V 5. Convergence

Given a filtered chain complex (C, F), every $H_n(C)$ is also filtered, via

$$F_p H_n(C) := \operatorname{im}\left(H_n(F_p C_n) \longrightarrow H_n(C)\right).$$

Thm 196 (Classical convergence theorem)

1. Suppose F is bounded. Then

$$E^1_{pq} = H_{p+q}\left({}^{F_p C}/_{F_{p-1} C}\right) \Rightarrow H_{p+q}(C)$$

2. Suppose F is bounded below and exhaustive.
Then the spectral sequence converges to $H_*(C)$.
If $f : C \longrightarrow C'$ is a map of filtered complexes, then $f_* : H_*(C) \to H_*(C')$ is compatible with the spectral sequences.

Proof: 2. implies 1. So we only need to prove 2.

$(E^r)_{r \geq 0}$ is bounded below (because the filtration of C is) and $(F_p H_n(C))_p$ is exhaustive.

Thus we only need to show that E^r weakly converges to H_*.

Choose $n \in \mathbb{Z}$, $p \in \mathbb{Z}$ and put $q := n - p$.

To show $E^\infty_{pq} \cong \dfrac{F_p H_n(C)}{F_{p-1} H_n(C)}$.

The filtration on C_n is bounded below

$\Rightarrow$ For r big enough (depending on p and q) we have

$$A^r_p = \{c \in F_p C_n \mid d(c) \in F_{p-r} C_{n-1}\}$$
$$= \{c \in F_p C_n \mid d(c) = 0\}$$
$$=: A^\infty_p$$

<u>Note:</u> $A_p^\infty = \ker\left(d\big|_{F_pC_n} : F_pC_n \to F_pC_{n-1}\right)$

By definition of $F_p(H_n(C))$ we have

$$\begin{array}{ccc} A_p^\infty & \xrightarrow{\pi} & F_pH_n(C) \subseteq H_n(C) \\ c & \longmapsto & [c]_{im(d_{n+1})} \end{array}$$

<u>$\ker(\pi)$:</u> $c \in \ker(\pi) \iff c \in F_pC_n \wedge d(c)=0$ and $c \in im(d_{n+1})$

$\iff c \in A_p^\infty$ and $\left(\exists_{p' > p} : \exists c' \in F_{p'}C_{n+1} : d_{n+1}(c') = c\right)$

$\uparrow$ F exhaustive

$\iff c \in A_p^\infty$ and $\exists_{p'>p} : c \in d\left(A_{p'}^{p'-p}\right)$

$\iff c \in \bigcup_{r=0}^{\infty} d\left(A_{p+r}^r\right)$

So we have

$$\begin{array}{ccc} A_p^\infty \big/ \bigcup_r d(A_{p+r}^r) & \xrightarrow{\sim} & F_p(H_n(C)) \\ \uparrow & \circlearrowleft & \cup\!| \\ A_{p-1}^\infty \big/ \bigcup_r d(A_{p+r-1}^r) & \xrightarrow{\sim} & F_{p-1}(H_n(C)) \end{array}$$

So $\dfrac{F_p(H_n(C))}{F_{p-1}(H_n(C))} \cong A_p^\infty \big/ A_{p-1}^\infty + \bigcup_r d(A_{p+r}^r)$

On the other hand we have the map

$$A_{pq}^\infty \xrightarrow{\overline{\eta}_p} Z_{pq}^\infty \big/ B_{pq}^\infty = E_{pq}^\infty.$$

<u>$\ker(\overline{\eta}_p)$:</u> Take $c \in A_p^\infty$.

$\overline{\eta}_p(c) = 0 \iff \eta_p(c) \in B_p^\infty = \bigcup_{r=1}^{\infty} \eta_p\left(d\left(A_{p+r-1}^{r-1}\right)\right)$

$\iff c \in F_{p-1}C_n + \bigcup_{r=1}^{\infty} d\left(A_{p+r-1}^{r-1}\right)$

$\underset{c \in A_p^\infty}{\iff}$ $\in\, \in A_{p-1}^\infty + \bigcup_{r=1}^{\infty} d\left(A_{p+r-1}^{r-1}\right)$ $\square$

We have a bit more general theorem.

<u>Theorem 197</u> (Complete Convergence Theorem) Let (C, F) be a filtered chain complex and suppose that it is complete and exhaustive. Suppose that the spectral sequence is regular. Then

1. The spectral sequence converges weakly to $H_*(C)$
2. The spectral seq. converges to $H_*(C)$ if the spectral sequence is bounded above.

<u>Proof</u>: [Weibel, Theorem 5.5.10] □

For derived categories we need the concept of hyperhomology.

V.6. Hyperhomology

We need for a chain complex A a double complex which works like a "projective resolution".

<u>Def 198</u> (Cartan - Eilenberg resolution) Let $\mathcal{A}$ be an abelian category with enough projectives and A be a chain complex.

A <u>Cartan - Eilenberg resolution of A</u> is a double complex P_{**} only consisting of projectives and supported on the upper half plane (i.e. $P_{*q} = 0$ for $q < 0$) together with a chain map $P_{*0} \xrightarrow{\epsilon} A$ such that for all $p \in \mathbb{Z}$

(i) If $A_p = 0$ then $P_{p*} = 0$.

(ii)

$$B_p(P, d^h) \longrightarrow B_p(A)$$
$$H_p(P, d^h) \longrightarrow H_p(A)$$

are projective resolutions in $\mathcal{A}$.

<u>Rk 199</u>: Given a Cartan + Eilenberg resolution of A then

(1) $P_{p*} \longrightarrow A_p$

and $Z_p(P, d^h) \longrightarrow Z_p(A)$

are projective resolutions.

(2) $\mathrm{Tot}^{\oplus}(P) \longrightarrow A$ is a quasi-isomorphism.

(Here we assume that P_{**} is bounded below, so that $\mathrm{Tot}^{\oplus}(P)$ exists.)

<u>Proof</u>: This is a good exercise □

<u>Lemma 200</u>: Every chain complex A has a Cartan – Eilenberg resolution.

The proof of Lemma 200 is essentially the Horseshoe lemma.

<u>Proof</u>: <u>Step 1</u>: Take projective resolutions of $B_p(A)$ and $H_p(A)$ and use Horseshoe to get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & P^B_{p*} \arrow[r] \arrow[d, Rightarrow] & P^Z_{p*} \arrow[r] \arrow[d, Rightarrow] & P^H_{p*} \arrow[r] \arrow[d, Rightarrow] & 0 \\
0 \arrow[r] & B_p(A) \arrow[r] & Z_p(A) \arrow[r] & H_p(A) \arrow[r] & 0
\end{tikzcd}$$

Note that $P^Z_{p*} / P^B_{p*} \cong P^H_{p*}$.

<u>Step 2</u>: Use Horseshoe again to obtain an exact sequence

$$0 \longrightarrow P^Z_{p*} \longrightarrow P^A_{p*} \longrightarrow P^B_{p-1*} \longrightarrow 0$$

lying over

$$0 \longrightarrow Z_p(A) \longrightarrow A_p \longrightarrow B_{p-1}(A) \longrightarrow 0$$

<u>Step 3</u>: P_{**} is the double complex with objects P^A_{**} and vertical differentials given by $\tilde{\varepsilon}^A_{p*}$ multiplied with $(-1)^p$ and horizontal differentials given by

$$P^A_{p+1\,*} \twoheadrightarrow P^B_{p*} \hookrightarrow P^Z_{p*} \hookrightarrow P^A_{p*}.$$

<u>Now</u>: $B_p(P, d^h) = P^B_{p*}$ gives

a projective resolution $P^B_{p*} \Rightarrow B_p(A)$,

and $H_p(P, d^h) = P^Z_{p*} / P^B_{p*} \simeq P^H_{p*}$

give a projective resolution

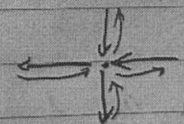
$$P^H_{p*} \longrightarrow H_p(A) \qquad \square$$

<u>Exercise 201</u>: Let $f: A \to B$ be a chain map and $P \to A$, $Q \to B$ be two Cartan–Eilenberg resolutions. Then $\exists$ morphism $\tilde{f}: P \to Q$ of double complexes lying over f.

<u>Hint</u>: Think about the box version of the Horseshoe lemma.

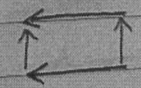
<u>Def 202</u>: A chain homotopy between two morphisms $f, g: D \to E$ of double complexes is a family of maps $(s^h_{pq}, s^v_{pq})_{pq}$ such that

$$g - f = s^h d^h + d^h s^h + s^v d^v + d^v s^v$$



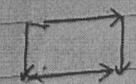
and

$$d^h s^v + s^v d^h = 0$$



and

$$d^v s^h + s^h d^v = 0$$



(s^h, s^v) is called chain homotopy from f to g.

Rk 203: If (s^h, s^v) is a chain homotopy from f to g.
Then $s^h + s^v$ is a chain homotopy from $\mathrm{Tot}^{\oplus}(f)$ to $\mathrm{Tot}^{\oplus}(g)$.

Exercise 204: (i) Let $f, g : A \to B$ two homotopic maps of chain complexes and $P \to A$, $Q \to B$ Cartan – Eilenberg res., with $\tilde{f}, \tilde{g} : P \longrightarrow Q$ lying over f, g resp.
Prove that $\tilde{f}$ is homotopic to $\tilde{g}$.

(ii) Taking $f, g = id$ in (i)
One can show that there is a homotopy equivalence lying over id_A, for two C–E–resolutions $P \longrightarrow A$, $Q \longrightarrow A$.

Def 204: Let $F : \mathcal{A} \to \mathcal{B}$ be right exact and assume that $\mathcal{A}$ has enough projectives. Let $P \longrightarrow A$ be a C–E–resolution of A, A a chain complex.
$\mathbb{L}_i F(A) := H_i(\mathrm{Tot}^{\oplus}(F(P)))$.
$\mathbb{L}_i F$ is called the i^{th} left hyper – derived functor of F.

$$\left(\mathbb{L}_i F(f) := H_i(\mathrm{Tot}^{\oplus}(\tilde{f})) \right)$$

Exercise 205: (i) Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor
Then $F : Ch_{\geq 0}(\mathcal{A}) \longrightarrow Ch_{\geq 0}(\mathcal{B})$ is right exact and thus $H_0 \circ F \; Ch_{\geq 0}(\mathcal{A}) \longrightarrow \mathcal{B}$ is right exact.
Show that $L_i(H_0 \circ F)$ and $\mathbb{L}_i F$ $Ch_{\geq 0}(\mathcal{A}) \longrightarrow \mathcal{B}$ are isomorphic.

(ii) $\mathbb{L}_i F(A[n]) = \mathbb{L}_{i+n} F(A)$
Here $A[n]_i := A_{n+i}$.

<u>Prop 206:</u> Let $0 \to A \to B \to C \to 0$ be an exact sequence of bounded below chain complexes. Then there is a long exact sequence

$$\cdots \to \mathbb{L}_{i+1}F(C) \xrightarrow{\delta} \mathbb{L}_i F(A) \to \mathbb{L}_i F(B) \to \mathbb{L}_i(C) \xrightarrow{\delta} \mathbb{L}_{i-1}(A) \to \cdots$$

<u>Proof:</u> Use dimension shifting to work in $Ch_{\geq 0}(\mathcal{A})$.
Take the long exact sequence from $L_i(H_0F) : Ch_{\geq 0}(\mathcal{A}) \to \mathcal{B}$, $i = 0, 1, 2, \ldots$.

<u>Prop 207:</u> (a) There is a convergent spectral sequence

$$''E^2_{pq} = (L_pF)(H_q(A)) \Rightarrow \mathbb{L}_{p+q}F(A)$$

(b) Suppose A is bounded below. Then there is a convergent spectral sequence

$$'E^2_{pq} = H_p(L_qF(A)) \Rightarrow \mathbb{L}_{p+q}F(A).$$

<u>Proof:</u> Take the 2 spectral sequences of the double complex $F(P)$ □

<u>Cor 208</u> 1. A exact $\Rightarrow (\mathbb{L}_i(A) = 0 \ \forall_i)$

2. Let $f : A \to B$ be a quasi-isom. of chain complexes.
$\Rightarrow \mathbb{L}_* F(A) \cong \mathbb{L}_* F(B)$.

3. Suppose each A_p is F-acyclic, i.e. $L_iF(A_p) = 0 \ \forall_{i>0}$.
Suppose further that A is bounded below.
Then $\mathbb{L}_p F(A) = H_p(F(A))$ for all $p \in \mathbb{Z}$.

<u>Example 209:</u> $''E^2_{pq} = \mathrm{Tor}^R_p(H_q(A), B) \Rightarrow \mathbb{T}\mathrm{or}^R_{p+q}(A, B)$

$A \in Ch(\text{mod-}R)$, $B \in R\text{-mod}$.

Suppose A is bounded below

$$ {}^{I}E^{2}_{pq} = H_p\left(\mathrm{Tor}_q^R(A_*, B)\right) \Longrightarrow \mathrm{Tor}^R_{p+q}(A, B) $$

VI Derived categories

VI The category $K(\mathcal{A})$.

Let $\mathcal{A}$ be an abelian category and $Ch(\mathcal{A})$ the category of cochain complexes in $\mathcal{A}$.

We define the <u>homotopy category of $\mathcal{A}$</u>, $K(\mathcal{A})$, as follows:

$$ \mathrm{Obj}(K(\mathcal{A})) := \mathrm{Obj}(Ch(\mathcal{A})) $$
$$ \mathrm{Hom}_{K(\mathcal{A})}(A, B) := \mathrm{Hom}_{Ch(\mathcal{A})}(A, B)/\sim $$

where $\sim$ means "chain homotopy".

We obtain $K^b(\mathcal{A})$, $K^+(\mathcal{A})$, $K^-(\mathcal{A})$.

Prop. 2.10: (universal property)
Let $\mathcal{D}$ be a category and
$F : Ch(\mathcal{A}) \longrightarrow \mathcal{D}$ be a functor which sends chain homotopy equivalences to isomorphisms. Then F factors through $ch(\mathcal{A}) \longrightarrow K(\mathcal{A})$ uniquely, i.e.

$$\begin{tikzcd} Ch(\mathcal{A}) \arrow[r, "F"] \arrow[d] & \mathcal{D} \\ K(\mathcal{A}) \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

Proof: Step 1: Behaviour of F on cylinders.
Recall: $cyl(B) = cyl(id_B) = B^n \oplus B^{n+1} \oplus B^n$
We have two maps
$$\alpha : B \longrightarrow cyl(B) \quad \alpha(b) = (0,0,b)$$
$$\alpha' : B \longrightarrow cyl(B) \quad \alpha'(b) = (b,0,0)$$

From an exercise before we know that α is a ch. homotopy equivalence with an homotopy inverse given by

$$\beta(b', b'', b) := b' + b.$$

We have $\beta \circ \alpha = id_B$ and $\alpha \circ \beta \sim id_{cyl(B)}$.
Further $\beta \circ \alpha' = id_B$.

$$\Rightarrow \quad F(\beta \circ \alpha') = F(id_B) = F(\beta \circ \alpha) = F(\beta) F(\alpha)$$
$$\qquad \| $$
$$F(\beta) F(\alpha')$$

$F(\beta)$ is an isomorphism $\Rightarrow$ $F(\alpha') = F(\alpha)$.

Step 2: Same value on homotopic maps:
Suppose $f, g : B \longrightarrow C$ are homotopic
Then $\exists\, s$: $(f, s, g) : cyl(B) \xrightarrow{\gamma} C$ is a chain map.
$$\Rightarrow \quad F(f) = F(\gamma \circ \alpha') = F(\gamma) F(\alpha')$$
$$\| $$
$$F(g) = F(\gamma \circ \alpha) = F(\gamma) F(\alpha)$$

□

We don't have kernels and cokernels anymore in $K(\mathcal{A})$. But still we have long exact sequence of cohomology.
They are produced by so called exact triangles, replacing short exact

sequences.

Def 2.11: (triangles in $K(\mathcal{A})$)
Let $u : A \longrightarrow B$ be a morphism in $Ch(\mathcal{A})$.
Then we have an exact sequence in $Ch(\mathcal{A})$:

$$0 \longrightarrow B \xrightarrow{v} \mathrm{cone}(u) \xrightarrow{\delta} A[-1] \longrightarrow 0$$
$$\mathrm{cone}(u) = \left(B^n \oplus A^{n+1}\right)_{\mathbb{Z}}$$

$$b \longmapsto (b, 0)$$
$$(b, a) \longmapsto -a$$

We call (u, v, δ) a strict triangle on u in $K(\mathcal{A})$.
We write

$$\begin{tikzcd} & \mathrm{cone}(u) \arrow[dl, "\delta"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd}$$

A triple of maps in $K(\mathcal{A})$
$(u : A \to B,\ v : B \to C,\ w : C \longrightarrow A[-1])$
is called exact triangle if
$\exists$ strict triangle (u', v', δ')
and a in $K(\mathcal{A})$ commutative diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & A[-1] \arrow[d, "f[-1]"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & \mathrm{cone}(u) \arrow[r, "\delta'"] & A'[-1] \end{tikzcd}$$

s.th. f, g, h are isomorphisms in $K(\mathcal{A})$, i.e. comming from homotopy equivalences in $Ch(\mathcal{A})$.

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd}$$

Rk 2.12: An exact triangle (u, v, w) defines a long exact sequence of cohomology

$$\hookrightarrow H^i(A) \xrightarrow{u_*} H^i(B) \xrightarrow{v_*} H^i(C) \xrightarrow{w_*} H^{i+1}(A) \longrightarrow H^{i+1}(B) \longrightarrow H^{i+1}(C) \hookrightarrow$$

<u>Example 213:</u> (a)
$$\begin{tikzcd}[column sep=small] & A \oplus A[-1] \arrow[dr] & \\ A \arrow[ur, leftarrow] \arrow[rr, "0"] & & A \end{tikzcd}$$

is a strict triangle.

as $\mathrm{cone}(0) = A \oplus A[-1]$

(b)
$$\begin{tikzcd}[column sep=small] & 0 \arrow[dl] & \\ A \arrow[rr, "id"] & & A \arrow[ul] \end{tikzcd}$$

is an exact triangle

as $\mathrm{cone}(id)$ is split exact,

so $\mathrm{cone}(id) \simeq 0$ in $K(A)$,

by Prop. 35.

<u>Pf:</u> $\mathrm{cone}(id_A) = \begin{matrix} A^{n+1} \\ \oplus \\ A^n \end{matrix}$

with $d_{cone} = \begin{pmatrix} -d & \\ -id_A & d \end{pmatrix}$.

Take $s : \begin{matrix} A^{n+1} \\ \oplus \\ A^n \end{matrix} \longrightarrow \begin{matrix} A^n \\ \oplus \\ A^{n-1} \end{matrix}$

$$s = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

Then $d_{cone}\, s\, d_{cone} =$

$$\begin{pmatrix} -d & \\ -id & d \end{pmatrix}\begin{pmatrix} & -1 \\ & \end{pmatrix}\begin{pmatrix} -d & \\ -id & d \end{pmatrix} = \begin{pmatrix} -d \\ -id\, d \end{pmatrix}\begin{pmatrix} id & -d \end{pmatrix}$$

$$= \begin{pmatrix} -d & \\ -id & d \end{pmatrix}$$

So $\mathrm{cone}(id_A)$ is split.

id_A is a quasi-isomorphism in $Ch(A)$

So $\mathrm{cone}(id_A)$ is acyclic. □