ∂) on (X', X, Z).

Then $t \in S$, as $H^i(Z) = 0\ \forall i \in \mathbb{Z}$, and $(f - g)t = u\underbrace{ht}_{0} = 0$.

(b) exercise $\square$

(The exact Δ of $S^{-1}K$ are the images of exact Δs of K.)

This proves 233 (a)
For 233(b), see [Weibel, 10.4.4]

We need to prove 233 (c) and (d).

<u>Lemma 237:</u> Let I be a bounded below cochain complex of injectives. Then every quasi-isom. $t : I \longrightarrow Z$ is a split mono in $K(\mathcal{A})$.

<u>Proof:</u> $\varphi : \mathrm{cone}(t) \longrightarrow TI$, $\varphi(i,z) := i$

$$\mathrm{cone}(t) = TI \oplus Z$$

is null-homotopic, by the comparison theorem for using injectives, see Thm 66.
(Note: $\mathrm{cone}(t)$ is acyclic and TI is bounded below.)
Say with chain homotopy

$v := (k, s) : \mathrm{cone}(t) \longrightarrow I$

$$d_{\mathrm{cone}(t)} = \begin{pmatrix} d_{TI} & \\ -t & d_Z \end{pmatrix} = \begin{pmatrix} -d_I & \\ -t & d_Z \end{pmatrix}$$

$$d_{TI} = -d_I.$$

$$-i = -i - 0 = -d_I\left((k,s)\begin{pmatrix} i \\ z \end{pmatrix}\right) + (k,s)\, d_{\mathrm{cone}(t)} \begin{pmatrix} i \\ z \end{pmatrix}$$

$$\Leftrightarrow \quad i = dki + dsz + kdi + sti - sdz$$
$$= (dk + kd + st)i + (ds - sd)z$$

Put $i = 0$ or $z = 0$.
Thus $d_I s = s d_Z$ and $st = id_I$ in $K^+(\mathcal{A})$.

Note that the proof doesn't use elb. It uses the universal property of $\oplus$. □

<u>Cor. 238:</u> $\mathrm{Hom}_{D(\mathcal{A})}(X, I) = \mathrm{Hom}_{K(\mathcal{A})}(X, I)$

for I as in Lemma 237.

Proof : • $X \xrightarrow{g} Z \underset{s}{\overset{t}{\rightleftarrows}} I$

$t^{-1}g = id_I\, t^{-1}g = (st)\,t^{-1}g = sg.$

• Suppose $f, g \in \mathrm{Hom}_{K(\mathcal{A})}(X, I)$ become equal in $D(\mathcal{A})$.
$\Rightarrow \exists\, t : I \underset{s}{\rightleftarrows} X' : tf = tg$

$\Rightarrow \quad f = (st)f = s(tf) = s(tg) = (st)g = g$

□

Sketch of the proof of 233(a):

Let X, Y be bounded below cochain complexes and $X \to I$ a Cartan-Eilenberg resolution of X.
$\Rightarrow \quad X \to \mathrm{Tot}^{\oplus}(I)$ is a quasi-isomorphism in $Ch^+(\mathcal{A})$.

Cor 238 $\Rightarrow$ $\mathrm{Hom}_{D(\mathcal{A})}(Y, X)$

$\simeq \mathrm{Hom}_{D(\mathcal{A})}(Y, \mathrm{Tot}^{\oplus}(I))$

$\simeq \mathrm{Hom}_{K(\mathcal{A})}(Y, \mathrm{Tot}^{\oplus}(I))$

is a set.
Thus $D^+(\mathcal{A})$ exists.

$K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})$ is fully faithfull and essentially surj. □

VI 5. Derived functors

Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be additive functor between abelian categories.
Then F induces a morphism of Δ-categories
$$F: K(\mathcal{A}) \longrightarrow K(\mathcal{B})$$
but not $D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ as F doesn't need to preserve quasi-isomorphisms.

We want to find RF such that

$$\begin{tikzcd}
K(\mathcal{A}) \arrow[r, "q_{\mathcal{A}}"] \arrow[d, "F"'] & D(\mathcal{A}) \arrow[d, "RF"] \\
K(\mathcal{B}) \arrow[r, "q_{\mathcal{B}}"] & D(\mathcal{B})
\end{tikzcd}$$

commutes.

<u>Problem</u>: In general it would not commute.
It commutes if F is exact.

<u>Notation 239</u>: $F \Longrightarrow G$ for a natural transformation between two functors $F, G: \mathcal{C} \longrightarrow \mathcal{D}$.

<u>Def 240</u>: (a) A (total) <u>right derived</u> functor of $F: K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ is a pair $RF: D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ and $q_{\mathcal{B}} F \overset{\xi}{\Longrightarrow} RF q_{\mathcal{A}}$

such that for all ($D(\mathcal{A}) \xrightarrow{G} D(\mathcal{B})$ morph. of Δ-cat., $q_{\mathcal{B}} F \overset{\zeta}{\Longrightarrow} G q_{\mathcal{A}}$)

$\exists!\ RF \overset{\eta}{\Longrightarrow} G : \eta \circ \xi = \zeta$.

$$\begin{tikzcd}
K(\mathcal{A}) \arrow[r, "q_{\mathcal{A}}"] \arrow[d, "F"'] & D(\mathcal{A}) \arrow[d, "RF"'] \arrow[dr, bend left, "G"] & \\
 & D(\mathcal{B}) \arrow[r, Rightarrow, "\exists!"] & D(\mathcal{B}) \\
K(\mathcal{B}) \arrow[r, "q_{\mathcal{B}}"] & D(\mathcal{B}) \arrow[u, Rightarrow, "\xi"'] \arrow[ur, Rightarrow, "\zeta"'] &
\end{tikzcd}$$

(b) Total left <u>derived functor</u>

$$\begin{tikzcd}
K(\mathcal{A}) \arrow[r, "q_{\mathcal{A}}"] \arrow[d, "F"'] & D(\mathcal{A}) \arrow[d, "LF"] \arrow[dr, "G"] & \\
K(\mathcal{B}) \arrow[dr, "q_{\mathcal{B}}"'] & D(\mathcal{B}) \arrow[d, Rightarrow, "\xi"] & D(\mathcal{B}) \arrow[l, Rightarrow, "\exists!\eta"'] \arrow[dl, Rightarrow, "\xi \circ \zeta"] \\
 & D(\mathcal{B}) &
\end{tikzcd}$$

Example 2.41: $F : \mathcal{A} \longrightarrow \mathcal{B}$ additive.

In Def. 2.40 we can replace K in K^b, K^+, K^- and D with D^b, D^+, D^-, and write $R^bF, R^+F, R^-F, L^bF, L^+F, L^-F$.

(a) Assume $\mathcal{A}$ has enough injectives. Then, for $X \in D^+(\mathcal{A})$,

$$R^+F(X) = F(\mathrm{Tot}^{\oplus}(I)),$$

in particular, for all $i \in \mathbb{Z}$,

$$\underbrace{\mathbb{R}^i F}_{\text{right hyper-derived functor}}(X) \simeq H^i(R^+F(X))$$

(b) Assume $\mathcal{A}$ has enough projectives. Then, for $X \in D^-(\mathcal{A})$,

$$L^-F(X) = F(\mathrm{Tot}^{\oplus}(P)),$$

in particular, for all $i \in \mathbb{Z}$,

$$\mathbb{L}_i F(X) \simeq H^{-i}(L^-F(X)).$$