Consider 187(c) for $F = \Omega B$, $E = PB$.
Note: PB is contractible, so $H_n(PB) = 0$ for $n > 0$.

① From 187(c) we get

$$\underset{\overset{\shortparallel}{0}}{H_2(E)} \longrightarrow H_2(B) \xrightarrow{\sim} H_1(F) \longrightarrow \underset{\overset{\shortparallel}{0}}{H_1(E)}$$

② Note that $E^3_{21} = E^\infty_{21} = 0$, as $H_3(PB) = 0$.
So

$$H_4(B) \xrightarrow{d^2} H_2(B) \otimes_{\mathbb{Z}} H_2(B) \xrightarrow{d^2} H_2(\Omega B) \to H_3(B) \to 0$$

is exact.
(For the exactness on $H_2(\Omega B)$ and $H_3(B)$, note that

$$H_3(B) \xrightarrow{\sim} H_2(\Omega B) \big/ \mathrm{im}\, d^2$$

by the 3^{rd} page.)

Exercise 189: Suppose $n \geq 2$. Then

$$H_p(\Omega S^n) \simeq \begin{cases} \mathbb{Z}, & n-1 \mid p \\ 0, & \text{else} \end{cases}$$

for all $p \in \mathbb{N}_0$.

Theorem 190: (Wang sequence)
Suppose $F \overset{i}{\hookrightarrow} E \xrightarrow{\pi} S^n$ is a fibration, $n \notin \{0,1\}$. Then there exists a long exact sequence

$$\cdots \to H_q(F) \xrightarrow{i_*} H_q(E) \to H_{q-n}(F) \xrightarrow{d^n} H_{q-1}(F) \xrightarrow{i_*} H_{q-1}(E) \to H_{q-1-n}(F) \xrightarrow{d^n} \cdots$$

Proof: $E^2_{pq} = H_p(S^n; H_q(F)) \overset{\text{UCT}}{\cong} H_p(S^n) \otimes_{\mathbb{Z}} H_q(F)$

$$= \begin{cases} H_q(F), & p = 0, n \\ 0, & p \neq 0, n. \end{cases}$$

$$E^n_{pq} \qquad \begin{tikzcd}[row sep=small]
H_n(F) & & \\
H_{n-1}(F) & & \vdots \\
\vdots & & \\
H_2(F) & & H_2(F) \\
H_1(F) & & H_1(F) \\
H_0(F) & & H_0(F) \arrow[uuuull, "d^n"'] \\
0 \quad 1 \quad 2 \quad \cdots & & n
\end{tikzcd}$$

$$E^{n+1}_{pq} = E^{\infty}_{p,q}.$$

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(F) \arrow[r, "i_*"] \arrow[dr, two heads] & H_q(E) \arrow[r] \arrow[dr, two heads] & H_{q-n}(F) \arrow[r, "d^n"] & H_{q-1}(F) \arrow[r] & \cdots \\
 & & E^{n+1}_{0q} \arrow[u, hook] & E^{n+1}_{n,q-n} \arrow[u, hook] & &
\end{tikzcd}$$

$\square$

(Gysin sequence)

Thm 191: Let $S^n \xrightarrow{i} E \xrightarrow{\pi} B$ be a fibration s.t. $n \neq 0$ and $\pi_1(B) = \pi_0(B) = 0$. Then there is an exact sequence

$$\cdots \xrightarrow{d^{n+1}} H_{p-n}(B) \to H_p(E) \xrightarrow{\pi_*} H_p(B) \xrightarrow{d^{n+1}} H_{p-1-n}(B) \to \cdots$$

Proof: (exercise) $\square$

V 4. Spectral sequence of a filtration

Let $\mathcal{A}$ be a subcategory of mod-R. Suppose $\mathcal{A}$ is abelian.

Thm 192: (construction theorem)
Let F be a filtration on a chain complex C. Then there is a homology spectral sequence $(E^r)_{r\geq 0}$ such that

$$E^0_{pq} = F_p C_{p+q} \big/ F_{p-1} C_{p+q}$$

and $E^1_{pq} = H_q(E^0_{p*}) = H_{p+q}\left(F_p C_* \big/ F_{p-1} C_*\right)$

Def 193: (filtration on a complex)
Let C be a chain complex.
A filtration $(F_p C)_{\mathbb{Z}}$ is a sequence of subcomplexes of C:

$$\cdots \subseteq F_{p-1} C \subseteq F_p C \subseteq F_{p+1} C \subseteq \cdots$$

$(F_p C)_{\mathbb{Z}}$ is called <u>exhaustive</u> if $C = \bigcup_{\mathbb{Z}} F_p C$.

$(F_pC)_{\mathbb{Z}}$ is called

- bounded, if $\forall_n \exists_{s<t \in \mathbb{Z}}: \ 0 = F_s C_n \subseteq F_t C_n = C_n$
- bounded below, if $\forall_n \exists_{s \in \mathbb{Z}}: \ 0 = F_s C_n$
- bounded above, if $\forall_n \exists_{t \in \mathbb{Z}}: \ F_t C_n = C_n$

$(F_pC)_{\mathbb{Z}}$ is called

- Hausdorff, if $\bigcap_{\mathbb{Z}} F_p C = 0$
- complete, if $C = \varprojlim C/F_pC$.

Given a filtered complex (C, F) the complex $\hat{C}$:

$$\hat{C}_n := \varprojlim_{p'} \frac{C_n}{F_{p'}C_n} \quad \text{with}$$

$$F_p\hat{C}_n := \varprojlim_{p'} F_pC_n / F_{p'}C_n, \quad p \in \mathbb{Z},$$

is a filtered complex $(\hat{C}, F)$ called the completion of (C, F).

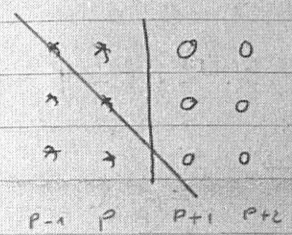
Example 194: Consider a double complex (E, d^v, d^h)

$$C := \mathrm{Tot}^{\oplus}(E)$$

We have 2 filtrations

1: (by columns) $\displaystyle ({}'\tau_{\le p} E)_{p'q'} := \begin{cases} E_{p'q'}, & p' \le p \\ 0, & p' > p \end{cases}$

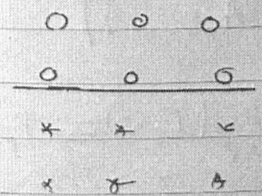
${}'\tau_{\le p} E$:



$${}'F_p C := \mathrm{Tot}^{\oplus}({}'\tau_{\le p} E)$$

$$\Big(\text{i.e. } {}'F_p C_n = \bigoplus_{p' \le p} E_{p', n-p'} \Big)$$

2: (by rows) $\displaystyle ({}''\tau_{\le q} E)_{p'q'} := \begin{cases} E_{p'q'}, & q' \le q \\ 0, & q' > q \end{cases}$



$${}''F_q C := \mathrm{Tot}^{\oplus}({}''\tau_{\le q} E).$$

<u>Proof of the construction theorem</u>:

During this construction it is worth to have a double complex with filt. by columns in mind. See the construction in § V.1 and Example 161.

We skip the subscript q.

$$\eta_p : F_pC \longrightarrow F_pC/F_{p-1}C \quad , p \in \mathbb{Z}$$

$$A_p^r := \{ c \in F_pC \mid d(c) \in F_{p-r}C \} \, , r \geq -1$$

cycles modulo $F_{p-r}C$, i.e. $\bar{c} \in F_pC/F_{p-r}C$ is a cycle.

<u>approximately cycle</u>

$$Z_p^r := \eta_p(A_p^r) \quad , p \in \mathbb{Z}, r \geq 0$$

$$B_p^r := \eta_p\left(d\left(A_{p+r-1}^{r-1}\right)\right)$$

Then $Z_p^0 = \eta_p(F_pC) = F_pC/F_{p-1}C$

and $B_p^0 = \eta_p\left(d\left(A_{p-1}^{-1}\right)\right) = \eta_p(F_{p-1}C) = 0$.

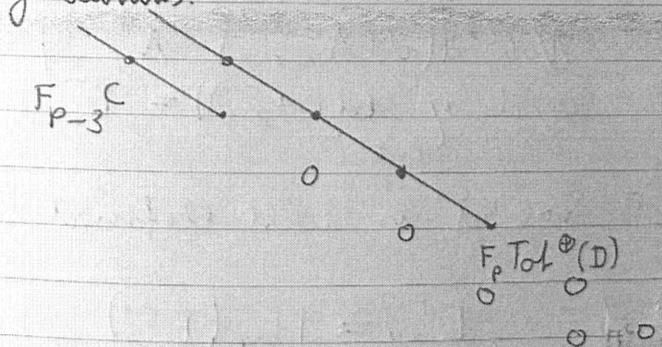
We obtain a nested family

$$0 = B_p^0 \subseteq B_p^1 \subseteq B_p^2 \subseteq \ldots \subseteq Z_p^2 \subseteq Z_p^1 \subseteq Z_p^0 = F_pC/F_{p-1}C$$

Put $E_p^r := Z_p^r / B_p^r$ and d^r is defined by d:

$$d^r([\bar{c}]_{B_p^r}) := \left[\overline{d(c)}\right]_{B_{p-r}^r} \, , c \in A_p^r.$$

<u>Picture for</u> $C = \mathrm{Tot}^{\oplus}(D)$ with filtration by columns.



So A_p^r encodes the idea of defining sequence in the case of $C = \mathrm{Tot}^{\oplus}(D)$.

<u>Claim:</u> $(E^r)_{r\geq 0}$ is a spectral sequence.

<u>Part 1:</u> d^r is well-defined:

<u>Pf:</u> $E_p^r = Z_p^r / B_p^r \simeq \dfrac{A_p^r + F_{p-1}(C)}{d(A^{r-1}_{p+r-1}) + F_{p-1}(C)}$

$$\simeq \frac{A_p^r}{A_p^r \cap (\underbrace{d(A^{r-1}_{p+r-1})}_{\subseteq A_p^r} + F_{p-1}C)}$$

$$= A_p^r / (d(A^{r-1}_{p+r-1}) + A^{r-1}_{p-1})$$

Now $d(d(A^{r-1}_{p+r-1}) + A^{r-1}_{p-1}) = d(A^{r-1}_{p-1})$
and $\eta_{p-r}(d(A^{r-1}_{p-1})) = B^r_{p-r}$.

So d^r is well-defined $\square$ (Part 1)

<u>Part 2:</u> $E^{r+1}_{pq} \cong H_{pq}(E^r)$, $r \geq 0$

<u>Pf:</u> We will prove in the next lemma that d induces an

isomorphism $Z_p^r / Z_p^{r+1} \xrightarrow{\sim} B^{r+1}_{p-r} / B^r_{p-r}$

Then d^r decomposes as follows.

$$\begin{tikzcd} E_p^r = Z_p^r / B_p^r \arrow[r, two heads] \arrow[rrr, bend right, "d_p^r"'] & Z_p^r / Z_p^{r+1} \arrow[r, "d", "\sim"'] & B^{r+1}_{p-r} / B^r_{p-r} \arrow[r, hook] & Z^r_{p-r} / B^r_{p-r} = E^r_{p-r} \end{tikzcd}$$

$\Rightarrow$ $\operatorname{im}(d^r_{p+r}) = B_r^{r+1} / B_p^r$

and $\ker(d_p^r) = Z_p^{r+1} / B_p^r$

$\Rightarrow$ $E_p^r = Z_p^r / B_p^r \simeq \ker(d_p^r) / \operatorname{im}(d^r_{p+r})$

$\square$ (Part 2)

$\square$ (Thm 192)

<u>Lemma 195:</u> d induces an isomorphism $\frac{Z_p^r}{Z_p^{r+1}} \xrightarrow{\sim} \frac{B_{p-r}^{r+1}}{B_{p-r}^r}$

<u>Proof:</u> We have

$$\frac{Z_p^r}{Z_p^{r+1}} \overset{\psi}{\twoheadleftarrow} A_p^r \overset{\varphi}{\twoheadrightarrow} \frac{B_{p-r}^{r+1}}{B_{p-r}^r}$$

$$\varphi(c) := [\eta_{p-r}(d(c))]_{B_{p-1}^r}$$

$$\psi(c) := [\eta_p(c)]_{Z_p^{r+1}}.$$

We compute the kernels.

$$\ker\varphi = \{c \in A_p^r \mid \eta_{p-r}(d(c)) \in \eta_{p-r}(d(A_{p-1}^{r-1}))\}$$

$$\text{So } c \in \ker\varphi \iff \exists c' \in A_{p-1}^{r-1} : \eta_{p-r}(d(c)) = \eta_{p-r}(d(c'))$$

$$\iff \exists c' \in A_{p-1}^{r-1} : d(c-c') \in F_{p-r-1}C$$

$$\iff c \in A_{p-1}^{r-1} + A_p^{r+1}$$

$$\ker(\psi) = \{c \in A_p^r \mid \eta_p(c) \in \eta_p(A_p^{r+1})\}$$

$$\text{So } c \in \ker(\psi) \iff \exists c' \in A_p^{r+1} : c - c' \in \underbrace{A_p^r \cap F_{p-1}C}_{= A_{p-1}^{r-1}}$$

$$\iff c \in A_p^{r+1} + A_{p-1}^{r-1}.$$

□