

I Basic Topology Theory.

Def 1.1 A Topological Space is a set X with a collection of open sets $\mathcal{O}$ satisfying the following axioms:

(i) X and $\emptyset$ are in $\mathcal{O}$.

(ii) If $\{U_i\}_{i \in I}$ is a family of open sets, then $\bigcup_{i \in I} U_i$ is open.

(iii) If $\{U_i\}_{i \in I}$ is a family of open sets, then $\bigcap_{i \in I} U_i$ is open.

Def 1.2 Let X be a topological space. A subset A of X is called closed if its complement $X \setminus A$ is open.

Def 1.3 Let X be a topological space. A subset A of X is called compact if every open cover of A has a finite subcover.

Def 1.4 Let X be a topological space. A subset A of X is called connected if it cannot be written as the union of two disjoint non-empty open sets.

Def 1.5 Let X be a topological space. A subset A of X is called locally compact if every point $x \in A$ has a compact neighborhood.

Def 1.6 Let X be a topological space. A subset A of X is called separable if it contains a countable dense subset.

Example 1.1 Let X be a topological space. Then X is compact if and only if X is closed and bounded in $\mathbb{R}^n$.

Example 1.2 Let X be a topological space. Then X is connected if and only if X is not the union of two disjoint non-empty open sets.

Text book:
 p-adic Representation,
 Correspondence and the
 Langlands - Mordell Theory.

Science from.

Similar for F non-arch local field:

$$\mathcal{O}_F := \{x \in F \mid |x| \leq 1\}$$

$$\mathcal{P}_F := \{x \in F \mid |x| < 1\} = \mathcal{O}_F \setminus \mathcal{O}_F^\times$$

$$(\mathcal{O}_F : \mathcal{O}_F^\times = \mathbb{Z}_p, \mathcal{P}_F = p\mathbb{Z}_p)$$

Find add basis

$$e_i := \mathcal{O}_F / \mathcal{P}_F \text{ residue field of } F$$

$$(\mathcal{O}_F : \mathbb{Z}_p / \mathbb{Z}_p = \mathbb{Z}_p)$$

$$(iv) F^\times = \{1 + \mathcal{P}_F^n \mid n \geq 1\}$$

(vii) Let E be a f.d. F -v.s.

$$E \simeq F \oplus \dots \oplus F$$

basis

Product top. gives a loc. prof. gp $GL_n(F)$ (indep. of basis)

$$SO_n(F) = F^{n \times n} \text{ and}$$

$$GL_n(F) = \{A \in M_n(F) \mid \det A \neq 0\}$$

are loc. prof. gps.

add system in $GL_n(F)$:

$$\{1 + \mathcal{P}_F^n M_n(\mathcal{O}_F) \mid n \geq 1\}$$

$$=: K_n$$

Prop 4: G loc. prof., $H \leq G$, $N \trianglelefteq G$

not closed subgp. Then H and G/H are loc. prof. gps.

Proof: Take a sys of add $\{K_i \mid i \in I\}$ of open subgps.

Then $\{H \cap K_i \mid i \in I\}$ and $\{\pi(K_i) \mid i \in I\}$ form a add basis of H and G/H

resp.

Def 5: If G prof. and $K \leq G$ open

Then K is compact and

G/K is finite

Convention 6: We fix a field R and

we denote $\chi := \text{char}(R)$.

I.1.2. Basic representation theory.

Def 6: An R-representation of a group G (π, V) is an R - vsn V together with a gr. isomorphism
 $\pi: G \rightarrow \text{Aut}_R(V)$

A morphism $\varphi: (\pi, V) \rightarrow (\pi', V')$ is an R -linear map or
 $\varphi(\pi(g)v) = \pi'(g)\varphi(v) \quad \forall g \in G, v \in V$

Def 7: (π, V) is an $R[G]$ -module.

Def 8: A subrepresentation of (π, V) is a G -stable sub-vsn of V .

(Notation: $(\pi|_W, W)$).
 (π, V) is called indecomposable if V is not the direct sum of proper sub-repr.

(π, V) is called irreducible if $V \neq 0$ and $\nexists$ proper non-zero sub-repr. of (π, V) .

Example 9: (a) $G = \mathbb{Q}_p^\times, R = \mathbb{C}, V = \mathbb{C}$
 $\pi(g)z := |g|z.$

(b) $G = \text{GL}_2(\mathbb{Z}_p), R = \mathbb{F}_p, V = \mathbb{F}_p^2$
 $\pi: G \rightarrow \text{GL}_2(\mathbb{F}_p)$
 $(\pi: \mathbb{Q}_p^\times \xrightarrow{\text{Aut}_{\mathbb{C}}(\mathbb{C})}$

Def 10: (π, V) is called of finite type if V is finitely generated as an $R[G]$ -module.

Prop 11: Let (π, V) be non-zero. Then

- (i) (π, V) has an irreducible sub-repr.
- (ii) (π, V) has an irred. quotient if it is of finite type.
- (iii) (π, V) has an irred. subrepr. if it is of finite length.

Proof: (iii) ✓

(ii) Take a minimal generating set $\{v_1, \dots, v_n\}$ of V .
 $M := \langle W \text{ subrepr of } V \mid v_1, \dots, v_n \in W \rangle$
 and $v_n \notin W$

open $\{v_1, \dots, v_t\} \in \mathcal{M}$, and $\mathcal{M}$ is

inductive. $\mathcal{Z}(\mathcal{M}) \Rightarrow \exists W \subseteq \mathcal{M}$ maximal in $\mathcal{M}$.

Claim: V_W is indecomposable.

pf: Assume $\exists W' \subsetneq W$ subrep.

$\Rightarrow W' \notin \mathcal{M}$ and thus $v_i \in W' \subseteq W$

$\Rightarrow W' = W$ $\square$

(ii) Take $v \in V \setminus \{0\}$. $W = \text{open}_{\mathcal{R}(\mathcal{G})} \{v\}$ is 1-g. Apply (ii) $\square$

Prop 12: Let (π, V) be a rep of G . T.a.e.

1° (π, V) is a direct sum of indecomposables.

2° (π, V) is a sum of indecomposables.

3° Any indecomposable has a G -complement.

Proof: $2^\circ \Rightarrow 1^\circ$ $V = \sum_{i \in I} U_i$, U_i indecomposable.

$\mathcal{M} := \{J \subseteq I \mid \sum_{j \in J} U_j \text{ is direct}\}$

$\mathcal{Z}(\mathcal{M}) \Rightarrow \exists \text{ max. } J \in \mathcal{M}$.

$\Rightarrow \sum_{j \in J} U_j = V$.

$1^\circ \Rightarrow 3^\circ$ $W \leq_G V = \bigoplus_{i \in I} U_i$

$\mathcal{M} := \{J \subseteq I \mid W \cap (\sum_{i \in J} U_i) = \{0\}\}$

$\mathcal{Z}(\mathcal{M}) \Rightarrow \exists J \in \mathcal{M}$ max.

$\Rightarrow W \oplus (\bigoplus_{j \in J} U_j) = V$

$3^\circ \Rightarrow 2^\circ$ Every indecomposable U also satisfies 3° .

Thus every subquotient of W is isomorphic to a subrep. of U .

Prop 11 (ii) $\Rightarrow W$ contains an indecomposable if $W \neq 0$.

$V_0 := \sum U$ has a direct summand U indecomposable.

G -complement W and W doesn't contain an indecomposable.

$\Rightarrow W = 0$. $\square$

We call such a rep. (π, V) completely reducible.

I 1.3. Smooth map.

 G loc. prof.Def 13: (π, V) is called smooth if $\forall v \in V$: the stabilizer $G_v = \{g \in G \mid gv = v\}$ is open.

(In Example 9 they are smooth.)

Prop: If G is profinite and V of finite type, any generated by $\{v_1, \dots, v_r\}$ put $H = G_v \cap \dots \cap G_{v_r}$. Then G/H is finite and V is a f.d. R - G/H -mod.Take $N \leq H$ open, normal in G $\Rightarrow \bar{G} = G/N$ is a finite group and $(\bar{\pi}, V)$ is a repr. of $\bar{G}$.Thm 15: (Mackey) let G be finite and $\pi \in \text{Irr}(G)$. Let (π, V) be a repr. of π . Then (π, V) is completely reducible.Proof: $T: V \xrightarrow{\pi} W$ ($W \leq_G V$)

$$T(x) := \frac{1}{|G|} \sum_{g \in G} \pi(g) P(\pi(g^{-1})x)$$

(P some linear projection $W \oplus W^\perp \rightarrow W$)
 W an R -linear complement) T is a G -map and $T \circ T = T$ and $\text{im}(T) = W$ $\Rightarrow \ker(T) \oplus W = V$ and $\ker(T) \leq_G V$ $\square$ Example 16: (a) $G = \mathbb{Q}_p^\times$, $R = \mathbb{Q}_p$, $V = \mathbb{Q}_p$. $\pi(x) \cdot v := x \cdot v$. (π, V) is not smooth. $v \neq 0$ $G_v = \{1\}$ is not open in $\mathbb{Q}_p^\times$. $G_0 = G$ is open.

The only smooth vector is 0.

(b) $G = \mathbb{Q}_p^\times$, $R = \mathbb{F}$, $V := \text{Ad}(\mathbb{Q}_p, \mathbb{Q})$. $(\pi(x)f)(y) = f(xy)$, $x \in G, y \in \mathbb{Q}$. (π, V) is not smooth. $G_1 = \mathbb{Z}_p$ $G_1 \trianglelefteq G$ (π, V) has smooth vectors. $G_1 \trianglelefteq \mathbb{Z}_p^\times$ is open in G .

We write $V^\infty = \bigcup_{K \text{ open}} V^K$ in G -stable
 (π^∞, V^∞) is a smooth representation.

Example 17:

Same R - U_D . $V = \text{Alg}(G, S)$
 $(\pi(g)f)(u) := f(ug)$.

(π, V) is a representation of G

$$(\pi(g_1 g_2)f)(u) = f(ug_1 g_2)$$

$$(\pi(g_1)(\pi(g_2)f))(u) = (\pi(g_2)f)(ug_1) \\ = f(ug_1 g_2)$$

$\mathcal{C}(G, S) := \{f \in \text{Alg}(G, S) \mid f \text{ is locally constant}\}$

f loc. constant if $\forall g \in G \exists H$ open
 $(f(yh) = f(y) \quad \forall h \in H)$

$$\mathcal{C}^\infty(G, S) := V^\infty \\ = \{f \in \text{Alg}(G, S) \mid \exists K \text{ open:} \\ \forall g \in K \exists \kappa: f(g\kappa) = f(g)\}$$

Space of smooth functions of G on S

Def 18: (support) $f \in \text{Alg}(G, S)$
 $\text{supp}(f) = \{g \in G \mid f(g) \neq 0\}$

(closure)

For $f \in \mathcal{C}(G, S)$: $\text{supp}(f) = \{g \in G \mid f(g) \neq 0\}$
 $\mathcal{C}_c^\infty(G, S) := \{f \in \mathcal{C}^\infty(G, S) \mid \text{supp}(f) \text{ compact}\}$

$$= \{f \in \mathcal{C}(G, S) \mid \text{supp}(f) \text{ compact}\}$$

Def 19: $\mathcal{C}$ contragredient representation

(π, V) smooth repr., $V^* = \text{Hom}_R(V, R)$

$$(\pi^*(g)f)(v) := f(g^{-1}v)$$

The smooth part

$(\hat{\pi}, \hat{V})$ is called contragredient representation of (π, V) .

I 1.4 Induced representations

$H \leq G$ closed.

We look for the left and right adjoint of $\text{Res}_H^G: R(G) \rightarrow R(H)$

Prop. 20: Let G be finite

$$R[G] \otimes_{R[H]} -: R(H) \rightarrow R(G)$$

is a right and left adjoint of Res_H^G

Proof: $\text{Hom}_{R[H]}(\text{Res}_H^G(\pi), s) \simeq \text{Hom}_{R[G]}(\pi, R[G] \otimes_{R[H]} s)$

for $(s, w) \in R(H)$ and $(\pi, v) \in R(G)$.

$$\alpha \mapsto \Lambda_\alpha$$

$$\Lambda_\alpha(\pi) = \sum_{g \in G/H} g \otimes \alpha(g^{-1}\pi)$$

$$\pi \circ \Lambda =: \delta \Lambda \longleftarrow 1 \Lambda$$

$$(R[G] \otimes_{R[H]} w) = \left(\bigoplus_{g \in G/H} (g \otimes w) \right) \oplus (1 \otimes w)$$

$\downarrow \text{perm}$

$$\text{Hom } R[G] (R[G] \otimes_{R[H]} w, \pi) \simeq \text{Hom}_{R[G]}(s, \text{Res}_H^G(\pi))$$

$$\Gamma^1 \longleftarrow \Gamma^1 \xrightarrow{\quad} \Gamma^1$$

$$\left(\Gamma^1(g \otimes w) := \pi(g) \pi(w) \right) \longleftarrow 1 \quad R$$

□

Generalization for loc. profinite groups.

Def 21: $(s, w) \in R_R(H) :=$ category of H -modules
R-rep. of H .

$(\text{Ind}_H^G(s), \text{Ind}_H^G(w))$ repr. induced by s .

$$\text{Ind}_H^G(w) := \sum_{g \in G/H} g \cdot w \mid \psi : \psi(g \cdot w) = g \cdot \psi(w)$$

and

$$\exists \psi \leq G \text{ open } \forall g \in H \exists k \in G \text{ s.t. } \psi(g \cdot w) = \psi(w)$$

$$(\text{Ind}_H^G(s)(g) \mid \psi(g) := \psi(g \cdot s))$$

G acts by right translation.

Def 22: $(C\text{-}\mathcal{A}nd_H^G S, C\text{-}\mathcal{A}nd_H^G W)$

$$C\text{-}\mathcal{A}nd_H^G W := \{f \in \mathcal{A}nd_H^G W \mid \exists \Omega \in G$$

$\Omega \text{ compact: } \text{supp}(f) \subseteq H\Omega\}$

$C\text{-}\mathcal{A}nd_H^G(S)$ action by right translation.

Hypercompactly induced by S

Thm 23: (a) W is a quotient of $\mathcal{A}nd_H^G(S)$

$(\text{Res}_H^G, \mathcal{A}nd_H^G)$ is a adjoint pair

H_{open}

(b) W is a subquotient of $C\text{-}\mathcal{A}nd_H^G(S)$
 $(C\mathcal{A}nd_H^G, \text{Res}_H^G)$ is an adjoint pair.

Proof: (a) $\mathcal{A}nd_H^G W \rightarrow W$
 $f \mapsto f(1)$

$$(a) \quad W \rightarrow C\text{-}\mathcal{A}nd_H^G W$$

$$w \mapsto f_w \quad f_w(g) = \begin{cases} 0 & \text{if } g \notin H \\ g w & \text{if } g \in H \end{cases}$$

Other actions: extensions $\square$

Thm 24: $(C\text{-}\mathcal{A}nd_H^G S)^K \simeq \bigoplus_{g \in \Lambda} W^{H \cap gKg^{-1}}$

$$(\mathcal{A}nd_H^G S)^K \simeq \prod_{g \in \Lambda} W^{H \cap gKg^{-1}}$$

for K compact. (A sys of representatives of Λ/K)

Proof: $f \in (\mathcal{A}nd_H^G W)^K \mapsto (f(g))_{g \in \Lambda}$

(Take $g \in \Lambda$, $u = gkg^{-1} \in H \cap gKg^{-1}$
 Then $uf(g) = f(ug) = f(gk) = f(g)$)

$\square$

Exercise 25: Thm 24 shows that

$\mathcal{A}nd_H^G$ and $C\text{-}\mathcal{A}nd_H^G$ are left-adjoint.

And they are exact if S does not divide the infinite part of the pro-order of G .

I 2. Admissible representations

I 2.1. Admissible repr. & first definition

From now on all repr. are smooth.

Def 6: (π, V) is called admissible

if $\forall K \leq G$ open: V^K is finite dim.

Convention (*): $\exists K \leq G$ copen:

$\mathcal{L} = \text{char}(\mathcal{E})$ does not divide the index of any ^{open} subgroup of K in K .

(i.e. $\forall K' \leq K$ open: $\ell(K/K')$)

$K^*(G) := \text{set of such } K \leq G$.

Prop 27: Suppose $(R, \mathcal{G})$ satisfies (*). Take $K \in K^*(G)$.

(i) $(\pi|_K, V|_K = \text{Res}_K(\pi))$ is completely reducible and

$$V = V^K \oplus V(K)$$

(ii) $V(K) := \text{span}_R \{ \pi(k)v - v \mid k \in K, v \in V \}$

(iii) $V^K \cong V(K)^*$ as R - $\mathcal{G}$ - $\mathcal{O}$.

(iv) $(\pi|_K : R(G) \rightarrow R\text{-mod.})$ is exact.

(v) $(\pi|_K : R_p(G) \rightarrow R_p(G))$ is exact.

Proof: (i) $\bullet$ $V|_K$ is the sum of repr. of K of finite type $(V = \sum_{v \in V} \text{span}_R \langle v \rangle)$

In $R_p(K)$ repr. of finite type are completely reducible, e.g. Maschke.

Take any K -complement W of V^K . Then $V(K) \subseteq \text{ker}(\text{pr}_{V^K \oplus W, V^K})$

$$\Rightarrow V(K) \subseteq W.$$

Take any $U \subseteq W$ irreducible.

Then $U^K = 0$ and thus $U(K) \neq 0$

and $U = U(K)$ as U irred.

So $U = U(K) \subseteq V(K)$.

Thus $W \subseteq V(K)$.

(ii) $V = V^K \oplus V(K)$

$$V^K \cong (V^*)^K \xrightarrow{f} (V^K)^*$$

$$f \longmapsto f|_{V^K}$$

is isomorphism as $f|_{V(K)} = 0$.

(iii) surjective

$$(iv) \quad 0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0 \text{ exact}$$

$$\xrightarrow{(\pi|_K)} 0 \rightarrow V_1^K \rightarrow V_2^K \rightarrow V_3^K \rightarrow 0 \quad \parallel \quad \forall K \in K^*(G)$$

$$\xrightarrow{=} 0 \rightarrow (V_1^K)^* \rightarrow (V_2^K)^* \rightarrow (V_3^K)^* \rightarrow 0 \text{ exact}$$

R irreducible in R -mod. as R is a field $V_K \in K^*(G)$

$$(ii) \quad 0 \rightarrow \tilde{V}_3^K \rightarrow \tilde{V}_2^K \rightarrow \tilde{V}_1^K \rightarrow 0 \text{ exact}$$

$$\forall K \in K^*(h)$$

$$\Rightarrow 0 \rightarrow \tilde{V}_3 \rightarrow \tilde{V}_2 \rightarrow \tilde{V}_1 \rightarrow 0 \text{ exact} \quad \square$$

Corollary 28: Suppose (X) for (K, G) .

Let $(\pi, V) \in R_K(G)$ (i) T. a. e.:

1° (π, V) admissible

2° $(\tilde{\pi}, \tilde{V}) \xrightarrow{-1} \tilde{V}$

3° $V \rightarrow \tilde{V}, v \mapsto \delta(v) \quad (\delta(v)(t) = t(v))$

is bijection.

(ii) Let (π, V) be admissible. Then

(π, V) is ind. iff $(\tilde{\pi}, \tilde{V})$ is indecom.

Proof: (i) $1^\circ \Leftrightarrow 2^\circ$ follows from 27(ii)

$1^\circ \Rightarrow 3^\circ$ For $K \in K^*(h)$ δ restricts

$$\text{to } \delta^K: V^K \hookrightarrow (\tilde{V}^K)^K \simeq (\tilde{V}^K)^K \simeq (V^K)^*$$

which is surjective, so $\dim V^K < \infty$

K arbitrary $\Rightarrow \delta$ is surjective.

$3^\circ \Rightarrow 1^\circ$ obvious

(ii) follows from the statement of (i) and 3° . $\square$

Remark 29: One can prove for $R = \mathbb{C}$ that all ind. repr. are admissible.

Example 30: Take a non-trivial

repr. $(\pi, V) \in R_{\mathbb{C}}(G)$.

Then $U := \bigoplus_{i=1}^{\infty} (\pi, V)$ is not admissible, because

$$U^K = \bigoplus_{i=1}^{\infty} V^K \text{ is not finite dimensional for small enough } K$$

I 2.2. Haar measure.

Def 31: An $\mathbb{R}$ -linear map $\mu: \mathcal{C}_c(G, \mathbb{R}) \rightarrow \mathbb{R}$ is called a measure on G .
 μ is called a Haar measure if it is invariant under G -left translation.

Def 32: μ induces an additive map on the set of open subsets of G :

$$X \subseteq G \text{ open} \\ \mu(X) := \mu(1_X)$$

(e) For $f \in \mathcal{C}_c^\infty(G, \mathbb{R})$, $g \in G$.

$$(\ell(g)f)(x) := f(g^{-1}x).$$

μ Haar-measure means

$$\mu(\ell(g)f) = \mu(f) \quad \forall f \in \mathcal{C}_c^\infty(G, \mathbb{R}) \\ \forall g \in G.$$

(c) $\mu(f)$ is just a finite sum:

$$f \in \mathcal{C}_c^\infty(G, \mathbb{R}) \quad f = \sum_{i=1}^k a_i \cdot 1_{g_i K}$$

$$\text{Then } \mu(f) = \sum_{i=1}^k a_i \mu(K).$$

(d) And for $K' \geq K$ a.t. $\ell[K':K]$ we get $\mu(K') = [K':K] \mu(K)$.

(e) More general K_1, K_2 open subp
 $[K_1:K_2] := [K_1:K_1 K_2] / [K_2:K_1 K_2]$

$$\exists K_1, K_2 \in \mathcal{K}^*(G) \text{ then} \\ \mu(K_1) = [K_1:K_2] \mu(K_2)$$

$$\text{Notation 33: } \mu(f) := \int_G f(g) d\mu(g).$$

Note: Supposing $(\mathfrak{x})(G, \mathbb{R})$ a Haar-measure always exist and is unique up to scalar by Rk 32 (c).

Given μ we define for $g \in G$ and $f \in \mathcal{C}_c^\infty(G, \mathbb{R})$: $\mu'(f) := \mu(\tau(g)f)$.

Then μ' is a Haar measure, i.e.
 $\exists \delta_G(g) \in \mathbb{R}^+$ such that $\mu' = \delta_G(g) \mu$

modulus character
 of G over $\mathbb{R}$.

For $\kappa \in K^*(G)$ we have

$$\begin{aligned} \delta_G(g) \mu(\mathbb{1}_\kappa) &= \mu(\text{reg}(\mathbb{1}_\kappa)) \\ &= \mu\left(\frac{\mathbb{1}_\kappa}{\text{reg}(g)}\right) \\ &= [\text{reg}(g^{-1} : \kappa)] \mu(\kappa) \end{aligned}$$

$$\text{So } \delta_G(g) = [\text{reg}(g^{-1} : \kappa)].$$

Example 34: $G = \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}^\times, x \in \mathbb{F} \right\}$
metaplectic group

Let μ be the Haar measure
 o.t. $\mu\left(\underbrace{\begin{pmatrix} \sigma & \sigma \\ 0 & 1 \end{pmatrix}}_\kappa\right) = 1.$

$$\delta_G\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \left[\begin{pmatrix} \sigma & a\sigma \\ 0 & 1 \end{pmatrix} : \kappa \right]$$

$$= [a\sigma : \sigma] = \underset{\uparrow}{q}^{-\nu_F(a)}$$

$$q := |R_F|$$

$$= |a|_F.$$

Def 35: (unimodular) G is called
 unimodular if $\delta_{G, \mathbb{Q}}$ is the trivial
 character, i.e. $\delta(g) = 1 \forall g \in G.$

Example 36: $G_n/\mathbb{F}$ reductive then
 $G_n = \text{GL}_n(\mathbb{F})$ is unimodular.
 The metaplectic group is not
 unimodular.

Remark 37: Suppose $(\mathcal{G}, (R, G))$.

We have a map

$$\mathcal{G} := \left\{ \frac{a}{g} \mid a, g \in \mathbb{Z}, g \neq 0, \exists! a.g \right\}$$

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{a} & \mathbb{R}^\times \\ \downarrow & & \downarrow \\ \mathbb{Q} & \xrightarrow{\quad} & (\mathbb{A}_F) (\mathbb{A}_F)^{-1}. \end{array}$$

and $\delta_{G,R} = \varphi \circ \delta_{G,0}$.

So we only need $\delta_{G,0}$ and write δ_G for $\delta_{G,0}$.

I 2.3. Hecke algebra of a locally-profinite group.

We suppose $(X)(R, G)$. We fix a Haar measure μ .

Def 3.8: (Hecke algebra of G over R .)

$$H_R(G) := \mathcal{L}^\infty(G, R) = \left\{ f: G \rightarrow R \mid \int_Y f(x) dx \text{ conv. } \forall Y \subseteq G \text{ compact} \right\}$$

The product on $H_R(G)$ is called convolution and defined as follows:

$$(f_1 * f_2)(g) := \int_G f_1(x) f_2(x^{-1}g) d\mu(x).$$

Prop 3.9: $(H_R(G), *)$ is an R -algebra.

Proof: We only show the associativity of $*$.

$$\begin{aligned} ((f_1 * f_2) * f_3)(g) &= \int_G (f_1 * f_2)(x) f_3(x^{-1}g) d\mu(x) \\ &= \int_G \left(\int_G f_1(y) f_2(y^{-1}x) d\mu(y) \right) f_3(x^{-1}g) d\mu(x). \end{aligned}$$

$$\begin{aligned}
 &= \int_G \int_G f_1(y) f_2(y^{-1}x) f_3(x^{-1}g) d\mu(x) d\mu(y) \\
 &\stackrel{\text{Fubini}}{=} \int_G f_1(y) \int_G f_2(z) f_3(z^{-1}y^{-1}g) d\mu(z) d\mu(y) \\
 &\quad \stackrel{z=y^{-1}x}{=} \int_G f_1(y) (f_2 * f_3)(y^{-1}g) d\mu(y) \\
 &= \int_G (f_1 * (f_2 * f_3))(g) d\mu(g) \quad \square
 \end{aligned}$$

Prop 4.0: We will see that $(\pi, V) \in \mathcal{R}_R(G)$ is an $\mathcal{H}_R(G)$ -module and a $R[G]$ -module.

$\mathcal{H}_R(G)$ can be better computed than $R[G]$ (by a filtration of operator Hecke algebras)

For G finite we have $\mathcal{H}_R(G) = R[G]$

Lemma 4.1: If G is infinite then $\mathcal{H}_R(G)$ has a lot of idempotents, namely $e_K := \frac{1}{\mu(K)} \mathbb{1}_K$, $K \leq G$ open.

$$\begin{aligned}
 \text{Proof: } (e_K * e_K)(g) &= \int_K \frac{1}{\mu(K)^2} \mathbb{1}_K(x^{-1}g) d\mu(x) \\
 &= \int_K \frac{1}{\mu(K)^2} \mathbb{1}_K(x) d\mu(x) \\
 &= \frac{1}{\mu(K)^2} \mu(K \cap gK) = e_K(g) \quad \square
 \end{aligned}$$

A smooth representation $(\pi, V) \in \mathcal{R}_R(G)$ can be interpreted as a module over $\mathcal{H}_R(G)$: $v \in V$ takes $K \in \mathcal{K}^*(G)$ out. $v \in V^R$

$$\pi(l) \cdot v := \sum_{g \in \text{support}(l)} \mu(K) l(g) \pi(g) v$$

for $f \in \mathcal{H}_R(G)$ right K invariant (does not depend on K)
We write

$$\pi(f) := \int_G f(g) \pi(g) d\mu(g).$$

Example 4.2: $K \in \mathcal{K}^*(G)$, $(\pi, V) \in \mathcal{R}_R(G)$

$$\begin{aligned}
 \pi(e_K) : V &\rightarrow V \\
 v \in V^K, \pi(e_K) v &= \int_G \frac{1}{\mu(K)} \mathbb{1}_K(g) \pi(g) v d\mu(g)
 \end{aligned}$$

$$= \frac{1}{\mu(K)} \int_K v d\mu(g) = v.$$

Take $v \in V(K)$, i.e. $\exists v_1, \dots, v_r \in V \text{ s.t. } v = v_1 - k_1 v_1 + \dots + (v_r - k_r v_r)$.

On the first term we obtain

$$\pi(e_k)(v_1 - k_1 v_1) = \int_{\mu(K)} \frac{1}{\mu(K)} \frac{1}{k} (g) \pi(g)(v_1 - \pi(k)v_1) d\mu(g)$$

$$= \frac{1}{\mu(K)} \sum_{g \in K} \pi(g) v_1 - \frac{1}{\mu(K)} \sum_{g \in K} \pi(g k) v_1$$

$$= 0.$$

So $\pi(e_k)v = 0$. So $\pi(e_k)$ is the projection onto V^K .

Prop 43:

(a) Let $(\pi, V) \in R_E(G)$. Then

V is an $H_E(G)$ module

n.b. $\forall v \in V \exists e \in H_E(G)$ idempotent

n.b. $eV = V$.

(non-degenerate, $H_E(G)$ -module, or smooth, or unital)

(b) $R_E(G) \longrightarrow H_E(G)\text{-Mod}$
(cat. of unital $H_E(G)$ modules)

$(\pi, V) \longmapsto V$
is an equivalence of categories

Proof: (a) Exercise: Shows that

$$\pi\left(\frac{1}{k_1} s \frac{1}{k_2}\right) v = \pi(k_1)(\pi(k_2)v).$$

For $v \in V^K$ take $e_k: \pi(e_k)v = v$.

$$(k \in K^*(G))$$

(b) Let W be a unital $H_E(G)$ -module.

We define a representation on W .

$w \in W$. Take $k \in K^*(G)$ s.t. $e_k w = w$.

We define:

$$\pi(g)w := \left(\frac{1}{\mu(K)} \frac{1}{gk} \right) \cdot w.$$

Then $(\pi, W) \in R_E(G)$ $\square$

Def 44: $K \in K^*(G)$.

$$\mathcal{H}_P(G, K) := e_K * \mathcal{H}_P(G) * e_K$$

$= \{f \in \mathcal{L}^\infty(G, R) \mid f \text{ is left and right } K\text{-invariant}\}$

Then V^K is an $\mathcal{H}_P(G, K)$ module, because $e_K * v = v \quad \forall v \in V^K$.

Prop 45: Let $(\pi, V) \in R_P(G)$ be irreducible and $K \in K^*(G)$. Suppose $V^K \neq 0$.

Then V^K is a simple $\mathcal{H}_P(G, K)$ module.

Proof: Let W be a non-zero $\mathcal{H}_P(G, K)$ -submodule of V^K .

Then $\mathcal{H}_P(G)W = V$ as V is irreducible.

$$\begin{aligned} \Rightarrow V^K &= e_K * V = e_K * \mathcal{H}_P(G)W \\ &\stackrel{\uparrow}{=} e_K * \mathcal{H}_P(G) * e_K W = W \\ &e_K W = W \end{aligned}$$

□

Example 46: We compute $\mathcal{H}_P(\mathbb{Q}_p)$

$$\mathcal{H}_P(\mathbb{Q}_p, \mathbb{Q}_p) = \mathbb{C}[\mathbb{Q}_p/\mathbb{Z}_p]$$

$$(Note: \mathcal{H}_{x+\mathbb{Z}_p} * \mathcal{H}_{y+\mathbb{Z}_p} = \mathcal{H}_{x+y+\mathbb{Z}_p}.)$$

$$We have \mathcal{H}_P(\mathbb{Q}_p, \mathbb{Q}_p^*) \simeq \mathbb{C}[\mathbb{Q}_p/\mathbb{Z}_p]$$

$$V \uparrow$$

$$\mathcal{H}_P(\mathbb{Q}_p, \mathbb{Z}_p) \simeq \mathbb{C}[\mathbb{Q}_p/\mathbb{Z}_p]$$

$$\mathcal{Q}([a]_{\mathbb{Z}_p}) := \sum_{i=0}^{p-1} [a+i]_p \mathbb{Q}_p$$

This is a sum in the group algebra $\mathbb{C}[\mathbb{Q}_p/\mathbb{Z}_p]$

and NOT in $\mathbb{Q}_p/\mathbb{Z}_p$.

$\mathcal{Q}$ is then extended to $\mathbb{C}[\mathbb{Q}_p/\mathbb{Z}_p]$

via

$$\mathcal{Q}\left(\sum_{i=1}^t \lambda_i [a_i]_{\mathbb{Z}_p}\right) := \sum_{i=1}^t \lambda_i \mathcal{Q}([a_i]_{\mathbb{Z}_p})$$

$$(\lambda_i \in \mathbb{C})$$

$$\text{Now } H_G(\mathbb{Q}_p) = \bigcup_{k=1}^{\infty} H_G(\mathbb{Q}_p, p^k \mathbb{Z}_p) \\ = \varprojlim_{\mathbb{R}} \mathbb{Q}[\mathbb{Q}_p/p^k \mathbb{Z}_p]$$

Exercise 47: Try to compute $H_G(\mathbb{Q}_p^{\times})$.

Ex. 2.4. Commutators.

$(\pi, \nu) \in R_G(G)$. $H \leq G$ closed.

H -invariants: V^H largest subspace of V on which H acts trivially.

H -coinvariants: V_H largest quotient of V on which H acts trivially.

$$V_H := V / \text{span}_{\mathbb{R}} \{v - \sigma(v) \mid \sigma \in H, v \in V\} \\ = V/H.$$

Prop 48: Let $H \in K^*(G)$. Then $V^H \rightarrow V_H$ $v \mapsto [\sigma]v(H)$

is an $\mathbb{R}$ -linear isomorphism.

Example 49: (a) Let $(\pi, \nu) \in R_G(\text{Gal}(\mathbb{F}))$ be a finite dim. irreducible representation. Then one can show that (π, ν) is

a character, i.e. $\exists \chi: \mathbb{F}^\times \rightarrow \mathbb{C}^\times$ smooth,
(open kernel) n.b.

$$\pi = \chi \cdot \det \quad \text{and } V \simeq \mathbb{C}.$$

$$\text{Take } H := \begin{pmatrix} 1 & F \\ & 1 \end{pmatrix}.$$

$$\forall z \in \mathbb{C} \forall u \in H: \pi(u)z = \chi(\det u)z = z$$

$$\text{So } V(H) = 0 \text{ and } V_H = V/V(H) = V = \mathbb{C}.$$

$$(e) \quad G_1 = \text{Gal}_2(\mathbb{F}_2) \simeq G_3 \text{ (quot param-} \\ \text{Analog on 3 elts)}$$

Let $(s, \phi) \in R_{\mathbb{C}}(G)$ be the sign
character, e.g.

$$s\left(\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\right) = -1.$$

$$H := \left\{ \begin{pmatrix} 1 & \mathbb{F}_2 \\ & 1 \end{pmatrix} \right\}$$

G is finite, so H is open in G , so

$$V_H \simeq V^H = \underbrace{\{z \in \mathbb{C} \mid \text{sign}\left(\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\right)z = z\}}_{=0} = 0.$$

$$(c) \quad G = \text{Gal}_2(\mathbb{Q}_2)$$

$$K := \text{Gal}_2(\mathbb{Z}_2), \quad Z := \mathbb{Q}_2 \cdot I_2 \quad (I_2 = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix})$$

Take $(s, \phi) \in R_{\mathbb{C}}(\text{Gal}_2(\mathbb{F}_2))$ from (e).

$(s, \phi) \in R_{\mathbb{C}}(K)$ is defined via

$$\tilde{s}(g_0) v := s(\bar{g}) \cdot v \quad \left(K \xrightarrow{\bar{g} \mapsto g} \mathbb{F}_2 \right)$$

$\tilde{s}$ is called inflation of s to K

Define $(\Lambda, \phi) \in R_{\mathbb{C}}(\mathbb{Z}_K)$

$$\Lambda(zkv) := \tilde{s}(v)v$$

$$\pi := \text{cind}_{\mathbb{Z}_K}^G \Lambda.$$

We compute for $H := \begin{pmatrix} 1 & \mathbb{Q}_2 \\ & 1 \end{pmatrix}$ the space V_H

$$u := \begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix}$$

$$f = f_1$$

$$\in \text{cind}_{\mathbb{Z}_K}^G \Lambda \text{ with supp}(f) = \mathbb{Z}_K$$

$$f - u.f = 2f$$

$$(f_0, f)(g) = f(gu) = \begin{cases} 0 & , g \notin \mathbb{Z}_K \\ \tilde{s}(g)\tilde{s}(u), g \in \mathbb{Z}_K \end{cases}$$

$$= -f(g)$$

so $f \in V(H)$. Analogously $(\tilde{u}^a, t), t \in V(H)$.

for all $a \in \mathbb{Z}$. As $G = \mathbb{Z}K(\mathbb{Q}^2)$
 we obtain that $V \subseteq V_{H_1}$.
 Thus $V_{H_1} = \mathcal{O}$.

Exercise 50: Take $\pi := \text{cind} \uparrow \uparrow_{\mathbb{Z}K}^{GL_2(\mathbb{Q})}$

for $K := GL_2(\mathbb{Z})$, $\mathbb{Z} = \mathbb{Q}_p \cap \mathbb{Z}$
 Put $H := (\uparrow \mathbb{Q}_p)$

Compute V_H .

Prop 51: (i) The functor $(\uparrow)_{\mathbb{Z}K}^{R_K(G)} \rightarrow R_{\mathbb{Z}K}$
 is right exact.

(ii) Suppose that $G = \bigcup_{i=1}^n K_i$
 with open subgroups $K_i \subseteq K_{i+1}$
 $A \in \mathcal{N}$, $K_1 \in K^s(G)$.
 Then $(\uparrow)_G$ is exact.

Proof: (i) $G/V \in R_K(G)$
 $\text{Hom}_G(V, 1) \xrightarrow{\sim} \text{Hom}_R(V_G, R)$
 $\text{Hom}_G(V, 1) \xrightarrow{q} \bar{q} \quad (\bar{q}(x) = q(x))$

Take $0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$ exact.

$$\Rightarrow 0 \rightarrow \text{Hom}_G(V_3, 1) \xrightarrow{\uparrow} \text{Hom}_{G_1}(V_{G_1}, 1) \xrightarrow{\uparrow} \text{Hom}_G(V_{G_1}, 1)$$

$$0 \rightarrow (V_3)_G \xrightarrow{\uparrow} (V_2)_G \xrightarrow{\uparrow} (V_1)_G \text{ exact.}$$

$$\Rightarrow V_{H_1} \rightarrow V_{H_2} \rightarrow V_{H_3} \rightarrow 0 \text{ exact.}$$

(ii) Let $V \hookrightarrow V_2$ in $R_K(G)$.

We need to show that $V_1 \cap V_2(G) = V_1(G)$.

Let $v \in V_1 \cap V_2(G)$. Then

$$\exists A \in \mathcal{N}: v \in V_2(K_A).$$

$$\text{So } v \in V_1 \cap V_2(K_A) = V_1(K_A) \subseteq V_1(G)$$

$(\uparrow)_{K_A}$ is exact

□

Example 52: $G = GL_n(F)$.

$$H = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} = \bigcup_{i=1}^{\infty} \begin{pmatrix} 1 & p^{-i} & & \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

K_A

K_A is open in H . (not in G)

I.3. $\mathbb{Z}$ -compact representations

I.3.1. Characters

Def 53: A character of G is a group homomorphism $\chi: G \rightarrow \mathbb{R}^\times$.
 χ is called smooth if χ has an open kernel.

Prop 54: A smooth character of G can be reinterpreted as a 1-dim smooth representation of G .
 Indeed: $\chi: G \rightarrow \mathbb{R}^\times$ smooth.

Define $(\pi, \mathbb{R}) \in \text{Rep}_\mathbb{R}(G)$ via
 $\pi(g).r := \chi(g) \cdot r$.

Prop 55: Let G be ^{loc.} profinite and
 $\chi: G \rightarrow \mathbb{C}^\times$ be a character.
 T.a.e.: 1° χ is smooth
 2° χ is continuous.

Proof: 1° $\Rightarrow$ 2° because χ is continuous w.r.t. discrete top. on $\mathbb{C}^\times$.

$$2^0 \Rightarrow 1^0 \text{ Let } \mathbb{Q} = \frac{1}{2}. B_{\frac{1}{2}}(1) = \{z \in \mathbb{C} \mid |z - 1| \leq \frac{1}{2}\}$$

$B_{\frac{1}{2}}(1)$ doesn't contain any non-trivial compact subgroup.

$\chi(B_{\frac{1}{2}}(1))$ is open $\Rightarrow \exists k \leq G$ open $\chi(k) \subseteq B_{\frac{1}{2}}(1)$. Thus $\chi(k) = \{1\}$

So χ is smooth. $\square$

Def 56: A character $\chi: G \rightarrow \mathbb{C}^\times$ with $\text{im}(\chi) \leq S^1$ is called unitary.

Prop 57: If G is profinite then every smooth character of G is unitary.

Example 58: F non-arch local field.

$$(a) \quad G_1 = F^\times = \mathbb{Q}^\times, \quad \mathcal{O}_F^\times \subseteq \mathbb{Z} \times \mathcal{O}_F^\times \\ \cong \mathbb{Z} \times \langle 5 \rangle \times (1 + \mathfrak{p}_F)$$

$\mathbb{Q}^\times$ the primitive root of unity.

Thus $\chi: G \rightarrow \mathbb{C}^\times$ smooth character is given by a triple (γ, ζ, χ_1)

where $\gamma \in \mathbb{R}^\times, \zeta \in \mu_{q-1}(F), \chi_1: 1 + \mathfrak{p}_F \rightarrow \mathbb{C}^\times$ a smooth character.

$$\chi(\mathbb{Q}^\times, \mathbb{Q}^\times(1+x)) = \gamma^a \zeta^b \chi_1(1+x)$$

(b) $G = (F, +)$. $\rho = \text{rechar}(F)$.

(c) If $\rho = \mathbb{Z}$. Then the only smooth character of G is trivial character.

(d) If $\rho \neq \mathbb{Z}$. We construct a non-trivial character of $F/\mathfrak{p}_F$ in R is algebraically closed.

Pr: Take a non-trivial character of $k_F = \mathcal{O}_F/\mathfrak{p}_F$, say $\chi: k_F \rightarrow \mathbb{C}^\times$. It exists as $k_F \cong \mathbb{F}_q \oplus \dots \oplus \mathbb{F}_q$ as $\mathbb{F}_q \subset \mathcal{O}_F/\mathfrak{p}_F$, and $\mu_p(R) \neq \emptyset$

Suppose we have defined $\chi_i: \mathfrak{p}_i/\mathfrak{p}_F \rightarrow \mathbb{C}^\times$

extending χ_0 .

Take $y \in \mathfrak{p}_F^{-1} \setminus \mathfrak{p}_F^{-1}$. Then $y \in \mathfrak{p}_F^{-1}$. Take a pth root of y $\chi_1(y)$, and put $\chi_1(y) = \gamma$.

— 41 —

This extends χ_1 to $\mathbb{Z}^r + \frac{p^i}{p}$
 continue this way until we reach $\frac{p^r}{p}$.

So by induction we get $\chi: F_{\frac{p^r}{p}} \rightarrow \mathbb{R}^r$
 non-trivial
 and we inflate to $\chi: F \rightarrow \mathbb{R}^r$.

□

Prop 59: Let A be an abelian group
 and $\chi: B \rightarrow \mathbb{R}^r$ be a character
 on a subgroup B of A .

Then $\exists \chi_A: A \rightarrow \mathbb{R}^r$ character such that
 $\chi_A|_B = \chi_B$ if B is algebraically closed.

Proof: Similar to Example 58 (a). □

— 42 —

Theorem 60: (Pontryagin - van Kampen
 duality theorem)
 Let A be an LCA group (locally
 compact, abelian, T^2)

Let $A^{pd} = \{ \chi: A \rightarrow S^1 \mid \chi \text{ continuous} \}$
 character with. End. top.

on S^1], called

"Pontryagin dual of A "

Then the map

$$A \longrightarrow (A^{pd})^{pd}$$

$$a \longmapsto (\chi \mapsto \chi(a))$$

is a topological group isomorphism.

Re 61: In Thm 60 we consider on
 A^{pd} the compact open topology.
 It is generated by

$$B(K, U) := \{ \chi \in A^{pd} \mid \chi(K) \subseteq U \}$$

for $K \subseteq A$ compact and $U \subseteq S^1$ open.
 It is the topology of uniform con-
 vergence on compact sets.

Example 6.1 (a) $A = \mathbb{Z}$, $A^{\text{pd}} = S^1$

$$\begin{aligned} \text{(b)} \quad A &= \mathbb{Z}/m\mathbb{Z}, \quad A^{\text{pd}} = (\mathbb{Z}/m\mathbb{Z})^{\text{pd}} \\ &= \left\{ e^{\frac{2\pi i k}{m}} \mid k=0, \dots, m-1 \right\} \\ &\simeq \mathbb{Z}/m\mathbb{Z} \end{aligned}$$

(c) $A = (F, t)$ with $||\cdot|| = \text{topology}$

Then

$$\{x: F \rightarrow \mathbb{C}^1 \mid x \text{ smooth}\} = F^{\text{pd}}$$

by Rk 57. Take $\psi \in F^{\text{pd}} \setminus \{1\}$

Claim: $F^{\text{pd}} = \{\psi_a \mid a \in F\}$

$$\psi_a(x) := \psi(ax)$$

Proof: Put $B = \{\psi_a \mid a \in F\} \subseteq F^{\text{pd}}$

Step 1: Then $F \xrightarrow{\simeq} B$ as top. groups
 $a \mapsto \psi_a$

(uniform conv. on compact sets)
 $\hat{=}$ convergence in F)

$$\begin{array}{ccc} \text{Step 2:} & & \\ F & \xrightarrow{a \mapsto \psi_a} & (x \mapsto \psi_a(x)) \\ & \searrow \text{Res} & \\ & B^{\text{pd}} & \\ F & \xrightarrow{\parallel} & B^{\text{pd}} \\ & \searrow & \\ a & \xrightarrow{\psi_a} & (\psi_a(a) = \psi(aa)) \end{array}$$

Res is surjective as F^{pd} is the closure of B . Res is bijective, as all other maps are injective.

$$\text{So } B^{\text{pd}} \xrightarrow{\text{Res}} F^{\text{pd}}$$

$\parallel$ Step 1

$$\{\psi_a \mapsto \psi(aa) \mid a \in F\}$$

$$\begin{aligned} \text{So } F^{\text{pd}} &= \{\psi_a \mapsto \psi(aa) \mid a \in F\} \\ &= \{\psi_a \mid a \in F\} = B. \end{aligned}$$

$\square$

Prop. 6.2: Let $\psi(F, t) \rightarrow R^X$ nontrivial smooth character. Then $\mu_{\text{pro}}(R) \neq \{1_R\}$ and every smooth character has the form $\psi(a \cdot)$ for some $a \in F$.

Proof: We have $\mu_{\text{pro}}(R) \hookrightarrow S^1$.
Now apply 6.1(c). $\square$

For $p = \ell$ has a big difference to $p \neq \ell$:

Lemma 5.3: Suppose G is a p -group and $p = \ell$. Suppose $(\pi, V) \in \text{Rep}(G)$ is irreducible.

Then Π is the trivial character.

Proof: G compact and f.v. (finite volume),
so $\dim_{\mathbb{R}} V < \infty$ and $G/\ker(\pi)$ is finite.

Thus w.l.o.g. G is a p -group and $\pi: G \rightarrow \text{Aut}(V)$ is injective.

Assume $G \neq \{1\}$.

G p -group $\Rightarrow Z(G) \neq \{1\}$.
Take $g \in Z(G) \setminus \{1\}$ s.t. $g^p = 1$.

$$\begin{array}{ccc} R[X] & \xrightarrow{\varphi} & \text{End}_{R[G]} V \\ \Sigma \mapsto & & \Pi(g) \end{array}$$

parallel: φ in $\mathbb{Q}$ is an integral domain (as $\text{End}_{R[G]} V$ is a skew-field, as V is irreducible).

φ in $\mathbb{Q}$ is an R -module of finite type, generated by $\varphi(1), \varphi(X), \varphi(X^2), \dots, \varphi(X^{p-1})$.

Thus $\ker(\varphi)$ is a maximal ideal containing $(X^p - 1) = (X - 1)^p$.

$$\text{So } \ker(\varphi) = (X - 1) \text{ and } \frac{R[X]}{\ker(\varphi)} = R, \text{ or}$$

$\varphi(X)$ acts as a scalar, i.e. $\Pi(g) = r \cdot \text{id}_V$ for some $r \in R^\times$.

$g^p = 1 \Rightarrow 0 = r^p - 1 = (r - 1)^p \Rightarrow r = 1$
+ contradiction, because Π is an injective map. $\square$

I 3.2. Sklar's Lemma

Here G is loc. profinite

Lemma 64: (Sklar)

Let $(\pi, V), (\pi', V') \in R_R(G)$ irreducible.

(i) $\text{Hom}_G(\pi, \pi') \neq 0 \Leftrightarrow \pi \cong \pi'$

(ii) Suppose R is alg. closed and $\exists k$ copns S_k such that $0 < \dim_R V^k < |R|$

Then $\text{End}_G(\pi) = R$.

Proof: (i) " $\Rightarrow$ " $f \in \text{Hom}_G(\pi, \pi')$

$\ker(f) \leq V$ and $\text{im}(f) \leq V'$ are subrep.

V, V' irred and $f \neq 0 \Rightarrow \ker(f) = 0$ and $\text{im}(f) = V'$

(ii) Take $v \in V^k$, ref.

$$\text{End}_G(\pi) \xrightarrow{f} V^k$$

$$f \mapsto f(v)$$

$$\dim_R(\text{End}_G(\pi)) \leq |\dim_R V^k| < |R|$$

Hence $\exists f \in \text{End}_G(\pi) \sim R$.

$$R[X] \xrightarrow{\varphi} \text{End}_G(\pi)$$

$$Q(X) \mapsto Q(f)$$

is not injective, because

The quotient field $R(X)$ has R -dimension $\geq |R|$, because $\{\frac{1}{x-a} \mid a \in R\}$ is linearly independent.

Thus $\ker \varphi$ is a maximal ideal of $R[X]$.

R algebraically closed

$\Rightarrow m = (X - \lambda)$ for some $\lambda \in R$

$\Rightarrow \forall v \in V: f(v) = \lambda v$. $\square$

Corollary 65: Let R be alg. closed,

$(\pi, V) \in R_R(G)$ irreducible,

Suppose one of the conditions (a) or (b).

(a) (π, V) is admissible

(b) $|R| > 1/M$ and

$\exists k \leq G$ copm: G/k is countable.

Then $\text{End}_G(\pi) = R$.

Example 66: 65(a) is satisfied for

$$G = GL_n(F) \text{ and } R = F$$

$$\{g: n \geq 1, F = \mathbb{Q}_p\}$$

Take $K := GL_2(\mathbb{Q}_p)$

Inverse decomposition $K/B = G$

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \text{ Borel subgroup}$$

$$K \left(\begin{smallmatrix} a & u \\ 0 & a \end{smallmatrix} \right) \stackrel{\substack{D=P \\ D^{-1} = P^{-1} D P}}{=} K \left(\begin{smallmatrix} a & u \\ 0 & a \end{smallmatrix} \right)$$

$$\Leftrightarrow \begin{pmatrix} u a^{-a} - v a^{-a} + a + u a^{-a} \\ 0 \end{pmatrix} \in K$$

$$\Leftrightarrow a = a' \wedge b = b' \wedge u - v \in \mathbb{Z}^{\mathbb{Z}^p}$$

We have only countably many choices for $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ and $[a]_{\mathbb{Z}^p} \in \mathbb{Z}^p / \mathbb{Z}^p$.

In the abelian case an irreducible representation often must be a character.

Prop. 67: Suppose G is abelian and

R is alg. closed

(i) $\exists \chi \in \text{Irr}(G)$ is f.d. and irred. Then χ is a character

(ii) Suppose $G \leq G_0 \times \mathbb{Z}^r$ with

G_0 profinite

Then every irred repr of G is a character.

Example 68: (for 67(ii))

The center of $\text{GL}_n(\mathbb{F})$ is $\mathbb{F}^\times \cong \mathbb{F}^\times \times \mathbb{Z}$

So every irred repr of $\mathbb{F}^\times$ is a character.

(cf 67)

Proof: (i) follows from 65(a)

(ii) Take $v \in V \setminus \{0\}$ and $k_0 \leq n_0$

choose $\lambda, \mu \in V^{k_0}$.

Then $G_0 \mathbb{F}^{k_0}$ is finite, so

$R[G_0 \mathbb{F}^{k_0}]$ is finite dimensional.

So $R[G_0 \mathbb{F}^{k_0}]$ contains an irreducible

G_0 repr, which is a character by (i).

$$V(x, G_0) := \{v \in V \mid \forall g \in G_0, \pi(g)v = \chi(g)v\}$$

$V(x, G_0)$ is a non-zero G_0 subrepr of V . $\Rightarrow V(x, G_0) = V$.

So G_0 acts as a character. So

$$u, \lambda \in \mathbb{F}^\times, G = \mathbb{Z}^r = \mathbb{Z} g_1 \oplus \mathbb{Z} g_2 \oplus \dots \oplus \mathbb{Z} g_r$$

Consider $R[\mathbb{Z}^{r_1}, \dots, \mathbb{Z}^{r_r}]$

$\downarrow$

$P(\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_r) \mapsto P(\beta_1, \beta_2, \dots, \beta_r)$
 Then $m \in V$ is V generates V
 $\ker \varphi = M$ a maximal ideal
 of $R \subseteq \mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_r$.
 R also closed
 $\Rightarrow \exists \alpha_1, \alpha_2, \dots, \alpha_r \in R$

$$M = (\mathcal{X}_1 - \alpha_1, \dots, \mathcal{X}_r - \alpha_r)$$

Thus $g_1, \dots, g_r$ act as a scalar
 on V .
 So (π, V) is a character $\square$

I 3.3. $\mathbb{Z}$ -compact representations

$\mathbb{Z}$ -compact is the interior description of compactibility.

Note: For $(\pi, V) \in R_p(G)$ we have
 the natural pairing
 $\langle \cdot, \cdot \rangle : V \times V \rightarrow R$

Def 6.9: (matrix coeff.)

$$(\pi, V) \in R_p(G), \quad v \in V, \quad \tilde{v} \in V$$

The map

$$g \mapsto \langle \tilde{v}, \pi(g)v \rangle$$

is called matrix coefficient
 associated to $\tilde{v}, v$.

Pl 7.0: Take K o.t. $v \in V^K, \tilde{v} \in V^{*K}$
 Then $\text{supp}(\varphi_v, \tilde{v}) = UKg, K$
 $g \in I$

Def 7.1: $f: G \rightarrow R$ is called

compactly supported mod center

$Z = \text{center}(G)$, if

$\exists \in \text{compact} : \text{supp}(f) \subseteq Z \in G$

Def 7.2: $(\pi, V) \in R_{\mathbb{R}}(G)$ is called Z -compact ($Z = \text{center of } G$), if

$\forall v \in V \exists f \in \mathcal{F} : \phi_{v, f}$ is compactly supported mod Z .

Z -compact helps us study admissibility.

Prop 7.3: Suppose $(\pi)(K, G)$ and let

$(\pi, V) \in R_{\mathbb{R}}(G)$ with central character

(i) T.a.l. $\chi \circ (\pi, V)$ Z -compact

$\Rightarrow \forall \chi \in K^*(G) \forall v \in V$

The map $g \mapsto \pi(e_g) \pi(g)v$ is compactly supported mod center

(ii) If (π, V) is Z -compact and $\chi \neq g$.

Then (π, V) is admissible.

Pr 1: $G) \chi \circ \pi \Rightarrow \chi \circ \pi$ Take $\chi \in K^*(G)$ but

$E := \text{span}_{\mathbb{R}} \{ \pi(e_v)gv \mid g \in G \}$

Take an $\mathbb{R}$ -basis of $E : \{ \pi(e_v)gv \mid i \in I \}$

$\underbrace{=}_{\substack{\text{is } U_g \\ \text{is } U_g}}$

and extend to a basis B of V^{χ} .

Now take $\chi \in \mathcal{F}_{0, h}$ $\chi \mid V(\chi) \equiv 0$

and $\chi \mid B \equiv 1$. (Note $\chi \in \mathcal{F}^{\chi} = (V^{\chi})^{\circ}$).

Then $\text{supp}(\phi_{v, \chi})$ is a union of double cosets $KgK'Z$ with $K' \in K^{\circ}(G)$ and $v \in K^1$.

(π, V) Z -compact $\Rightarrow$ only finitely many such double cosets occur.

$\Rightarrow B$ is finite

$\Rightarrow \dim_{\mathbb{R}} E < \infty$

(This already shows (ii))

Let $\chi_i, i \in I$, be elts of $\mathcal{F}^{\chi}$ with $\chi_i \mid E$ in the dual basis of $\{ U_g, i \in I \}$.

Then $\text{supp}(\phi_{v, \chi_i}) \rightarrow \pi(e_v)gv$

$= \bigcup_{i \in I} \text{supp}(\phi_{v, \chi_i})$

is compact mod center

$2^0 \Rightarrow 1^0$ Take $v \in V, v' \in V'$ and $k \in K(\mathbb{Q}_p)$ fixing v and v' .

Then $\text{supp}(\varphi_{v,v'}) \subseteq \text{supp}(g \mapsto \pi(e_{v,v'}))$

□

Cor 74: Suppose that

- R is abelian, closed and uncountable and archimedean (*) and
 - $\exists k \leq G$ open: G/k is countable
- Then every π -compact $\mathbb{Z}$ -compact repr. is admissible.

Example 75: $R = \mathbb{C}$ and $G = GL_n(\mathbb{F})$

Pl 76: The assumptions in Cor 74 are there to ensure that an π -compact repr. has a central character.

Most useful prop in this context for p -adic reductive groups is the following

Prop 76: Suppose (*) (G, R) and that

G contains an open normal subgroup G_0 s.t. $[G:G_0, \mathbb{Z}] < \infty$ and G_0 has compact center. Let $(\pi, V) \in \text{Re}(G)$

(i) True:

1° (π, V) is $\mathbb{Z}$ -compact

2° $\forall k \in K(\mathbb{Q}_p) \forall v \in V: g \mapsto \pi(e_{v,v'})v$ has support compact mod center

3° $\pi|_{G_0}$ is a compact repr, i.e.

all matrix coeff of $\pi|_{G_0}$ have compact support

(ii) Suppose that all π -reps. of $\mathbb{Z}$ are characters.

If (π, V) is $\mathbb{Z}$ -compact and irreducible then (π, V) is admissible and has a central character.

Example 77: For $G = GL_n(\mathbb{F})$

Take $G_0 = \{g \in G \mid \det(g) \in \mathbb{O}_F^\times\}$

$\mathbb{Z} = \left\{ \begin{pmatrix} x & & \\ & x & \\ & & x \end{pmatrix} \mid x \in \mathbb{F}^\times \right\}$

$$[G : ZG_0] < \infty$$

(You can find a set of representatives

$$\text{in } \left\{ \begin{pmatrix} \sigma^{a_1} & & \\ & \ddots & \\ & & \sigma^{a_n} \end{pmatrix} \mid n > a_1 \geq a_2 \geq \dots \geq a_n \geq 0 \right\}$$

Proof (cf Prop 7.6)

(i) $2^0 \Rightarrow 1^0$ as in Prop 7.3.

$1^0 \Rightarrow 3^0$ $\text{supp}(\mathcal{Q}_{\mathcal{U}})$ is a disjoint
finite union of double cosets

$KgZK$ with $K \leq G_0 \text{ open}$
 $\cap \Delta$. $v \in V^K$ and $\tilde{v} \in V^K$.

$\exists (KgZK) \cap G_0 \neq \emptyset$ Then

we can assume $g \in G_0$

Then $(KgZK) \cap G_0 = KgK(\underbrace{Z \cap G_0}_{\text{compact}})$

is compact.

$3^0 \Rightarrow 2^0$ W.l.o.g. G_0 is normal in G

Indeed: Take system of representatives

$\{g_1, \dots, g_s\}$ of G/ZG_0

and replace G_0 by

$$g_1 G_0 g_1^{-1} \cap \dots \cap g_s G_0 g_s^{-1} = \bigcap_{g \in G} g G_0 g^{-1}$$

$\Pi|_{G_0}$ is compact.

So for $K \in K^*(G)$ and $v \in V$

$G_0 \ni g \mapsto \Pi(\varepsilon_K)gv$ has compact
support, denoted by $S(K, v)$.

Then the support of

$G \ni g \mapsto \Pi(\varepsilon_K)gv$ is equal to

$$\bigcup_{i=1}^s g_i S(g_i^{-1} K g_i, v) \cdot Z.$$

(ii) (Madelung)

$(\Pi|_V)$ invd, $[G : G_0 Z] < \infty$

as $\Pi|_{G_0 Z}$ is f.g. by

$X = \{v_1, \dots, v_n\}$. Take $K \in K^*(G_0) \setminus \{0\}$.

$W := \text{span}_\mathbb{R} \{ \Pi(\varepsilon_K)gv_i \mid g \in G_0, i=1, \dots, n \}$

Then $V^K = \Pi(\varepsilon_K)V = \Pi(Z)V$.

So V^K is f.g. Z -repr. has invd
quasi-invariant which is a character χ

(by assumption in (ii)).

So $V^k(X) := \{ \sigma(0) \sigma - \chi(0) \sigma \mid \sigma \in V^k, \sigma \in Z \}$
is a proper $\mathcal{H}(G, k)$ sub-module
of V^k .
 $(\pi, V) \text{ irr.} \Rightarrow V^k \text{ mult. } 1, 1$

So $V^k(X) = 0$.

$\Rightarrow Z$ act on V^k as X .

$\Rightarrow \dim_k V^k < \infty$.

Further

$V(X)$ is G -stable and non-zero $\Rightarrow V(X) = V$.

So (π, V) has a central character which is χ . $\square$

The next proposition explains how to construct Z -compact representations.

Def 7.8: Let G be a loc. profinite gr.
(or group)

$H \leq G$ open is called
a compact mod center subgroup of G
if $Z = \text{center of } G \subseteq H$ and
 H/Z is compact.

Example: $G = GL_2(F)$

1) $F^\times \cdot GL_2(O_F^\times)$ is a compact
mod center subgroup of G .

2) E/F quadratic field or -
extension of F .

Then $E \subseteq \text{End}_F E \cong M_2(F)$
and $E^\times \subseteq \text{Aut}_F E \cong GL_2(F)$
is a comp. mod Z subgroup of G .

Prop 7.9: Let H be an open comp
mod Z subgroup of G and

$(\chi, \rho) \in \mathcal{R}_P(H)$. Assume $(\chi)(\mathcal{R}(G))$.

Suppose $\pi = \text{c-ind}_H^G \rho$ is irr. and adm.

Then π is Z -compact.

Proof: $\tilde{\pi}$ is also irred. and assem.

So we only need to check one matrix coeff.

Take $w \in W \setminus \{0\}$ and $\tilde{w} \in \tilde{W} \setminus \{0\}$ such that $\tilde{w}(w) \neq 0$.

$$f_w(g) := \begin{cases} \pi(g)w, & g \in H \\ 0, & g \notin H \end{cases}$$

$v := f_w$ lies in $c\text{-ind}_H^G W$.

Take $\tilde{v} \in \tilde{\pi}$ defined via

$$f \in c\text{-ind}_H^G W \mapsto \tilde{v}(f) := \tilde{w}(f(1))$$

The matrix coeff $\varphi_{v, \tilde{v}}$ satisfies

$$\begin{aligned} \varphi_{v, \tilde{v}}(g) &= \tilde{v}(\pi(g)v) = \tilde{w}((\pi(g)f_w)(1)) \\ &= \tilde{w}(f_w(g)) \end{aligned}$$

So $\text{supp}(\varphi_{v, \tilde{v}}) \subseteq H$, so is compact mod center. $\square$

Example 80: $M := \left[\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \mid \begin{matrix} a \in F^\times \\ x \in F \end{matrix} \right]$

$$M = SN$$

$$S = \left\{ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \mid a \in F^\times \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mid x \in F \right\}$$

Let $\chi: N \rightarrow \mathbb{C}^\times$ be a smooth character

(This is the same as an additive character of F via $N \cong F$)

$$\pi_c = c\text{-ind}_N^M \chi, \quad \pi = \text{ind}_N^M \chi$$

Claim: 1) π_c is irreducible (no proof)
[Bushnell-Henniart, "GL(2)", §8.2.]

2) π_c is a proper subrepresentation of π .

So π is not irreducible.

3) π_c is not admissible

Proof: (Just the idea)

2) As linear space we have

$$\begin{array}{ccc} \Pi_c = C\text{-ind}_N^M \mathcal{V} & \xrightarrow{\quad \varphi \quad} & C_c^\infty(F^X) \\ \cap & \searrow \alpha & \cap \\ \Pi = \text{ind}_N^M \mathcal{V} & \xrightarrow{\quad \varphi \quad} & C^\infty(F^X) \end{array}$$

$$f \longmapsto f|_S$$

$$\begin{aligned} \text{Im}(\varphi) &= \{ f \in C^\infty(F^X) \mid \exists m \in N \\ &\quad f|_{F^X \setminus (m^{-1}m\sigma_F)} \equiv 0 \} \\ &\neq C_c^\infty(F^X). \end{aligned}$$

So $\Pi/\Pi_c \neq 0$ and thus Π is not irreducible.

$$3) \quad C\text{-ind}_N^M \mathcal{V} \simeq \text{ind}_N^M \tilde{\mathcal{V}}$$

$$\begin{aligned} &\xrightarrow{\text{GL}(C)} \text{ind}_N^M \mathcal{V} = \Pi \\ &\downarrow \\ &\mathcal{V} \text{ is } \tilde{\mathcal{V}} \\ &\text{conjugate to } \tilde{\mathcal{V}} \end{aligned}$$

is not irreducible.

Cor. 28 $\Rightarrow \Pi_c$ is not admissible $\square$

Rk 81: The group M is not a group of F -points of a connected reductive group defined over F .

We will see that for all latter groups that all complex irreducible representations are admissible.

I.4. Cupical representations of reductive p-adic groups

Steps in this section:

- 1) Def of cupical representations for reductive p-adic groups using parabolic induction and restriction
- 2) Introducing cupical support
- 4) ($e \neq 1$) smooth mod repr are admissible
- 3) ($e \neq 1$) $\mathbb{Z}$ -compact = cupical

I.4.1. Reductive p-adic groups and parabolic induction

In the whole I.4. let G be

the set of F -points of a connected reductive algebraic group G defined over F

(Example: $G = GL_n(F)$, $G = GL_n$)

Each: (a) $(*) (G, G) \Leftrightarrow p \neq 2$

(e) For $K = G(F)$ we have that G/K is countable.

(c) $Z(G) = \text{center of } G \simeq \text{primitive } p \times \mathbb{Z}^r$
Terminology 82: G is called a reductive p-adic group

Let P be a parabolic subgroup of G .

$P = LN$ with

L is the centralizer of a split torus (a closed subgroup $S \simeq F^r \times \mathbb{Z}^r$ of G).

N is the maximal normal unipotent subgroup of P .

(Example: $G = GL_n(F)$, $P = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$
① $\begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$

$$L = \begin{pmatrix} x & & \\ & x & \\ & & x \end{pmatrix} \quad N = \begin{pmatrix} I_{n_1} & & \\ & I_{n_2} & \\ & & I_{n_n} \end{pmatrix}$$

(For $G_{L_3}(F)$:

$$\begin{pmatrix} x & x & x \\ x & x & x \\ 0 & 0 & x \end{pmatrix}, \begin{pmatrix} x & x & x \\ x & x & x \\ 0 & 0 & x \end{pmatrix}$$

and all G -conjugates of P ,

$$\begin{pmatrix} x & x & x \\ x & x & x \\ 0 & 0 & x \end{pmatrix}$$

Ex 83: L is a reductive p -adic group

Def 84: (parabolic induction functor)

$$i_{L,P}^G: R_P(L) \rightarrow R_P(G)$$

inflation $\nearrow$ $\searrow$ $\downarrow$
 $\downarrow$ $\nearrow$ $\searrow$
 $R_P(P)$ $\xrightarrow{\text{ind}_P^G}$ $R_P(P)$

inflation: pullback via $P \rightarrow P_N \subseteq L$

Def 85: (parabolic restriction functor)

$$r_{L,P}^G: R_P(G) \rightarrow R_P(L)$$

$\swarrow$ $\searrow$
 $r_{L,P}^G$ $\searrow$ $\swarrow$
 $R_P(P)$ $\xrightarrow{N\text{-conv}}$ $R_P(P)$

$r_{L,P}^G$ is also called Jacquet restriction functor

Example 86: (a) $R = \mathbb{Q}$ $G = GL_2(\mathbb{Q}_2)$

$$K := GL_2(\mathbb{Z}_2)$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = L_N, L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Gamma_{L, \beta}^G (\underbrace{\text{c-ind}_{\mathbb{I}K}^G \text{sign}}_{=1}) = 1_N = 0,$$

(e) G, β as in (a).

By 49(c).

$$\Gamma_{L, \beta}^G (\underbrace{\text{ind}_{\beta}^G 11}_V) = 1_N \neq 0,$$

by 50.

Prop 87: For $\pi \in R_p(G)$ and $\rho \in R_p(L)$ we have a natural isomorphism of R -mod.

$$\text{Hom}_L(\Gamma_{L, \rho}^G(\pi), \rho) \simeq \text{Hom}_G(\pi, \Gamma_{L, \rho}^G(\rho))$$

$$\text{Proof: } \text{Hom}_L(\Gamma_{L, \rho}^G(\pi), \rho) \simeq \text{Hom}_p(\text{res}_p^G(\pi), \text{inf}_p^G(\rho))$$

$$\simeq \text{Hom}_G(\pi, \Gamma_{L, \rho}^G(\rho)) \quad \square$$

Thm 23.

Lem 88 1) $i_{L, p}^G$ is left exact and

$r_{L, p}^G$ is right exact.

Both are exact if $p \neq l$.

2) $r_{L, p}^G$ preserves finite type

Proof: 1) $i_{L, p}^G$ is a right adjoint and

$$\Gamma_{L, p}^G = 11 - \text{left} \quad \text{---} \quad \text{---}$$

So $i_{L, p}^G$ is left exact and

$r_{L, p}^G$ is right exact.

The exactness statement, in case of $p \neq l$ follows from 25, 51(ii) (see 50).

2) Let $(\pi, \nu) \in R_p(G)$ of finite type
 $V = \text{span}_R[G] \{ \nu_1, \dots, \nu_s \}$

Take $K \leq G$ open s.t. $\nu_1, \dots, \nu_s \in V^K$
 ρ_K is compact, so ρ_K^K is finite

$$\{ [g_1, g_2], \dots, [g_{s-1}, g_s] \}$$

$$\Rightarrow \{ g_i, \nu_j \mid 1 \leq i \leq s, 1 \leq j \leq s \}$$

generates $\text{Res}_p^G(\pi, \nu)$. So

$$\{ g_i, \nu_j \mid 1 \leq i \leq s, 1 \leq j \leq s \} \text{ generates } \Gamma_{L, p}^G(\pi, \nu). \quad \square$$

Def 89 (a) (P, L) with a parabolic subgroup P of G and a Levi subgroup L of P , $P = LN$ ($N = \text{unipotent radical of } P$) is called a parabolic pair

$(P, L) \leq (P', L')$ iff $P \leq P'$ and $L \leq L'$

(b) For $(P_0, L_0) \leq \text{minimal}$

We call (P, L) standard (w.r.t. (P_0, L_0)) if $(P_0, L_0) \leq (P, L)$

Example 90: $G = GL_3(F)$

$$(P_0, L_0) = \left(\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}, \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ 0 & * & * \end{pmatrix} \right)$$

The standard parabolic pairs are

$$\left(\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}, \begin{pmatrix} * & * & 0 \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right), \left(\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}, \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right), (P_0, L_0)$$

and (G, G) .

$$\text{e.g. } (P_0, L_0) \leq \left(\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}, \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right) \leq (G, G)$$

Def 91 (superficial representation)

$(\pi, V) \in R_R(G)$ is called superficial if $\forall (P, L) \leq (G, G) : \Gamma_{L, P}^G(\pi, V) = 0$

Re 92: By Prop. 87 "superficial"

is equivalent to

$$\forall (P, L) \leq (G, G) : \nexists \varphi \in R_P(L)$$

$$\text{Hom}_G(\pi, L_{L, P}^G(S)) \neq 0$$

Def 93: An irreducible representation

$(\pi, V) \in R_R(G)$ is called supercuspidal

if $\forall (P, L) \leq (G, G) \nexists \varphi \in R_P(L) : (\pi, V)$ is not a subquotient of $L_{L, P}^G(S)$

Re 94: irreducible supercuspidal

$\Rightarrow$ irreducible superficial

When $R = \mathbb{C}$, then " $\Leftarrow$ " holds

In general " $\Leftarrow$ " doesn't hold

Thm 25: Let $(\pi, \nu) \in R_P(G)$.

Then $\exists (P, L)$ parabolic pair in G :

$\exists \beta \in R_P(L)$ irreducible cuspidal.

$\text{Hom}_G(\pi, \nu_{L,P}^{\beta}(\beta)) \neq 0$.

Proof: Take (P, L) minimal such

that $\Gamma_{L,P}^G(\pi) \neq 0$.

This exists, because the "algebraic dimension" of each parabolic subgrp of G is finite.

π is irreducible $\Rightarrow \Gamma_{L,P}^G(\pi)$ is of finite type

$\Rightarrow \Gamma_{L,P}^G(\pi)$ has an irreducible quotient (S, W) .

Claim: (S, W) is cuspidal.

Proof: We use lemma 27 for the proof.

A parabolic pair in L has

the form $(P_{\lambda L}, L')$ for some

$(P, L) \leq (P, L)$, by 27(ii).

Suppose $\Gamma_{L',P_{\lambda L}}^L(S) \neq 0$.

Lemma 27(iii) $\Rightarrow \Gamma_{L',P_{\lambda L}}^G(\pi) \cong \Gamma_{L',P_{\lambda L}}^L(S) \otimes \Gamma_{L,P}^G(\pi)$

$\cong \Gamma_{L',P_{\lambda L}}^L(S) \neq 0$.

Then $(P, L') = (P, L)$, by minimality.

Thus (S, W) is cuspidal. $\square$ (Claim)

Pl 26: Given $(\pi, \nu) \in R_P(G)$,

the pair (L, S) from Thm 25 is uniquely determined up to $\sim$ and G -conjugacy.

We call this equivalence class cuspidal support of (π, ν) .

Lemma 27: (i) All minimal parabolic pairs in G are G -conjugate.

(If we always can assume that

(P, L) is standard)

(ii) Let $(P, L') \leq (P, L)$. Then

$(P_{\lambda L}, L')$ is a parabolic pair in L , and every parabolic pair

in L can be obtained this way.

(iii) Let $(P, L') \leq (P, L)$ and

$(\pi, \nu) \in R_P(G)$. Then

$\Gamma_{L',P_{\lambda L}}^G(\pi) \cong \Gamma_{L',P_{\lambda L}}^L(S) \otimes \Gamma_{L,P}^G(\pi)$.

Proof: [Good-Tsalo, groups & fields] $\square$

Example 98: $G = GL_4(F)$.

$$(P, L) = \left(\left(\begin{array}{cc|c} * & * & \\ \hline 0 & & * \end{array} \right)^{(2,2)}, \left(\begin{array}{cc|c} * & 0 & \\ \hline 0 & & * \end{array} \right)^{(2,2)} \right)$$

$$(P', L') = \left(\left(\begin{array}{cc|c} \square & & * \\ \hline 0 & & \end{array} \right)^{(1,1,2)}, \left(\begin{array}{cc|c} * & \square & 0 \\ \hline 0 & & * \end{array} \right)^{(1,1,2)} \right)$$

Then $(P', L') \not\leq (P, L)$.

Further

$$(P' \cap L, L') = \left(\left(\begin{array}{cc|c} * & * & 0 \\ \hline 0 & * & \\ \hline 0 & & * & * \end{array} \right), \left(\begin{array}{cc|c} * & \square & 0 \\ \hline 0 & & * & * \\ \hline 0 & & * & * \end{array} \right) \right)$$

a parabolic pair in $L = \left(\begin{array}{cc|c} * & * & 0 \\ \hline * & * & \\ \hline 0 & & * & * \end{array} \right)$.