Text book:
p-adic Representations, Θ-Correspondence and the Langlands-Shahidi Theory.

Science Press.

I Basic repr. theory.

I 1. Smooth repr. of loc. profinite gps.

I 1.1. Loc. profinite gps.

Def 1: Let G be a top. gp. G is called

(i) profinite if G is compact and totally disconnected

(ii) locally profinite if G locally compact and totally disconnected.

Rk 2 (a) G top gp.: G loc. profinite iff every nbhd of 1_G contains a c-open subgp.

(b) G profinite: We find normal c-open subgroups.

$$G \xrightarrow{\sim} \varprojlim_{\substack{U \trianglelefteq G \\ \text{c-open}}} G/U$$

Example 3: (i) $(\mathbb{Q}_p, +)$ $\{p^n \mathbb{Z}_p \mid n \in \mathbb{N}_0\}$ nbhd basis of 0 of c-open subgroups.

Similar for F non-Arch local field:

$\mathcal{O}_F := \{x \in F \mid |x| \leq 1\}$

$\mathfrak{p}_F := \{x \in F \mid |x| < 1\} = \varpi\, \mathcal{O}_F$

$(\mathbb{Q}_p: \mathcal{O}_{\mathbb{Q}_p} = \mathbb{Z}_p,\ \mathfrak{p}_{\mathbb{Q}_p} = p\mathbb{Z}_p)$

Find nbhd basis $\{\varpi^n \mathcal{O}_F \mid n \in \mathbb{N}\}$

$k_F := \mathcal{O}_F/\mathfrak{p}_F$ residue field of F

$(\mathbb{Q}_p: \mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p)$

(ii) $F^\times$ $\{1+\mathfrak{p}_F^n \mid n \geq 1\}$.

(iii) Let E be a f.d. F-v.s.

$E \underset{\substack{\uparrow \\ \text{basis}}}{\cong} F \oplus \dots \oplus F$

Product top. gives a loc. prof. gp $(E,+)$ (indep. of basis)

So $M_n(F) = F^{n\times n}$ and

$GL_n(F) = \{A \in M_n(F) \mid \det A \neq 0\}$

are loc. prof. gps.

nbhd system in $GL_n(F)$:

$\{\underbrace{1+\mathfrak{p}_F^n M_n(\mathcal{O}_F)}_{=: K_n} \mid n \geq 1\}$

<u>Prop 4:</u> G loc. prof., $H \leq G$, $N \trianglelefteq G$ two closed subgps. Then H and G/H are loc. profinite.

<u>Proof:</u> Take a sys of nbhd $\{K_i \mid i \in I\}$ of open subgps.

Then $\{H \cap K_i \mid i \in I\}$ and $\{\pi(K_i) \mid i \in I\}$ form a nbhd basis of H and G/N resp. $\square$

<u>Rk 5:</u> If G prof. and $K \leq G$ open
Then K is compact and G/K is finite

<u>Convention 6:</u> We fix a field R and we denote $\ell := \mathrm{char}(R)$.

I 1.2. Basic repr. theory

Def 6: An *R-representation* of a group G (π, V) is an R-v.s. V together with a gp. homomorphism
$$\pi : G \longrightarrow \mathrm{Aut}_R(V)$$

A *morphism* $\varphi : (\pi, V) \longrightarrow (\pi', V')$ is an R-linear map s.t.
$$\varphi(\pi(g) v) = \pi'(g)\varphi(v) \quad \forall g, v.$$

Rk 7: (π, V) is an $R[G]$-module.

Def 8: A *subrepresentation* of (π, V) is a G-stable sub-v.s. of V.
(Notation: $(\pi|_W, W)$).
(π, V) is called *indecomposable* if V is not the direct sum of proper sub-repr.
(π, V) is called *irreducible* if $V \neq 0$ and $\nexists$ proper non-zero sub-repr. of (π, V).

Examples 9: (a) $G = \mathbb{Q}_p^\times$, $R = \mathbb{C} =: V$
$\pi(x) z := |x| z$.
$(\pi : \mathbb{Q}_p^\times \longrightarrow \mathrm{Aut}_{\mathbb{C}} \mathbb{C})$

(b) $G = GL_2(\mathbb{Z}_\ell)$, $R = \mathbb{F}_\ell$, $V = \mathbb{F}_\ell^2$
$\pi : G \longrightarrow GL_2(\mathbb{F}_\ell)$

Def 10: (π, V) is called of *finite type* if V is finitely generated as an $R[G]$-module

Prop 11: Let (π, V) be non-zero. Then
(i) (π, V) has an irreducible subquotient.
(ii) (π, V) has an irred. quotient if it is of finite type
(iii) (π, V) has an irred. subrepr. if it is of finite length.

Proof: (iii) ✓
(ii) Take a minimal generating system $\{v_1, \ldots, v_k\}$ of V.
$\mathcal{M} := \{ W \text{ subrep. of } V \mid \{v_2, \ldots, v_k\} \subseteq W \text{ and } v_1 \notin W \}$

$\mathrm{span}_{R[G]}\{v_1,\dots,v_k\} \in \mathcal{M}$, and $\mathcal{M}$ is inductive. Zorn $\Rightarrow \exists W \in$-maximal in $\mathcal{M}$.

<u>Claim:</u> V/W is irred.

Pf: Assume $\exists\ W \subsetneq U \subsetneq V$ subrepr.
$\Rightarrow U \notin \mathcal{M}$ and thus $v_i \in U \supseteq W$
$\Rightarrow U = V$ ↯ $\square$

(i) Take $v \in V \setminus \{0\}$. $W := \mathrm{span}_{R[G]}\{v\}$ is f.g.. Apply (ii) $\square$

<u>Prop 12:</u> Let (π, V) be a repr. of G. T.a.e.:

1° (π, V) is a direct sum of irred. subrepr.

2° (π, V) is a sum of irred subrep.

3° Any subrepr. has a G-complement.

<u>Proof:</u> $2^\circ \Rightarrow 1^\circ$ $\quad V = \sum_{i\in I} U_i$, U_i irred.

$$\mathcal{M} := \{ J \subseteq I \mid \sum_J U_i \text{ is direct} \}$$

Zorn $\Rightarrow \exists$ max. $\hat{J} \in \mathcal{M}$.
$\Rightarrow \bigoplus_{\hat{J}} U_i = V$.

$1^\circ \Rightarrow 3^\circ$ $\quad W \leq_G V = \bigoplus_I U_i$

$$\mathcal{M} := \{ J \subseteq I \mid W \cap (\textstyle\sum_{i\in J} U_i) = \{0\} \}$$

Zorn $\Rightarrow \exists\ \hat{J} \in \mathcal{M}$ max.
$\Rightarrow W \oplus (\bigoplus_{\hat{J}} U_i) = V$

$3^\circ \Rightarrow 2^\circ$ Every subrepr. W of V also satisfies 3°.
Thus every subquotient of W is isom. to a subrepr. of W.
Prop 11 (ii) $\Rightarrow W$ contains an irred subrepr. if $W \neq 0$.
$V_0 := \sum_{U \text{ irred}} U$ has a direct G-complement W, and W doesn't contain an irr. subrep.
$\Rightarrow W = 0$. $\square$

We call such a repr. (π, V) <u>completely reducible</u>.

I 1.3. Smooth repr.

G loc. prof.

<u>Def 13:</u> (π, V) is called <u>smooth</u> if $\forall_{v \in V}$: the stabilizer $G_v := \{g \in G \mid gv = v\}$ is open.

(In Example 9 they are smooth.)

<u>Rk 14:</u> If G is profinite and V of finite type, say generated by $\{v_1, \dots, v_k\}$, put $H := G_{v_1} \cap \dots \cap G_{v_k}$, then G/H is finite and V is a f.d. R-v.s..
Take $N \leq H$ open, normal in G
$\Rightarrow \bar{G} := G/N$ is a finite group and

$(\bar{\pi}, V)$ is a repr. of $\bar{G}$.

<u>Thm 15:</u> (Maschke) Let G be finite and $\ell \nmid |G|$. Let (π, V) be a repr. of G. Then (π, V) is completely reducible.

<u>Proof:</u> $T : V \longrightarrow W \quad (W \leq_G V)$

$$T(v) := \frac{1}{|G|} \sum_{g \in G} \pi(g) P(\pi(g^{-1}) v)$$

(P some linear projection $W \oplus W' \to W$, W' an R-linear complement)
T is a G-morphism and
$T \circ T = T$ and $\operatorname{im}(T) = W$
$\Rightarrow \ker(T) \oplus W = V$
and $\ker(T) \leq_G V$ $\square$

<u>Example 16:</u> (a) $G = \mathbb{Q}_\ell^\times$, $R = \mathbb{Q}_\ell$, $V = \mathbb{Q}_\ell$.
$\pi(x) v := x \cdot v$. (π, V) is not smooth.
$v \neq 0$ $\quad G_v = \{1\}$ is not open in $\mathbb{Q}_\ell^\times$.
$G_0 = G$ is open.
The only smooth vector is 0.

(b) $G = \mathbb{Q}_\ell^\times$, $R := \mathbb{C}$, $V := \mathrm{Abb}(\mathbb{Q}_\ell, \mathbb{C})$.
$(\pi(x) f)(y) = f(yx)$, $x \in G$, $y \in \mathbb{Q}_\ell$.
(π, V) is not smooth.
$G_{\mathbb{1}_{\{1\}}} = \{1\}$

(π, V) has smooth vectors
$G_{\mathbb{1}_{\mathbb{Z}_\ell}} = \mathbb{Z}_\ell^\times$ is open in G.

We write $V^\infty = \bigcup_{K \text{ copen}} V^K$. V^∞ is G-stable

(π^∞, V^∞) is a smooth representation.

Example 17: S an R-v.s. $V = \mathrm{Abb}(G, S)$

$(\pi(g) f)(h) := f(hg)$.

- (π, V) is a representation of G

$(\pi(g_1 g_2) f)(h) = f(h g_1 g_2)$

$(\pi(g_1)(\pi(g_2) f))(h) = (\pi(g_2) f)(h g_1)$

$= f((h g_1) g_2)$

- $\mathcal{C}(G, S) := \{ f \in \mathrm{Abb}(G,S) \mid f \text{ is locally constant} \}$

(f loc. constant if $\forall_{g \in G} \exists H$ copen ($f(gh) = f(g)\ \forall h \in H$))

- $\mathcal{C}^\infty(G,S) := V^\infty$

$= \{ f \in \mathrm{Abb}(G,S) \mid \exists K \text{ copen}: \forall_{g\in G} \forall_{k \in K}: f(gk) = f(g) \}$

space of smooth functions of G to S.

Def 18: (support) $f \in \mathrm{Abb}(G,S)$

$\mathrm{supp}(f) = \overline{\{ g \in G \mid f(g) \neq 0 \}}$ (closure)

For $f \in \mathcal{C}(G,S)$: $\mathrm{supp}(f) = \{ g \in G \mid f(g) \neq 0 \}$

$\mathcal{C}_c^\infty(G,S) := \{ f \in \mathcal{C}^\infty(G,S) \mid \mathrm{supp}(f) \text{ compact} \}$

$= \{ f \in \mathcal{C}(G,S) \mid \mathrm{supp}(f) \text{ compact} \}$

Def 19: (contragredient representation)

(π, V) smooth repr., $V^* := \mathrm{Hom}_R(V, R)$

(π^*, V^*) $(\pi^*(g) f)(v) := f(g^{-1} v)$

The smooth part

$(\tilde{\pi}, \tilde{V})$ is called contragredient representation of (π, V).

I 1.4. Induced representations

$H \leq G$ closed.

We look for the left and right adjoint of $\mathrm{Res}^G_H : \mathcal{R}_R(G) \longrightarrow \mathcal{R}_R(H)$

Prop. 20: Let G be finite

$$R[G] \otimes_{R[H]} - : \mathcal{R}_R(H) \longrightarrow \mathcal{R}_R(G)$$

is a right and left adjoint of Res^G_H.

Proof: $\mathrm{Hom}_{R[H]}(\mathrm{Res}^G_H(\pi), \rho) \simeq \mathrm{Hom}_{R[G]}(\pi, R[G] \otimes_{R[H]} W)$

for $(\rho, W) \in \mathcal{R}_R(H)$ and $(\pi, V) \in \mathcal{R}_R(G)$.

$$\alpha \longmapsto \Lambda_\alpha$$

$$\Lambda_\alpha(v) = \sum_{\bar{g} \in G/H} g \otimes \alpha(g^{-1}v)$$

$$\mathrm{pr}_W \circ \Lambda =: \alpha_\Lambda \longleftarrow\!\shortmid \Lambda$$

$$\left(R[G] \otimes_{R[H]} W = \left(\bigoplus_{\bar{g} \in G/H \setminus \{H\}} (g \otimes W) \right) \oplus \left(1 \otimes_{R[H]} W\right) \xrightarrow{\mathrm{pr}_W} W \right)$$

$$\mathrm{Hom}_{R[G]}(R[G] \otimes_{R[H]} W, \pi) \simeq \mathrm{Hom}_{R[H]}(\rho, \mathrm{Res}^G_H(\pi))$$

$$\Gamma \longmapsto \Gamma|_W$$

$$\left(\Gamma_\beta(g \otimes w) := \pi(g)\beta(w) \right) \longleftarrow\!\shortmid \beta$$

□

Generalization to loc. profinite groups.

Def 21: $(\rho, W) \in \mathcal{R}_R(H) :=$ category of smooth R-repr. of H.

$(\mathrm{Ind}^G_H(\rho), \mathrm{Ind}^G_H(W))$ <u>repr. induced by ρ.</u>

$$\mathrm{Ind}^G_H(W) := \{ f : G \to W \mid \forall_{g \in G, h \in H} : f(hg) = \rho(h) f(g) \text{ and } \exists\, K \leq G \text{ open } \forall_{g \in G, k \in K} : f(g) = f(gk) \}$$

$$\left((\mathrm{Ind}^G_H(\rho))(g) \right) f(\tilde{g}) := f(\tilde{g} g).$$

G acts by right translation.

<u>Def 22:</u> $(c\text{-}\mathrm{Ind}_H^G \rho, c\text{-}\mathrm{Ind}_H^G W)$

$$c\text{-}\mathrm{Ind}_H^G W := \{ f \in \mathrm{Ind}_H^G W \mid \exists\, \Omega \subset G \ \Omega \text{ compact}: \operatorname{supp}(f) \subseteq H\Omega \}$$

$c\text{-}\mathrm{Ind}_H^G(\rho)$ action by right translation.

<u>repr. compactly induced by ρ.</u>

<u>Thm 23:</u> (a) W is a quotient of $\mathrm{Ind}_H^G \rho|_H$

$(\mathrm{Res}_H^G, \mathrm{Ind}_H^G)$ is a adjoint pair

(b) H open: W is a subrepr. of $c\text{-}\mathrm{Ind}_H^G \rho|_H$

$(c\text{-}\mathrm{Ind}_H^G, \mathrm{Res}_H^G)$ is an adjoint pair.

<u>Proof:</u> (a) $\mathrm{Ind}_H^G W \to W$
$f \mapsto f(1)$

(b) $W \longrightarrow c\text{-}\mathrm{Ind}_H^G W$
$w \longmapsto f_w$

$$f_w(g) = \begin{cases} 0, & g \notin H \\ gw, & g \in H \end{cases}$$

Other assertions: exercise □

<u>Thm 24:</u> $$\left(c\text{-}\mathrm{Ind}_H^G \rho\right)^K \simeq \bigoplus_{g \in \Lambda} W^{H \cap gKg^{-1}}$$

$$\left(\mathrm{Ind}_H^G \rho\right)^K \simeq \prod_{g \in \Lambda} W^{H \cap gKg^{-1}}$$

for K c-open. (Λ sys of representative of $H \backslash G / K$)

<u>Proof:</u> $f \in \left(\mathrm{Ind}_H^G W\right)^K \mapsto (f(g))_{g \in \Lambda}$

(Take $g \in \Lambda$, $h = gkg^{-1} \in H \cap gKg^{-1}$
then $h f(g) = f(hg) = f(gk) = f(g)$)

□

<u>Exercise 25:</u> Thm 24 shows that Ind_H^G and $c\text{-}\mathrm{Ind}_H^G$ are left-exact.

And they are exact if ℓ does <u>not</u> divide the infinite part of the pro-order of G.

I 2. Admissible representations

I 2.1. Admissible repr., first definitions

From now on all repr. are smooth.

Def. 2.6: (π, V) is called admissible if $\forall K \leq G$ c.open: V^K is f. dim.

Convention (*): $\exists K \leq G$ c.open: $\ell = \operatorname{char}(R)$ does not divide the index of any open subgroup of K in K.

(i.e. $\forall K' \leq K$ open: $\ell \nmid |K/K'|$)

$\mathcal{K}^*(G) :=$ set of such $K \leq G$.

Prop 2.7: Suppose (R, G) satisfies (*). Take $K \in \mathcal{K}^*(G)$.

(i) $(\pi, V)|_K = \operatorname{Res}^G_K(\pi)$ is completely reducible. and

$V = V^K \oplus V(K)$

$V(K) := \operatorname{span}_R \{\pi(k)v - v \mid k \in K, v \in V\}$

(ii) $(\tilde{V})^K \simeq (V^K)^*$ as R-v.s..

(iii) $(\ )^K : \mathcal{R}_R(G) \longrightarrow R$-v.s. is exact.

(iv) $(\tilde{\ }) : \mathcal{R}_R(G) \longrightarrow \mathcal{R}_R(G)$ is exact.

Proof: (i) • $V|_K$ is the sum of repr. of K of finite type. $\left(V = \sum_{v \in V} \operatorname{span}_{R[K]}\{v\}\right)$

In $\mathcal{R}_R(K)$ repr. of finite type are completely reducible, by Maschke.

• Take any K-complement W of V^K

Then $V(K) \subseteq \ker(\operatorname{Pr}_{V^K \oplus W, V^K})$

$\Rightarrow V(K) \subseteq W.$

Take any $U \subseteq W$ irreducible.

Then $U^K = 0$ and thus $U(K) \neq 0$

and $U = U(K)$ as U is irred..

So $U = U(K) \subseteq V(K)$.

Thus $W \subseteq V(K)$.

(ii) $V = V^K \oplus V(K)$

$(\tilde{V})^K = (V^*)^K \longrightarrow (V^K)^*$

$f \longmapsto f|_{V^K}$

is bijective as $f|_{V(K)} = 0$.

(iii) exercise

(iv) $0 \to V_1 \to V_2 \to V_3 \to 0$ exact

$\overset{(iii)}{\Longrightarrow}$ $0 \to V_1^K \to V_2^K \to V_3^K \to 0$ exact $\forall K \in \mathcal{K}^*(G)$

$\Longrightarrow$ $0 \to (V_3^K)^* \to (V_2^K)^* \to (V_1^K)^* \to 0$ exact $\forall K \in \mathcal{K}^*(G)$

R injective in R-mod., as R is a field

$\overset{(ii)}{\Longrightarrow}$ $0 \to \tilde{V}_3^K \to \tilde{V}_2^K \to \tilde{V}_1^K \to 0$ exact $\forall K \in \mathcal{K}^*(G)$

$\Longrightarrow$ $0 \to \hat{V}_3 \to \hat{V}_2 \to \hat{V}_1 \to 0$ exact □

Corollary 28: Suppose $(*)$ for (R, G). Let $(\pi, V) \in R_R(G)$. (i) T.a.e.:

1° (π, V) admissible

2° $(\tilde{\pi}, \tilde{V})$ —"—

3° $V \to \tilde{\tilde{V}}$, $v \mapsto \delta(v)$ $\quad(\delta(v)(f) = f(v))$ is bijective.

(ii) Let (π, V) be admissible. Then (π, V) is irred. iff $(\tilde{\pi}, \tilde{V})$ is irreducible.

Proof: (i) $1^\circ \Leftrightarrow 2^\circ$ follows from 27 (ii)

$1^\circ \Rightarrow 3^\circ$ For $K \in \mathcal{K}^*(G)$ δ restricts to $\delta^K : V^K \hookrightarrow (\tilde{\tilde{V}})^K \simeq (\tilde{V}^K)^* \simeq ((V^K)^*)^*$ which is surjective, as $\dim_R V^K < \infty$.

K arbitrary $\Longrightarrow$ δ is surjective.

$3^\circ \Rightarrow 1^\circ$ exercise

(ii) follows from the exactness of $(\tilde{\ })$ and 3°. □

Remark 29: One can prove for $R = \mathbb{C}$ that all irred. repr. are admissible.

Example 30: Take a non-zero repr. $(\pi, V) \in R_R(G)$. Then $U := \bigoplus_{i=1}^{\infty} (\pi, V)$ is not admissible, because

$$U^K = \bigoplus_{i=1}^{\infty} V^K \quad \text{is not finite}$$

dimensional for small enough K.

I 2.2. Haar measure.

Def 31: An R-linear map
$\mu : \mathcal{C}_c^\infty(G, R) \longrightarrow R$ is called a measure on G.
μ is called a Haar measure if it is invariant under G-left translation.

Rk 32: (a) μ induces an additive map on the set of copen subsets of G:
$X \subseteq G$ copen
$\mu(X) := \mu(\mathbb{1}_X)$

(b) For $f \in \mathcal{C}_c^\infty(G, R)$, $g \in G$.
$(\ell(g) f)(h) := f(g^{-1} h)$.
μ Haar-measure means
$\mu(\ell(g) f) = \mu(f) \quad \forall f \in \mathcal{C}_c^\infty(G, R) \; \forall g \in G$.

(c) $\mu(f)$ is just a finite sum:

$f \in \mathcal{C}_c^\infty(G, R) \quad f = \sum_{i=1}^{k} a_i \mathbb{1}_{g_i K}$
Then $\mu(f) = \sum_{i=1}^{k} a_i \mu(K)$.

(d) And for $K' \geq K$ s.t. ex $[K' : K]$ we get $\mu(K') = [K' : K] \mu(K)$.

(e) More general K_1, K_2 copen subgp
$[K_1 : K_2] := [K_1 : K_1 \cap K_2] / [K_2 : K_1 \cap K_2]$

If $K_1, K_2 \in \mathcal{K}^*(G)$ then
$\mu(K_1) = [K_1 : K_2] \; \mu(K_2)$

Notation 33: $\mu(f) =: \int_G f(g) \, d\mu(g)$.

Note: Supposing (*) (G, R) a Haar-measure always exists and is unique upto scalar by Rk 32 (c).
Given μ we define for $g \in G$ and $f \in \mathcal{C}_c^\infty(G, R)$: $\mu'(f) := \mu(r(g) f)$.

Then μ' is a Haar measure, i.e.
$\exists\ \underbrace{\delta_G(g)}\in R^\times : \quad \mu' = \delta_G(g)\,\mu$

<u>modulus character</u>
of G to R.

For $K \in \mathcal{K}^*(G)$ we have

$$\delta_G(g)\,\mu(\mathbb{1}_K) = \mu(r(g)\mathbb{1}_K)$$

$$= \mu(\mathbb{1}_{gKg^{-1}})$$

$$= [gKg^{-1}:K]\,\mu(K).$$

So $\delta_G(g) = [gKg^{-1}:K]$.

<u>Example 34:</u> $G = \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \middle|\ a \in F^\times,\ x \in F \right\}$
<u>metaplectic group.</u>

Let μ be the Haar measure
s.t. $\mu\Big(\underbrace{\begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ 0 & 1 \end{pmatrix}}_{K}\Big) = 1.$

$$\delta_G\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \left[\begin{pmatrix} \mathcal{O}^\times & a\mathcal{O} \\ 0 & 1 \end{pmatrix} : K\right]$$

$$= [a\mathcal{O}:\mathcal{O}] \underset{\substack{\uparrow \\ q := |R_F|}}{=} q^{-\nu_F(a)}.$$

$$= |a|_F.$$

<u>Def 35:</u> (unimodular) G is called unimodular if $\delta_{G,\mathbb{Q}}$ is the trivial character, i.e. $\delta_{G,\mathbb{Q}}(g) = 1\ \forall g \in G$.

<u>Example 36:</u> $\mathbb{G}/_F$ reductive then $G = \mathbb{G}(F)$ is unimodular.
The metaplectic group is not unimodular.

<u>Remark 37:</u> Suppose $(*)(R, G)$.
We have a map

$$\mathbb{Q}_{\ell\nmid} := \left\{ \tfrac{a}{b} \,\middle|\, a,b \in \mathbb{Z},\ b \neq 0,\ \ell \nmid a\cdot b \right\} \xrightarrow{\varphi} R^\times$$

$$\tfrac{a}{b} \longmapsto (a\cdot 1_R)(b\cdot 1_R)^{-1}.$$

and $\delta_{G,R} = \varphi \circ \delta_{G,\mathbb{Q}}$.

So we only need $\delta_{G,\mathbb{Q}}$ and write δ_G for $\delta_{G,\mathbb{Q}}$.

I 2.3. Hecke algebra of a locally profinite group

We suppose $(*)(R,G)$. We fix a Haar measure μ.

<u>Def 38:</u> (Hecke algebra of G over R.)

$$\mathcal{H}_R(G) := \mathcal{C}_c^\infty(G,R) = \left\{ f: G \to R \;\middle|\; \begin{array}{l} f \text{ loc. const.} \\ \operatorname{supp}(f) \text{ compact} \end{array} \right\}$$

The product on $\mathcal{H}_R(G)$ is called <u>convolution</u> and defined as follows:

$$(f_1 * f_2)(g) := \int_G f_1(x) f_2(x^{-1}g)\, d\mu(x).$$

<u>Prop 39:</u> $(\mathcal{H}_R(G), *)$ is an R-algebra.

<u>Proof:</u> We only show the associativity of $*$.

$$((f_1 * f_2) * f_3)(g) = \int_G (f_1 * f_2)(x) f_3(x^{-1}g)\, d\mu(x)$$

$$= \int_G \left(\int_G f_1(y) f_2(y^{-1}x)\, d\mu(y) \right) f_3(x^{-1}g)\, d\mu(x).$$

$$\underset{\substack{\uparrow\\ \text{finite sums}}}{=} \int_G \int_G f_1(y) f_2(y^{-1}x) f_3(x^{-1}g)\, d\mu(x)\, d\mu(y)$$

$$\underset{\substack{\uparrow\\ z=y^{-1}x}}{=} \int_G f_1(y) \int_G f_2(z) f_3(z^{-1}y^{-1}g)\, d\mu(z)\, d\mu(y)$$

$$= \int_G f_1(y) (f_2 * f_3)(y^{-1}g)\, d\mu(y)$$

$$= (f_1 * (f_2 * f_3))(g). \qquad \square$$

<u>Rk 40:</u> We will see that $(\pi, V) \in R_R(G)$ is an $\mathcal{H}_R(G)$-module and a $R[G]$-module.
$\mathcal{H}_R(G)$ can be better computed than $R[G]$. (by a filtration of spherical Hecke algebras)
For G finite we have $\mathcal{H}_R(G) = R[G]$

<u>Lem 41:</u> If G is infinite then $\mathcal{H}_R(G)$ has a lot of idempotents, namely
$$e_K := \frac{1}{\mu(K)} \mathbb{1}_K, \quad K \leq G \text{ c open.}$$

<u>Proof:</u> $(e_K * e_K)(g) = \int_K \frac{1}{\mu(K)^2} \mathbb{1}_K(x^{-1}g)\, d\mu(x)$
$$= \int_K \frac{1}{\mu(K)^2} \mathbb{1}_{gK}(x)\, d\mu(x)$$
$$= \frac{1}{\mu(K)^2} \mu(K \cap gK) = e_K(g). \qquad \square$$

A smooth representation $(\pi, V) \in R_R(G)$ can be interpreted as a module over $\mathcal{H}_R(G)$: $v \in V$ Take $K \in K^*(G)$ s.t. $v \in V^K$

$$\pi(f) v := \sum_{gK \in \mathrm{supp}(f)} \mu(K) f(g) \pi(g) v$$

for $f \in \mathcal{H}_R(G)$ right K invariant (does not depend on K)
We write
$$\pi(f) := \int_G f(g) \pi(g)\, d\mu(g).$$

<u>Example 42:</u> $K \in K^*(G)$, $(\pi, V) \in R_R(G)$.

$$\pi(e_K) : V \longrightarrow V$$
$v \in V^K$: $\pi(e_K) v = \int_G \frac{1}{\mu(K)} \mathbb{1}_K(g) \pi(g) v\, d\mu(g)$

$$= \frac{1}{\mu(K)} \int_K v \, d\mu(g) = v.$$

Take $v \in V(K)$, i.e. $\exists v_1, \dots, v_t \in V \ \exists k_1, \dots, k_t \in K$

$v = (v_1 - k_1 v_1) + \dots + (v_t - k_t v_t)$.

On the first term we obtain

$$\pi(e_K)(v_1 - k_1 v_1) \overset{K' \trianglelefteq K \text{ s.t. } v_1 \in V^{K'}}{=} \int_G \frac{1}{\mu(K)} \mathbb{1}_K(g)\, \pi(g)(v_1 - \pi(k_1)v_1)\, d\mu(g)$$

$$= \frac{1}{\mu(K)} \sum_{gK' \subseteq K} \pi(g) v_1 - \frac{1}{\mu(K)} \sum_{gK' \subseteq K} \pi(gk_1) v_1$$

$= 0.$

So $\pi(e_K) v = 0$. So $\pi(e_K)$ is the projection onto V^K.

<u>Prop. 43:</u>

(a) Let $(\pi, V) \in R_R(G)$. Then V is an $\mathcal{H}_R(G)$ module s.t. $\forall v \in V \ \exists e \in \mathcal{H}_R(G)$ idempotent s.t. $ev = v$.

(<u>non-degenerate</u> $\mathcal{H}_R(G)$-module, or smooth, or unital)

(b) $$R_R(G) \longrightarrow \mathcal{H}_R(G)\text{-Mod}$$ (cat. of unital $\mathcal{H}_R(G)$ modules)

$$(\pi, V) \longmapsto V$$

is an equivalence of categories

<u>Proof:</u> (a) Exercise: Show that

$$\pi(f_1 * f_2) v = \pi(f_1)(\pi(f_2) v).$$

For $v \in V^K$ take e_K : $\pi(e_K) v = v$.

$(K \in \mathcal{K}^o(G))$

(b) Let W be a unital $\mathcal{H}_R(G)$-module. We define a representation on W. $w \in W$. Take $K \in \mathcal{K}^o(G)$ s.t. $e_K w = w$. We define:

$$\pi(g) w := \left(\frac{1}{\mu(K)} \mathbb{1}_{gK} \right) . w.$$

Then $(\pi, W) \in R_R(G)$ □

Def 44: $K \in K^*(G)$.

$$\mathcal{H}_R(G,K) := e_K * \mathcal{H}_R(G) * e_K$$

$$= \{ f \in C_c^\infty(G,R) \mid f \text{ is left and right } K\text{-invariant} \}$$

Then V^K is an $\mathcal{H}_R(G,K)$ module, because $e_K * v = v \ \forall v \in V^K$.

Prop 45: Let $(\pi, V) \in R_R(G)$ be irreducible and $K \in K^*(G)$. Suppose $V^K \neq 0$. Then V^K is a simple $\mathcal{H}_R(G,K)$ module

Proof: Let W be a non-zero $\mathcal{H}_R(G,K)$-submodule of V^K. Then $\mathcal{H}_R(G)W = V$ as V is irreducible.

$$\Rightarrow V^K = e_K * V = e_K * \mathcal{H}_R(G) W$$
$$\underset{\uparrow}{=} e_K * \mathcal{H}_R(G) * e_K W = W$$
$$e_K W = W$$

$\square$

Example 46: We compute $\mathcal{H}_{\mathbb{C}}(\mathbb{Q}_p)$

$$\mathcal{H}_{\mathbb{C}}(\mathbb{Q}_p, \mathbb{Z}_p) = \mathbb{C}[\mathbb{Q}_p/\mathbb{Z}_p]$$

(Note that $1_{x+\mathbb{Z}_p} * 1_{y+\mathbb{Z}_p} = 1_{x+y+\mathbb{Z}_p}$.)

We have $\mathcal{H}_{\mathbb{C}}(\mathbb{Q}_p, p^{+1}\mathbb{Z}_p) \simeq \mathbb{C}[\mathbb{Q}_p/p\mathbb{Z}_p]$

$$\cup \qquad \qquad \varphi \uparrow$$

$$\mathcal{H}_{\mathbb{C}}(\mathbb{Q}_p, \mathbb{Z}_p) \simeq \mathbb{C}[\mathbb{Q}_p/\mathbb{Z}_p]$$

$$\varphi([a]_{\mathbb{Z}_p}) := \underbrace{\sum_{i=0}^{p-1} [a+i]_{p\mathbb{Z}_p}}$$

This is a sum in the group algebra $\mathbb{C}[\mathbb{Q}_p/p\mathbb{Z}_p]$ and NOT in $\mathbb{Q}_p/p\mathbb{Z}_p$.

φ is then extended to $\mathbb{C}[\mathbb{Q}_p/\mathbb{Z}_p]$ via

$$\varphi\left(\sum_{j=1}^{A} \lambda_j [a_j]_{\mathbb{Z}_p}\right) := \sum_{j=1}^{A} \lambda_j \varphi([a_j]_{\mathbb{Z}_p}).$$

$(\lambda_j \in \mathbb{C})$

Now $\mathcal{H}_{\mathbb{C}}(\mathbb{Q}_p) = \bigcup_{k=1}^{\infty} \mathcal{H}_{\mathbb{C}}(\mathbb{Q}_p, p^k\mathbb{Z}_p)$

$$= \varinjlim_{k} \mathbb{C}[\mathbb{Q}_p/p^k\mathbb{Z}_p]$$

<u>Exercise 47:</u> Try to compute $\mathcal{H}_{\mathbb{C}}(\mathbb{Q}_p^\times)$.

I. 2.4. Coinvariants.

$(\pi, V) \in \mathcal{R}_R(G)$. $H \leq G$ closed.

H-invariants : V^H biggest subspace of V on which H acts trivially

H-coinvariants: V_H biggest quotient of V on which H acts trivially.

$$V_H := V/\mathrm{span}_R\{v - hv \mid h \in H, v \in V\}$$

$$= V/V(H).$$

<u>Rk 48:</u> Let $H \in \mathcal{K}^*(G)$. Then

$$V^H \longrightarrow V_H \qquad v \longmapsto [v]_{V(H)}$$

is an R-linear isomorphism.

<u>Example 49:</u> (a) Let $(\pi, V) \in \mathcal{R}_{\mathbb{C}}(GL_n(F))$ be a finite dim. irreducible representation. Then one can show that (π, V) is

a character, i.e. $\exists\ \chi : F^\times \longrightarrow \mathbb{C}^\times$ smooth (open kernel) s.t.

$$\pi = \chi \circ \det \quad \text{and } V \simeq \mathbb{C}.$$

Take $H := \begin{pmatrix}1 & F\\ & 1\end{pmatrix}$.

$\forall z \in \mathbb{C}\ \forall h \in H: \quad \pi(h)z = \chi(\det(h))z = z$

So $V(H) = 0$ and $V_H = V/V(H) = V = \mathbb{C}$.

(a) $G = GL_2(\mathbb{F}_2) \simeq \mathfrak{S}_3$ (gp of permutations on 3 elts.)

Let $(\rho, \mathbb{C}) \in R_{\mathbb{C}}(G)$ be the sign character, e.g.

$$\rho\left(\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}\right) = -1.$$

$H := \begin{pmatrix}1 & \mathbb{F}_2\\ & 1\end{pmatrix} = \left\{\begin{pmatrix}1 & \\ & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ & 1\end{pmatrix}\right\}$

G is finite, so H is open in G, so

$V_H \simeq V^H = \{ z \in \mathbb{C} \mid \underbrace{\mathrm{sign}\begin{pmatrix}1 & 1\\ & 1\end{pmatrix} z}_{-z} = z \} = 0.$

(c) $G = GL_2(\mathbb{Q}_2)$

$K := GL_2(\mathbb{Z}_2), \quad Z := \mathbb{Q}_2 I_2 \quad \left(I_2 = \begin{pmatrix}1 & \\ & 1\end{pmatrix}\right)$

Take $(\rho, \mathbb{C}) \in R_{\mathbb{C}}(GL_2(\mathbb{F}_2))$ from (a).

$(\tilde{\rho}, \mathbb{C}) \in R_{\mathbb{C}}(K)$ is defined via

$$\tilde{\rho}(k)v := \rho(\bar{k})v \quad \left(\begin{matrix} k \longmapsto \bar{k} \\ K \twoheadrightarrow GL_2(\mathbb{F}_2)\end{matrix}\right)$$

($\tilde{\rho}$ is called <u>inflation of ρ to K</u>)

Define $(\Lambda, \mathbb{C}) \in R_{\mathbb{C}}(ZK)$

$$\Lambda(zk)v := \tilde{\rho}(k)v$$

$$\pi := \operatorname{c-ind}_{ZK}^{G} \Lambda.$$

We compute for $H := \begin{pmatrix}1 & \mathbb{Q}_2\\ & 1\end{pmatrix}$ the space V_H.

$h := \begin{pmatrix}1 & -1\\ & 1\end{pmatrix}$

$f := f_1 \qquad \in \operatorname{c-ind}_{ZK}^{G} \Lambda$ with $\mathrm{supp}(f) = ZK$

$f - h.f = 2f$

$$((h.f)(g) = f(gh) = \begin{cases} 0_{\mathbb{C}}, & g \notin ZK \\ \tilde{\rho}(g)\underbrace{\tilde{\rho}(h)}_{-1}, & g \in ZK \end{cases}$$

$= -f(g)$)

So $f \in V(H)$. Analogously $\begin{pmatrix}\varpi^a & \\ & 1\end{pmatrix} f_1 \in V(H)$.

for all $a \in R$. As $G = ZK\begin{pmatrix}\varpi^2 & \\ & 1\end{pmatrix}H$
We obtain that $V \subseteq V(H)$.
Thus $V_H = 0$.

Exercise 50: Take $\pi := \text{c-ind}_{ZK}^{GL_2(\mathbb{Q}_p)} \mathbb{1}$

for $K := GL_2(\mathbb{Z}_p)$, $Z = \mathbb{Q}_p I_2$.
Put $H := \begin{pmatrix}1 & \mathbb{Q}_p \\ & 1\end{pmatrix}$.
Compute V_H.

Prop 51: (i) The functor $(\ )_G : R_R(G) \longrightarrow R\text{-v.n.}$ is right exact.

(ii) Suppose that $G = \bigcup_{t=1}^{\infty} K_t$ with compact open subgroups $K_t \subseteq K_{t+1}$, $t \in \mathbb{N}$, $K_t \in K^\circ(G)$.
Then $(\ )_G$ is exact.

Proof: (i) $(\pi, V) \in R_R(G)$

$$\mathrm{Hom}_G(V, \mathbb{1}) \cong \mathrm{Hom}_R(V_G, R)$$
$$a \longmapsto \bar{a} \qquad (\bar{a}(\bar{v}) := a(v))$$

Take $0 \to V_1 \to V_2 \to V_3 \to 0$ exact.

$\Rightarrow$ $0 \to \mathrm{Hom}_G(V_3, \mathbb{1}) \to \mathrm{Hom}_G(V_2, \mathbb{1}) \to \mathrm{Hom}_G(V_1, \mathbb{1})$

$\qquad\quad \wr\vert \qquad\qquad\qquad \wr\vert \qquad\qquad\qquad \wr\vert$

$0 \to ((V_3)_G)^* \to ((V_2)_G)^* \to ((V_1)_G)^*$ exact.

$\Rightarrow$ $\quad V_{1G} \to V_{2G} \to V_{3G} \to 0$ exact

(ii) Let $V_1 \hookrightarrow V_2$ in $R_R(G)$.
We need to show that $V_1 \cap V_2(G) = V_1(G)$.
Let $v \in V_1 \cap V_2(G)$. Then
$\exists\, t \in \mathbb{N}$: $v \in V_2(K_t)$.
So $v \in V_1 \cap V_2(K_t) \underset{\substack{\uparrow \\ (\ )^{K_t} \text{ is exact}}}{=} V_1(K_t) \subseteq V_1(G)$

$\square$

Example 52: $G = GL_n(F)$.

$$H = \begin{pmatrix}1 & & * \\ & 1 & \\ 0 & & 1\end{pmatrix} = \bigcup_{i=1}^{\infty} \underbrace{\begin{pmatrix}1 & p^{-i} & p^{-2i} & \cdots & p^{-(n-1)i} \\ & 1 & p^{-i} & \cdots & p^{-(n-2)i} \\ & & 1 & \ddots & \vdots \\ & & & \ddots & p^{-i} \\ 0 & & & & 1\end{pmatrix}}_{K_i}$$

K_i is compact open in H. (not in G)

I 3. Z-compact representations

I 3.1. Characters

Def 53: A characte of G is a group homomorphism $\chi : G \longrightarrow R^\times$.
χ is called smooth if χ has an open kernel.

Rk 54: A smooth character of G can be reinterpreted as a 1-dim smooth representation of G.
Indeed: $\chi : G \longrightarrow R^\times$ smooth.
Define $(\pi, R) \in \mathcal{R}_R(G)$ via
$\pi(g).r := \chi(g) \cdot r$.

Prop 55: Let G be loc. profinite and
$\chi : G \longrightarrow \mathbb{C}^\times$ be a characte
T.a.e.: 1° χ is smooth
2° χ is continuous.

Proof: 1° $\Rightarrow$ 2° because χ is continuous w.r.t. discrete top. on $\mathbb{C}^\times$.

$2^\circ \Rightarrow 1^\circ$ Let $\varepsilon := \frac{1}{2}$. $B_{\frac{1}{2}}(1) = \{z \in \mathbb{C} \mid |z-1| < \frac{1}{2}\}$

$B_{\frac{1}{2}}(1)$ doesn't contain any non-trivial compact subgroup.
$\chi^{-1}(B_{\frac{1}{2}}(1))$ is open $\Rightarrow \exists K \leq G$ open: $\chi(K) \subseteq B_{\frac{1}{2}}(1)$. Thus $\chi(K) = \{1\}$
So χ is smooth. $\square$

<u>Def 56:</u> A character $\chi : G \to \mathbb{C}^\times$ with $\mathrm{im}(\chi) \subseteq S^1$ is called <u>unitary</u>.

<u>Rk 57:</u> If G is profinite then every smooth character of G to $\mathbb{C}^\times$ is unitary.

<u>Example 58:</u> F non-Arch loc. field.

(a) $G := F^\times = \varpi^{\mathbb{Z}} \cdot \mathcal{O}_F^\times \cong \mathbb{Z} \times \mathcal{O}_F^\times$
$\cong \mathbb{Z} \times \langle \zeta \rangle \times (1 + \mathfrak{p}_F)$

$q-1$ th primitive root of unity.

Thus $\chi : G \to R^\times$ smooth character is given by a triple (r_1, ζ, χ_1)

where $r_1 \in R^\times$, $\zeta \in \mu_{q-1}(R)$, $\chi_1 : 1 + \mathfrak{p}_F \to R^\times$ a smooth character.

$\chi(\varpi^a \cdot \zeta^b \cdot (1+x)) := r_1^a \zeta^b \chi_1(1+x)$.

(b) $G = (F, +)$. $p := \mathrm{reschar}(F)$.

(b-1) If $p = \ell$. then the only smooth character of G is the trivial character

(b-2) If $p \neq \ell$. We construct a non-trivial character of $F/\mathfrak{p}_F$ if R is algebraically closed.

Pf: Take a non-trivial character of $k_F = \mathcal{O}_F/\mathfrak{p}_F$, say $\chi_0 : k_F \to R^\times$.
(It exists as $k_F \cong \mathbb{Z}/p \oplus \dots \oplus \mathbb{Z}/p$ as $\mathbb{F}_p$-v.s., and $\mu_p(R) \neq \emptyset$)

Suppose we have defined $\chi_{-i} : \mathfrak{p}_F^{-i}/\mathfrak{p}_F \to R^\times$

extending χ_0.
Take $y \in \mathfrak{p}_F^{-(i+1)} \setminus \mathfrak{p}_F^{-i}$ Then $py \in \mathfrak{p}_F^{-i}$. Take a pth root of r $\chi_{-i}(py)$. and put $\chi_{-i}(\bar{y}) := r$.

This extends χ_{i} to $\mathbb{Z}y + p^{-i}/p$

continue this way until we reach p^{i+1}/p

So by induction we get $\chi : F/_{\varphi_p} \to R^\times$ non-trivial

and we inflate to $\chi : F \longrightarrow R^\times$.

$\square$

<u>Prop 59:</u> Let A be an abelian group and $\chi_B : B \longrightarrow R^\times$ be a character on a subgroup B of A.

Then $\exists\ \chi_A : A \longrightarrow R^\times$ character such that $\chi_A|_B = \chi_B$ if R is algebraically closed.

<u>Proof:</u> Similar to Example 58 (b2). $\square$

<u>Theorem 60:</u> (Pontryagin – van Kampen duality theorem)

Let A be an LCA group (locally compact, abelian, T^2)

Put $A^{pd} := \{\chi : A \longrightarrow S^1 \mid \chi$ continuous character (w.r.t. Eucl. top. on S^1)$\}$, called

"Pontryagin dual of A".

Then the map

$$\begin{array}{ccc} A & \longrightarrow & (A^{pd})^{pd} \\ a & \longmapsto & (\chi \longmapsto \chi(a)) \end{array}$$

is a topological group isomorphism.

<u>Rk 61:</u> In Thm 60 we consider on A^{pd} the compact open topology.

It is generated by

$$B(K,U) := \{\chi \in A^{pd} \mid \chi(K) \subseteq U\}$$

for $K \subseteq A$ compact and $U \subseteq S^1$ open.

It is the topology of uniform convergence on compact sets.

<u>Example 61:</u> (a) $A = \mathbb{R}$, $A^{pd} = S^1$.

(b) $A = \mathbb{Z}/m\mathbb{Z}$, $A^{pd} = (\mathbb{Z}/m\mathbb{Z})^{pd}$
$$= \{e^{\frac{2\pi i}{m}k} \mid k = 0, \ldots, m-1\}$$
$$\simeq \mathbb{Z}/m\mathbb{Z}$$

(c) $A = (F, +)$ with $|\cdot|_F$ topology.
Then
$$\{\chi : F \to \mathbb{C}^\times \mid \chi \text{ smooth}\} = F^{pd}$$
by Rk 57. Take $\psi \in F^{pd} \setminus \{1\}$.
Claim: $\quad F^{pd} = \{\psi_a \mid a \in F\}$
$$\psi_a(x) := \psi(ax).$$

<u>Proof:</u> Put $B = \{\psi_a \mid a \in F\} \subseteq F^{pd}$

<u>Step 1:</u> Then $\quad F \xrightarrow{\sim} B$ as top. groups
$$a \longmapsto \psi_a$$

(uniform conv. on compact sets
$\hat{=}$ convergence in F)

<u>Step 2:</u>
$$\begin{tikzcd}
a \arrow[r, mapsto] & (x \mapsto x(a)) \\
F \arrow[r, "\sim"] \arrow[d, equal] & (F^{pd})^{pd} \arrow[d, "\mathrm{Res}"] \\
F \arrow[r, hook] & B^{pd} \\
a \arrow[r, mapsto] & (\psi_{a'} \longmapsto \psi_{a'}(a) = \psi(aa'))
\end{tikzcd}$$

Res is surjective as F^{pd} is the closure of B.
Thus Res is bijective, as all other maps are injective.

So $B^{pd} \overset{\uparrow}{\simeq} F^{pd}$
$\quad \| \text{ Step 1}$
$\{\psi_{a'} \mapsto \psi(aa') \mid a \in F\}$

So $F^{pd} = \{a' \mapsto \psi(aa') \mid a \in F\}$
$$= \{\psi_a \mid a \in F\} = B.$$
□

<u>Prop. 62:</u> Let $\psi : (F, +) \longrightarrow R^\times$ nontrivial smooth character. Then $\mu_{p^\infty}(R) \neq \{1_R\}$ and every smooth character has the form $\psi(a\,\cdot)$ for some $a \in F$.

<u>Proof:</u> We have $\mu_{p^\infty}(R) \hookrightarrow S^1$.
Now apply 61(c) □

For $p=\ell$ has a big difference to $p\neq\ell$:

<u>Lem 6.3:</u> Suppose G is a pro-p-group and $p=\ell$. Suppose $(\pi,V)\in R_R(G)$ is irreducible.
Then π is the trivial character

<u>Proof:</u> G compact and (π,V) irred, so $\dim_R V<\infty$ and $G/\ker(\pi)$ is finite.

Thus w.l.o.g. G is a p-group and $\pi: G\longrightarrow \mathrm{Aut}_R(V)$ is injective.
<u>Assume</u> $G\neq\{1_G\}$.

G p-group $\Rightarrow Z(G)\neq\{1_G\}$.
Take $g\in Z(G)\setminus\{1_G\}$ s.t. $g^p=1$.

$$R[X]\xrightarrow{\varphi}\mathrm{End}_{R[G]}V$$
$$X\longmapsto \pi(g)$$

satisfies: • $\mathrm{im}\,\varphi$ is an integral domain (as $\mathrm{End}_{R[G]}V$ is a skew-field, as V is irreducible)

• $\mathrm{im}\,\varphi$ is an R-module of finite type, generated by $\varphi(1),\varphi(X),\varphi(X^2),\dots,\varphi(X^{p-1})$

Thus $\ker(\varphi)$ is a maximal ideal containing $(X^p-1)=(X-1)^p$.
So $\ker(\varphi)=(X-1)$
and $R[X]/\ker(\varphi)=R$, so
$\varphi(X)$ acts a scalar, i.e. $\pi(g)=r\,\mathrm{id}_V$ for some $r\in R^\times$.
$g^p=1\Rightarrow 0=r^p-1=(r-1)^p\Rightarrow r=1_R$
A contradiction, because π is an injective map. □

I 3.2. Schur's lemma

Here G is loc. profinite

Lemma 64: (Schur)

Let $(\pi, V), (\pi', V') \in R_R(G)$ irreducible.

(i) $\mathrm{Hom}_G(\pi, \pi') \neq 0 \iff \pi \simeq \pi'$

(ii) Suppose R is alg. closed and $\exists K$ compen $\leq G$ such that $0 < |\dim_R V^K| < |R|$.

Then $\mathrm{End}_G(\pi) = R$.

Proof: (i) "$\Rightarrow$" $f \in \mathrm{Hom}_G(\pi, \pi')$

$\ker(f) \subseteq V$ and $\mathrm{im}(f) \subseteq V'$ are subrepr.

V, V' irred and $f \neq 0 \Longrightarrow \ker(f) = 0$ and $\mathrm{im}(f) \overset{\shortparallel}{V'}$

(ii) Take $v \in V^K \setminus \{0\}$.

$$\mathrm{End}_G(\pi) \overset{\alpha}{\hookrightarrow} V^K$$
$$f \longmapsto f(v).$$

Thus $\dim_R(\mathrm{End}_G(\pi)) \leq |\dim_R V^K| < |R|$.

Assume $\exists f \in \mathrm{End}_G(\pi) \setminus R$.

$$R[X] \xrightarrow{\varphi} \mathrm{End}_G(\pi)$$
$$Q(X) \longmapsto Q(f).$$

is not injective, because

the quotient field $R(X)$ has R-dimension $\geq |R|$, because $\{\frac{1}{X-\lambda} \mid \lambda \in R\}$ is linearly independent.

Thus $\ker \varphi$ is a maximal ideal of $R[X]$.

R algebraically closed

$\Rightarrow \quad m = (X - \lambda)$ for some $\lambda \in R$

$\Rightarrow \forall v \in V: \quad f(v) = \lambda v.$ ↯ $\square$

Corollary 65: Let R be alg closed, $(\pi, V) \in R_R(G)$ irreducible.

Suppose one of the conditions (a) or (b).

(a) (π, V) is admissible

(b) $|R| > |V|$ and

$\exists K \leq G$ compen: G/K is countable.

Then $\mathrm{End}_G(\pi) = R$.

Example 66: 65(b) is satisfied for $G = GL_n(F)$ and $R = \mathbb{C}$.

E.g.: $n = 2$, $F = \mathbb{Q}_p$.

Take $K := GL_2(\mathbb{Z}_p)$.

Iwasawa decomposition $KB = G$

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ Borel subgroup

$\varpi := p$

$$K\begin{pmatrix}\varpi^a & u\\ & \varpi^b\end{pmatrix} = K\begin{pmatrix}\varpi^{a'} & v\\ & \varpi^{b'}\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}\varpi^{a-a'} & -v\varpi^{-a'-b'+a} + u\varpi^{-b'}\\ 0 & \varpi^{b-b'}\end{pmatrix} \in K$$

$$\Leftrightarrow \quad a=a' \wedge b=b' \wedge u-v\in\varpi^{b'}\mathbb{Z}_p$$

We have only cöntabley many choices for $(a,b)\in\mathbb{Z}\times\mathbb{Z}$ and $[u]_{\varpi^b\mathbb{Z}_p} \in \mathbb{Q}_p/\varpi^b\mathbb{Z}_p$.

In the abelian case an irreducible representation often must be a character.

<u>Prop. 67:</u> Suppose G is abelian and R is alg. closed

(i) If $(\pi,V)\in R_R(G)$ is f. d. and irred. then π is a character

(ii) Suppose $G \cong G_0\times\mathbb{Z}^r$ with G_0 profinite. Then every irred repr. of G is a character.

<u>Example 68:</u> (for 67 (ii))

The center of $GL_n(F)$ is $F^\times \cong \mathcal{O}_F^\times\times\varpi^{\mathbb{Z}}$. So every irred repr. of $F^\times$ is a character.

(of 67)

<u>Proof:</u> (i) follows from 65 (a)

(ii) Take $v\in V\setminus\{0\}$ and $K_0\leq G_0$ open s.t. $v\in V^{K_0}$.

Then G_0/K_0 is finite, so $R[G_0]v$ is a finite dimensional. So $R[G_0]v$ contains an irreducible G_0 repr., which is a character χ by (i).

$$V(\chi,G_0) := \{v\in V \mid \forall_{g\in G_0}: \pi(g)v=\chi(g)v\}$$

$V(\chi,G_0)$ is a non-zero G subrepr. of V. $\underset{\text{irr.}}{\Longrightarrow}$ $V(\chi,G_0)=V$.

So G_0 acts as a character. So w.l.o.g. $G=\mathbb{Z}^r = \mathbb{Z}g_1\oplus\mathbb{Z}g_2\oplus\cdots\oplus\mathbb{Z}g_r$

Consider $R[X_1,\dots,X_r,Y_1,\dots,Y_r]$

$\quad q\downarrow$

$\quad V$

$P(X_1, X_2, \dots, X_r) \mapsto P(g_1, \dots, g_r, g_1^{-1}, \dots, g_r^{-1})v$

Then $\operatorname{im} \varphi = V$ is v generates V.

$\ker \varphi = \mathfrak{M}$ a maximal ideal of $R[X_i, Y_i, i = 1, \dots, r]$.

R alg. closed

$\underset{HNS}{\Longrightarrow} \exists\, s_1, \dots, s_r, t_1, \dots, t_r \in R$:

$$\mathfrak{M} = (X_1 - s_1, \dots, Y_r - t_r)$$

Thus $g_1, \dots, g_r$ act as a scalar on V.

So (π, V) is a character $\square$

I 3.3. Z - compact representations

Z-compact is the interior description of cuspidality.

Note: For $(\pi, V) \in R_R(G)$ we have the natural pairing

$$\langle\,,\rangle : \tilde{V} \times V \longrightarrow R$$

Def 69: (matrix coeff.)

$(\pi, V) \in R_R(G)$, $v \in V$, $\tilde{v} \in \tilde{V}$

The map

$$\varphi_{v,\tilde{v}} \; G \longrightarrow R$$
$$g \longmapsto \langle \tilde{v}, \pi(g) v \rangle$$

is called matrix coefficient associated to $v, \tilde{v}$.

Rk 70: Take K s.t. $v \in V^K$, $\tilde{v} \in \tilde{V}^K$

Then $\operatorname{supp}(\varphi_{v,\tilde{v}}) = \bigcup_{i \in I} K g_i K$

Def 7.1: $f : G \to R$ is called compactly supported mod center $Z = \text{center}(G)$, if $\exists\, E$ compact : $\text{supp}(f) \subseteq Z E$.
$\hat{G}$

Def 7.2: $(\pi, V) \in R_R G$ is called Z-compact ($Z = \text{center of } G$), if $\forall v \in V\ \forall \tilde{v} \in \tilde{V}$: $\varphi_{v,\tilde{v}}$ is compactly supported mod Z.

Z-compact helps to study admissibility.

Prop 7.3: Suppose $(*)(R, G)$ and let $(\pi, V) \in R_R(G)$ with central character.

(i) T.a.e. 1° (π, V) Z-compact

2° $\forall K \in K^\sigma(G)\ \forall v \in V$: the map $g \mapsto \pi(e_K)\pi(g)v$ is compactly supported mod center.

(ii) If (π, V) is Z-compact and f.g. then (π, V) is admissible.

Pf: (i) $1^\circ \Rightarrow 2^\circ$ Take $K \in K^\sigma(G)$ $\overset{\vee}{v \in V}$. Put $E := \text{span}_R\{\pi(e_K) g v \mid g \in G\}$

Take an R-basis of E: $\{\underbrace{\pi(e_K) g_i v}_{=: v_{g_i}} \mid i \in I\}$

and extend to a basis B of V^K.
Now take $\lambda \in \tilde{V}$ s.t. $\lambda|_{V(K)} \equiv 0$ and $\lambda|_B \equiv 1$. (Note $\lambda \in \tilde{V}^K = (V^K)^*$).
Then $\text{supp}(\varphi_{v,\lambda})$ is a union of double cosets $K g K' Z$ with $K' \in K^\sigma(G)$ s.t. $v \in K'$

(π, V) Z-compact $\Rightarrow$ only finitely many such double cosets occur.
$\Rightarrow$ B is finite
$\Rightarrow$ $\dim_R E < \infty$
(This already shows (ii))
Let λ_i, $i \in I$, be elts of $\tilde{V}^K$ s.t. $\lambda_i|_E$ is the dual basis of $\{v_{g_i} \mid i \in I\}$.

Then $\text{supp}(g \mapsto \pi(e_K) g v)$
$= \bigcup_{i \in I} \text{supp}(\varphi_{v,\lambda_i})$
is compact mod center.

$2^\circ \Rightarrow 1^\circ$ Take $v \in V$, $\tilde{v} \in \tilde{V}$ and $K \in K^\times(G)$ fixing v and $\tilde{v}$.
Then $\mathrm{supp}(c_{v,\tilde{v}}) \subseteq \mathrm{supp}(g \mapsto \pi(e_K) g v)$

$\square$

Cor 74: Suppose that
- R is alg. closed and uncountable and satisfies (*) and
- $\exists K \leq G$ cpt open : G/K is countable

Then every irred Z-compact repr. is admissible.

Example 75: $R = \mathbb{C}$ and $G = GL_n(F)$

Rk 76: The • assumptions in Cor 74 are there to ensure that an irred. repr. has a central character.

Most useful prop. in this context for p-adic reductive group is the following

Prop 76: Suppose (*) (G, R) and that G contains an open normal subgroup G_0 s.t. $[G : G_0 Z] < \infty$ and G_0 has comp. center.
Let $(\pi, V) \in \mathcal{R}_R(G)$.

(i) T.f.a.a.e.:

1° (π, V) is Z-compact

2° $\forall_{K \in K^\times(G)} \forall v \in V$: $g \mapsto \pi(e_K) g v$ has support compact mod center

3° $\pi|_{G_0}$ is a compact repr., i.e. all matrix coeff of $\pi|_{G_0}$ have compact support

(ii) Suppose that all irred R-repr. of Z are characters.
If (π, V) is Z-compact and irreducible then (π, V) is admissible and has a central character.

Example 77: For $G = GL_n(F)$.
Take $G_0 = \{g \in G \mid \det(g) \in \mathcal{O}_F^\times\}$
$Z = \left\{ \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix} \,\middle|\, x \in F^\times \right\}$

$[G : ZG_0] < \infty$

(You can find a set of representatives in $\left\{ \begin{pmatrix} \varpi^{a_1} & & \\ & \ddots \varpi^{a_{n-1}} & \\ & & 1 \end{pmatrix} \middle| \, n > a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq 0 \right\}$)

<u>Proof</u> (of Prop 76)

(i) $2^\circ \Rightarrow 1^\circ$ as in Prop 73.

$1^\circ \Rightarrow 3^\circ$ $\operatorname{supp}(\varphi_{v\tilde{v}})$ is a disjoint finite union of double cosets

$KgZK$ with $K \leq G_0$ c.open

s.t. $v \in V^K$ and $\tilde{v} \in \tilde{V}^K$.

If $(KgZK) \cap G_0 \neq \emptyset$ then we can assume $g \in G_0$.

Then $(KgZK) \cap G_0 = KgK\underbrace{(Z \cap G_0)}_{\text{compact}}$

is compact.

$3^\circ \Rightarrow 2^\circ$ W.l.o.g. G_0 is normal in G.

<u>Indeed</u>: Take system of representatives $\{g_1, \ldots, g_s\}$ of G/ZG_0 and replace G_0 by

$g_1 G_0 g_1^{-1} \cap \ldots \cap g_s G_0 g_s^{-1} = \bigcap_{g \in G} g G_0 g^{-1}$.

$\therefore \pi|_{G_0}$ is compact.

So for $K \in \mathcal{K}^c(G_0)$ and $v \in V$

$G_0 \ni g \longmapsto \pi(e_K) g v$ has compact support, denoted by $S(K, v)$.

Then the support of

$G \ni g \longmapsto \pi(e_K) g v$ is equal to

$\bigcup_{i=1}^{s} g_i S(g_i^{-1} K g_i, v) \cdot Z$.

(ii) (Waldspurger)

(π, V) irred, $[G : G_0 Z] < \infty$

so $\pi|_{G_0 Z}$ is f.g., say by

$X = \{v_1, \ldots, v_s\}$. Take $K \in \mathcal{K}^c(G_0)$, $V^K \neq 0$.

$W := \operatorname{span}_R \{\pi(e_K) g v_i \mid g \in G_0, i = 1, \ldots, s\}$

Then $V^K = \pi(e_K) V = \pi(Z) W$.

So V^K is f.g. Z-repr., has irred quotient which is a character χ (by assumption in (ii)).

So $V^K(\chi) := \{\pi(g)v - \chi(g)v \mid v \in V^K, g \in Z\}$ is a proper $\mathcal{H}(G,K)$ sub-module of V^K.
(π, V) irr. $\Rightarrow$ V^K simple !!!

So $V^K(\chi) = 0$.
$\Rightarrow$ Z acts on V^K as χ.
$\Rightarrow$ $\dim_R V^K < \infty$.
Further
$V(\chi)$ is G-stable and non-zero $\Rightarrow$ $V(\chi) = V$.
So (π, V) has a central character, which is χ. □

The next proposition explains how to construct Z-compact representations.

Def 78: Let G be a loc. profinite gp. (as granted)
$H \leq G$ open is called a compact mod center subgp of G if Z = center of G $\subseteq H$ and H/Z is compact.

Example: $G = GL_2(F)$

1) $F^\times GL_2(\mathcal{O}_F)$ is a compact mod center subgp of G.

2) E/F quadratic field extension of F.
Then $E \subseteq \mathrm{End}_F E \simeq M_2(F)$ and $E^\times \subseteq \mathrm{Aut}_F E \simeq GL_2(F)$ is a comp. mod Z subgp of G.

Prop 79: Let H be an open comp mod Z subgp of G and $(W, \rho) \in \mathcal{R}_R(H)$. Assume $(*)(R, G)$.
Suppose $\pi := c\text{-}\mathrm{ind}_H^G \rho$ is irred. and adm.
Then π is Z-compact.

<u>Proof:</u> $\tilde{\pi}$ is also irred. and adm.

So we only need to check one matrix coeff.
Take $w \in W \setminus \{0\}$ and $\tilde{w} \in \tilde{W} \setminus \{0\}$ such that $\tilde{w}(w) \neq 0$.

$$f_w(g) := \begin{cases} \rho(g) w, & g \in H \\ 0, & g \notin H \end{cases}$$

$v := f_w$ lies in $c\text{-}\mathrm{ind}_H^G W$.

Take $\tilde{v} \in \tilde{\pi}$ defined via
$$f \in c\text{-}\mathrm{ind}_H^G W \longmapsto \tilde{v}(f) := \tilde{w}(f(1))$$

The matrix coeff $\varphi_{v,\tilde{v}}$ satisfies
$$\varphi_{v,\tilde{v}}(g) = \tilde{v}(\pi(g)v) = \tilde{w}((\pi(g)f_w)(1)) = \tilde{w}(f_w(g))$$
So $\mathrm{supp}(\varphi_{v,\tilde{v}}) \subseteq H$, so is compact mod center. $\square$

Example 80: $M := \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \,\middle|\, \begin{matrix} a \in F^\times, \\ x \in F \end{matrix} \right\}$

$M = SN$ $\qquad S = \left\{ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \,\middle|\, a \in F^\times \right\}$

$$N = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \,\middle|\, x \in F \right\}$$

Let $\vartheta : N \longrightarrow \mathbb{C}^\times$ be a smooth character.
(This is the same as an additive character of F via $N \simeq F$)

$\Pi_c := c\text{-}\operatorname{ind}_N^M \vartheta$, $\Pi = \operatorname{ind}_N^M \vartheta$.

Claim: 1) Π_c is irreducible (no proof)
[Bushnell – Henniart, "GL(2)", §8.2]
2) Π_c is a proper subrepresentation of Π. So Π is not irreducible.
3) Π_c is not admissible

<u>Proof:</u> (Just the idea)

2) As linear space we have

$$\begin{array}{ccc} \pi_c = c\text{-}\mathrm{ind}_N^M \vartheta & \overset{\varphi}{\hookrightarrow} & C_c^\infty(F^\times) \\ \cap\!\!| & a & \cap\!\!| \\ \pi = \mathrm{ind}_N^M \vartheta & \overset{\varphi}{\hookrightarrow} & C^\infty(F^\times) \\ & f \longmapsto f|_S & \end{array}$$

$$\mathrm{im}(\varphi) = \{ f \in C^\infty(F^\times) \mid \exists_{m \in \mathbb{N}}\ f|_{F^\times \setminus (\varpi^{-m} \mathcal{O}_F)} \equiv 0 \}$$
$$\supsetneqq C_c^\infty(F^\times).$$

So $\pi / \pi_c \neq 0$ and thus π is not irreducible.

3) $$\overline{c\text{-}\mathrm{ind}_N^M \vartheta} \simeq \mathrm{ind}_N^M \tilde{\vartheta} \underset{\substack{\uparrow \\ \vartheta \text{ is } S \\ \text{conjugate to } \tilde{\vartheta}}}{\overset{61(c)}{\simeq}} \mathrm{ind}_N^M \vartheta = \pi$$

is not irreducible.

Cor. 28 $\Rightarrow$ π_c is not admissible $\square$

<u>Rk 81.</u> The group M is not a group of F-points of a connected reductive group defined over F.

We will see that for all latter groups that all complex irreducible representations are admissible

I.4 Cuspidal representations of reductive p-adic groups.

Steps in this section:

1) Def of cuspidal representations for reductive p-adic groups using parabolic induction and restriction
2) Introducing cuspidal support.
4) $(p \neq \ell)$ smooth irred repr. are admissible
3) $(p \neq \ell)$ Z-compact = cuspidal

I 4.1. reductive p-adic groups and parabolic induction

In the whole I.4. let G be the set of F-points of a connected reductive algebraic group $\mathbb{G}$ defined over F.
(Example: $G = GL_n(F)$, $\mathbb{G} = GL_n$)

Facts: (a) $(*)(R, G) \iff p \neq \ell$.
(b) For $K = \mathbb{G}(\mathcal{O}_F)$ we have that G/K is countable.
(c) $Z(G)$ = center of $G \simeq$ profinite gp $\times \mathbb{Z}^r$

Terminology 82: G is called a reductive p-adic group.

Let P be a parabolic subgroup of G.
$P = LN$ s.t.
L is the centralizer of a split torus (a closed subgroup $S \simeq F^\times \times \dots \times F^\times$ of G),
N is the maximal normal unipotent subgroup of P.

(Example: $G = GL_n(F)$, $P = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$

$$L = \begin{pmatrix} \boxed{*} & & & 0 \\ & \boxed{*} & & \\ & & \boxed{*} & \\ 0 & & & \boxed{*} \end{pmatrix}^{(n_1,\dots,n_d)} \qquad N = \begin{pmatrix} I_{n_1} & & & * \\ & I_{n_2} & & \\ & & \ddots & \\ 0 & & & I_{n_d} \end{pmatrix}$$

(For $GL_3(F)$: $\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$, $\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$, $\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$)

and all G-conjugates of P,

e.g. $\begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ & 0 & * & * \\ & & & * \end{pmatrix}^{(n_1,\dots,n_d)}$.

<u>Rk 83:</u> L is a reductive p-adic group

<u>Def 84:</u> (<u>parabolic induction functor</u>)

$$\begin{tikzcd} i^G_{L,P} : R_R(L) \arrow[rr] \arrow[dr, "\text{inflation}"'] & & R_R(G) \\ & R_R(P) \arrow[ur, "\mathrm{ind}^G_P"'] & \end{tikzcd}$$

· inflation: pullback via $P \longrightarrow P/N \simeq L$.

<u>Def 85:</u> (<u>parabolic restriction functor</u>)

$$\begin{tikzcd} r^G_{L,P} : R_R(G) \arrow[rr] \arrow[dr, "\mathrm{res}^G_P"'] & & R_R(L) \\ & R_R(P) \arrow[ur, "N\text{-coinv.}"'] & \end{tikzcd}$$

$r^G_{L,P}$ is also called <u>Jacquet</u> <u>restriction functor</u>.

<u>Example 86:</u> (a) $R = \mathbb{C}$, $G = GL_2(\mathbb{Q}_p)$
$K := GL_2(\mathbb{Z}_p)$
$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = LN$, $L = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$

$$r^G_{L,B}\big(\underbrace{\text{c-ind}^G_{ZK}\,\text{sign}}_{=:V}\big) = V_N = 0,$$

by 49(c).

(b) G, B as in (a).

$$r^G_{L,B}\big(\underbrace{\text{ind}^G_B \mathbb{1}}_{V}\big) = V_N \neq 0,$$

by 50.

<u>Prop 87</u>: For $\pi \in R_p(G)$ and $\rho \in R_R(L)$ we have a natural isomorphism of R-v.s.

$$\text{Hom}_L\big(r^G_{L,P}(\pi), \rho\big) \simeq \text{Hom}_G\big(\pi, i^G_{L,P}(\rho)\big)$$

<u>Proof</u>: $\text{Hom}_L(r^G_{L,P}(\pi), \rho) \simeq \text{Hom}_P(\text{res}^G_P(\pi), \text{infl}(\rho))$

$$\underset{\substack{\uparrow \\ \text{Thm 23}}}{\simeq} \text{Hom}_G(\pi, i^G_{L,P}(\rho)) \qquad \square$$

<u>Lem 88</u>: 1) $i^G_{L,P}$ is left exact and $r^G_{L,P}$ is right exact. Both are exact if $p \neq \ell$.

2) $r^G_{L,P}$ preserves finite type

<u>Proof</u>: 1) $i^G_{L,P}$ is a right adjoint and $r^G_{L,P}$ — " — left — " —
So $i^G_{L,P}$ is left exact and $r^G_{L,P}$ is right exact.

The exactness statement, in case of $\ell \neq p$, follows from 25, 51(ii) (see 52).

2) Let $(\pi, V) \in R_R(G)$ of finite type
$V = \text{span}_{R[G]}(\{v_1, \dots, v_s\})$

Take $K \leq G$ copen s.t. $v_1, \dots, v_s \in V^K$.
$P \backslash G$ is compact, so $P \backslash G / K$ is finite
$\quad = \{[g_1], \dots, [g_t]\}$

$\Rightarrow \{ g_i v_j \mid 1 \leq i \leq t,\ 1 \leq j \leq s \}$ generates $\text{Res}^G_P(\pi, V)$. So
$\{\overline{g_i v_j} \mid 1 \leq i \leq t,\ 1 \leq j \leq s\}$ generates $r^G_{L,P}(\pi, V)$. $\square$

<u>Def 89</u>: (a) (P, L) with a parabolic subgroup P of G and a Levi subgroup L s.t. $P = LN$ (N = unipotent radical of P) is called a <u>parabolic pair</u>.

$$(P, L) \preceq (P', L') \text{ iff } P \subseteq P' \text{ and } L \subseteq L'.$$

(b) Fix (P_0, L_0) $\preceq$-minimal.
We call (P, L) standard (w.r.t. (P_0, L_0)) if $(P_0, L_0) \preceq (P, L)$.

<u>Example 90</u>: $G = GL_3(F)$

$$(P_0, L_0) = \left(\begin{pmatrix} * & * & * \\ & * & * \\ 0 & & * \end{pmatrix}, \begin{pmatrix} * & & 0 \\ 0 & * & \\ & & * \end{pmatrix} \right)$$

The standard parabolic pairs are

$$\left(\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \right), \left(\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \begin{pmatrix} * & & \\ & * & * \\ & * & * \end{pmatrix} \right), (P_0, L_0)$$
and (G, G).

e.g. $(P_0, L_0) \preceq \left(\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix} \right) \preceq (G, G)$.

<u>Def 91</u> (cuspidal representation)

$(\pi, V) \in R_R(G)$ is called cuspidal if $\forall (P, L) \precneqq (G, G)$: $r^G_{L,P}((\pi, V)) = 0$.

<u>Rk 92</u>: By Prop. 87 "cuspidal" is equivalent to
$\forall_{(P,L) \precneqq (G,G)} : \nexists\, \rho \in R_R(L)$:

$$\mathrm{Hom}_G(\pi, i^G_{L,P}(\rho)) \neq 0.$$

<u>Def 93</u>: An irreducible representation $(\pi, V) \in R_R(G)$ is called supercuspidal if $\forall_{(P,L) \precneqq (G,G)}$ $\nexists\, \rho \in R_R(L)$: (π, V) is not a subquotient of $i^G_{L,P}(\rho)$.

<u>Rk 94</u>: irreducible supercuspidal
$\Rightarrow$ irreducible cuspidal
When $R = \mathbb{C}$, then "$\Leftarrow$" holds.
In general "$\Leftarrow$" doesn't hold

Thm 95: Let $(\pi, V) \in R_R(G)$.
Then $\exists$ (P,L) parabolic pair in G:
$\exists$ $\rho \in R_R(L)$ irreducible cuspidal:
$\mathrm{Hom}_G(\pi, i^G_{L,P}(\rho)) \neq 0$.

Proof: Take (P,L) minimal such that $r^G_{L,P}(\pi) \neq 0$.
This exists, because the "algebraic dimension" of each parabolic subgrp of G is finite.
π is irreducible $\Rightarrow$ $r^G_{L,P}(\pi)$ is of finite type
$\Rightarrow$ $r^G_{L,P}(\pi)$ has an irreducible quotient (ρ, W).

Claim: (ρ, W) is cuspidal.
Proof: We use Lemma 97 for the proof.
A parabolic pair in L has the form $(P' \cap L, L')$ for some $(P', L') \preceq (P, L)$, by 97 (ii).
Suppose $r^L_{L', P' \cap L}(\rho) \neq 0$.
Lemma 97 (iii) $\Longrightarrow$ $r^G_{L',P'}(\pi) \cong r^L_{L',P'\cap L} \circ r^G_{L,P}(\pi)$
$\cong r^L_{L',P'\cap L}(\rho) \neq 0$.

Thus $(P', L') = (P, L)$, by minimality.
Thus (ρ, W) is cuspidal. □ (Claim)
□

Rk 96: Given $(\pi, V) \in R_R(G)$,
the pair (L, ρ) from Thm 95 is uniquely determined upto $\simeq$ and G-conjugacy.
We call this equivalence class cuspidal support of (π, V).

Lemma 97: (i) All minimal parabolic pairs in G are G-conjugate.
(So we always can assume that (P,L) is standard)
(ii) Let $(P', L') \leq (P, L)$. Then $(P' \cap L, L')$ is a parabolic pair in L, and every parabolic pair in L can be obtained this way.
(iii) Let $(P', L') \preceq (P, L)$ and $(\pi, V) \in R_R(G)$. Then
$r^G_{L',P'}(\pi) \simeq r^L_{L', L\cap P'}(r^G_{L,P}(\pi))$.

Proof: [Borel-Tits, Groupes réductifs] □

Example 98: $G = GL_4(F)$.

$$(P,L) = \left(\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}^{(2,2)}, \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}^{(2,2)} \right)$$

$$(P',L') = \left(\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}^{(1,1,2)}, \begin{pmatrix} * & & 0 \\ & * & \\ 0 & & * \end{pmatrix}^{(1,1,2)} \right)$$

Then $(P',L') \precneqq (P,L)$.

Further

$$(P' \cap L, L') = \left(\begin{pmatrix} \begin{matrix} * & * \\ 0 & * \end{matrix} & 0 \\ 0 & \begin{matrix} * & * \\ * & * \end{matrix} \end{pmatrix}, \begin{pmatrix} \begin{matrix} * & \\ & * \end{matrix} & 0 \\ 0 & \begin{matrix} * & * \\ * & * \end{matrix} \end{pmatrix} \right)$$

a parabolic pair in $L = \begin{pmatrix} \begin{matrix} * & * \\ * & * \end{matrix} & 0 \\ 0 & \begin{matrix} * & * \\ * & * \end{matrix} \end{pmatrix}$.

<u>Thm 99:</u> Let G be a reductive p-adic group and $p \neq \ell$ and $\pi \in R_R(G)$ be irreducible. Then π is admissible.

(See Thm. 4.42)

We will only prove it for $G = GL_n(F)$ and R is algebraically closed. (and $p \neq \ell$).

Recall: (a) We have the sequence of open subgroups $(K_m)_{m \geq 0}$

$$K_0 := GL_n(\mathcal{O})$$

$$K_m := 1 + M_n(\mathfrak{p}^m) \quad , m > 0.$$

(b) $P_0 := \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = L_0 \cdot N_0$
$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$

$A_0 = L_0$ is a maximal split torus in G, i.e. $A_0 \overset{\text{alg.}}{\cong} F^\times \times \dots \times F^\times$ with maximal dimension.

If $P = LN \supseteq P_0 = L_0 N_0$ such that $L \supseteq L_0$, then

$$P = \begin{pmatrix} \ast & & & \ast \\ & \ast & & \\ & & \ddots & \\ 0 & & & \ast \end{pmatrix}$$

$$L = \begin{pmatrix} \ast & & & 0 \\ & \ast & & \\ & & \ast & \\ 0 & & & \ddots \end{pmatrix}$$

$$N = \begin{pmatrix} I & & & \ast \\ & I & & \\ & & I & \\ 0 & & & I \end{pmatrix}$$

$$N_- := \begin{pmatrix} I & & & \\ & I & & \\ & & I & \\ \ast & & & I \end{pmatrix}$$ opposite unipotent radical.

$P_- := L N_-$ opposite parabolic subgroups.

<u>Fact 100:</u>

(i) Cartan decomposition

$G = K_0 D K_0$ with

$$D := \left\{ \begin{pmatrix} \varpi^{n_1} & & 0 \\ & \ddots & \\ 0 & & \varpi^{n_n} \end{pmatrix} \,\middle|\, \begin{matrix} n_i \in \mathbb{Z} \\ n_1 \geq n_2 \geq \dots \geq n_n \end{matrix} \right\}$$

(ii) $\forall m \geq 1 : \forall_{(P,L) \geq (P_0, L_0)}$:

(standard parabolic pair)

$$K_m = (K_m \cap N_-)(K_m \cap L)(K_m \cap N)$$

<u>Example</u>:

In $GL_2(F)$: $1 + M_2(\mathfrak{p}) = \begin{pmatrix} 1 & \\ \mathfrak{p} & 1 \end{pmatrix} \begin{pmatrix} 1+\mathfrak{p} & \\ & 1+\mathfrak{p} \end{pmatrix} \begin{pmatrix} 1 & \mathfrak{p} \\ & 1 \end{pmatrix}$

<u>Lemma 101:</u> Put $A := \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \middle| \begin{array}{l} \text{all } a_i \in F^*, \\ \nu(a_1) \geq \dots \geq \nu(a_n) \end{array} \right\}$

<u>Then are equivalent:</u> for $\pi \in R_{\mathbb{C}}(G)$:

1° π is Z-compact

2° $\forall v \in V \; \forall_{m \geq 1}$: $\underset{\hat{A}}{g} \longmapsto \pi(e_{K_m})\pi(g)v$

has support compact mod Z.

<u>Proof:</u> $1^\circ \Rightarrow 2^\circ$ ✓

$2^\circ \Rightarrow 1^\circ$ Take $v \in V$ and $m \geq 1$.

<u>To show:</u> $g \longmapsto \pi(e_{K_m})\pi(g)v$
is compactly supported mod Z.

Put $X := \{ \pi(x)v \mid x \in K_0 \}$
(a finite set) Take $w \in X$.

The map $a \in A \longmapsto \pi(e_{K_m})\pi(a)w$
has compact support mod Z,
say in $H_w Z$ with $H_w \subseteq A_0^-$
compact.

Take $g \in G = K_0 A K_0$ (Cartan)
in the support of $\underset{\in G}{g} \longmapsto \pi(e_{K_m})\pi(g)v$.

Write
$g = k a k'$, $k, k' \in K_0$, $a \in A$
and put $w := \pi(k')v$.
Then $g \in K_0 H_w Z K_0$

(compact mod Z) $\square$

<u>Remark 102:</u> In Lemma 101 (G)
we can replace the sentence by

$\forall_{m \geq 1} \forall v \in V^{K_m}$: $\underset{\hat{A}}{g} \longmapsto \pi(e_{K_m})\pi(g)v$

has compact support mod Z.

The following lemma relates Z-compactness to cuspidality.

We get a criterion for a $v \in V$ to be in $V(N)$.

<u>Lemma 103:</u> Let $(P = LN, L)$ be a parabolic pair in $G = GL_n(F)$ and H_i, $i \geq 1$ be an increasing sequence of open subgroups of N, s.t. $N = \bigcup_{i \in \mathbb{N}} H_i$. Let $(\pi, V) \in \mathcal{R}_{\mathbb{R}}(G)$. Take $v \in V$. Then are equivalent:

$1°$ $v \in V(N)$

$2°$ $\exists_{i_0 \in \mathbb{N}} \forall_{i \geq i_0}: v \in V(H_i)$

$3°$ $\forall_{m \in \mathbb{N}} \exists_{k \in \mathbb{N}}: \forall_{d \in A}:$
$(d H_k d^{-1} \subseteq K_m \Longrightarrow \pi(e_{K_m}) \pi(d) v = 0)$

$4°$ $\forall m \in \mathbb{N} \; \exists_{k \in \mathbb{N}} \; \forall d \in A \cap C(L)$ (center of L):
$(d H_k d^{-1} \subseteq K_m \Longrightarrow \pi(e_{K_m}) \pi(d) v = 0)$

<u>Proof:</u> $1° \Rightarrow 2°$: $v \in V(N) \Rightarrow$

$\exists_t \exists n_1, \ldots, n_t \in N \; \exists v_1, \ldots, v_t \in V$:

$$v = \sum_{j=1}^{t} (v_j - \pi(n_j) v_j)$$

Take $i_0 \in \mathbb{N}$ s.t. $n_1, \ldots, n_t \in H_{i_0}$

$\Rightarrow \forall_{i \geq i_0}: v \in V(H_{i_0}) \subseteq V(H_i)$.

$2° \Rightarrow 1°$ ✓

$2° \Rightarrow 3°$ Let $m \geq 1$. Take $k \geq 1$ s.t.
$\pi(e_{H_k}) v = 0$ (Here we use $2°$)

Let $d \in A$ such that $d H_k d^{-1} \subseteq K_m$

$\Rightarrow \pi(e_{K_m}) \pi(d) v$
$= \pi(e_{K_m \cap N^-}) \pi(e_{K_m \cap L}) \pi(e_{K_m \cap N}) \pi(d) v$

$\underset{\uparrow}{=} \pi(e_{K_m}) \pi(d H_k d^{-1}) \pi(d) v$

$\pi(e_{K_m \cap N}) = \pi(e_{K_m \cap N}) \pi(e_{d H_k d^{-1}})$

$= \Pi(e_{K_m})\, \Pi(d)\, \Pi(e_{H_k})\, v = 0$.

$3^\circ \Rightarrow 4^\circ$ ✓

$4^\circ \Rightarrow 2^\circ$ Take $m \geq 1$ such that $v \in V^{K_m}$ and we use k from 4°.

For $d \in A \cap C(L)$ we have

$\Pi(e_{K_m})\, \Pi(d)\, v$.

$= \Pi(e_{K_m \cap N})\, \Pi(e_{K_m \cap L})\, \Pi(e_{K_m \cap N^-})\, \Pi(d)\, v$

$= \Pi(e_{K_m \cap N})\, \Pi(d)\, \Pi(e_{K_m \cap L})\, \Pi(e_{d^{-1}(K_m \cap N_-)d})\, v$

$\underset{\uparrow}{=} \Pi(e_{K_m \cap N})\, \Pi(d)\, v$

$d(K_m \cap N_-)d^{-1} \subseteq K_m \cap N_-$

$= \Pi(d)\, \Pi(e_{d^{-1}(K_m \cap N)d})\, v$

We can find $d \in A \cap C(L)$ such that $d H_k d^{-1} \subseteq K_m \cap N$.

Then, by 4°, $\Pi(e_{K_m})\, \Pi(d)\, v = 0$.

Thus $\Pi(e_{d^{-1}(K_m \cap N)d})\, v = 0$

$\Rightarrow$ $\forall_{i \in \mathbb{N}}$ with $H_i \supseteq d^{-1}(K_m \cap N)d$ we have $\Pi(e_{H_i})\, v = 0$, i.e. $v \in V(H_i)$

$\square$

<u>Theorem 104:</u> Let $\Pi \in R_R(G)$ be irr. and $R = \bar{R}$. Then are equivalent

1° Π is Z-compact

2° Π is cuspidal.

<u>Proof:</u> $2^\circ \Rightarrow 1^\circ$. Assume Π is not Z-compact.

Remark 102 + lemma 101 $\Rightarrow$

$\exists_{m \geq 1} \exists\, v \in V^{K_m}$:

$$a \in A \overset{\alpha}{\longmapsto} \Pi(e_{K_m})\, \Pi(a)\, v$$

has <u>not</u> compact support mod Z.

$\Rightarrow \exists_{r \in \{1, \ldots, n-1\}}$ $a \in Y := \mathrm{supp}(\alpha) \longmapsto \nu(a_{r+1}) - \nu(a_r)$ is unbounded below.

Take $(P = LN, L)$ $\quad P = \begin{pmatrix} * & * \\ & * \end{pmatrix} \begin{matrix} \}r \\ \}n-r \end{matrix}$

Take any $k \in \mathbb{N}$.
$\{ a H_k a^{-1} \mid a \in Y \}$ contains
arbitrary small subgroups

$\Rightarrow$ Lemma 103(3°) doesn't hold.
$\Rightarrow$ $\quad v \notin V(N)$
$\uparrow$
103(1° $\Rightarrow$ 3°)

$\Rightarrow$ $\quad r^G_{L,P}(\pi) = V / V(N) \neq 0.$

and $(P, L) \neq (G, G)$

$\Rightarrow$ (π, V) not cuspidal.

1° $\Rightarrow$ 2° Let (π, V) be Z-compact.
Take $v \in V$ and $m \geq 1$ s.t. $v \in V^{K_m}$
Lemma 102 $\Rightarrow$ The map α
$$a \in A \xmapsto{\alpha} \pi(e_{K_m})\, \pi(a)\, v$$
is comp. supported mod Z.

Put $Y := \operatorname{supp}(\alpha)$. Take $(P = LN, L) \neq (G, G)$
Thus
$$\bigcup_{a \in Y} a^{-1}(K_m \cap N) a =: S$$

is a compact set in N.

$\Rightarrow \exists_{k \in \mathbb{N}}: \quad H_k \supsetneq S.$

$\Rightarrow$ Lemma 103(3°) is satisfied.

$\Rightarrow$ $\quad v \in V(N).$
$\uparrow$
103(3° $\Rightarrow$ 1°)

Thus we have shown:
$\forall_{v \in V} \forall_{(P = LN, L) \neq (G, G)}$:

$\bar{v} \in r^G_{L,P}(\pi)$ is the zero vector.

Thus $\forall_{(P, L) \neq (G, G)}$: $\quad r^G_{L,P}(\pi) = 0$
Thus $\quad (\pi, V)$ is cuspidal. $\square$

for $GL_n(F)$, $R = \bar{R}$, $p \neq \ell$

<u>Proof (Theorem 99)</u>: Take (ρ, L) in the cuspidal support of π, i.e.
$\pi \subseteq i_{P,L}^{G}(\rho)$

ρ cusp. $\overset{104}{\Rightarrow}$ ρ Z-compact

$\overset{76}{\Longrightarrow}$ ρ admissible

$\overset{24}{\Longrightarrow}$ $i_{P,L}^{G}(\rho)$ admissible

$\Longrightarrow$ π is admissible $\square$