$$pr_1\left(\overline{([4y_0']_{32},[4y_1']_{32},[0]_{32})}\right)=[4y_1']_8\in 4\mathbb{Z}/8\mathbb{Z}$$
$$\cong\quad [y_1']_2\in\mathbb{Z}/2$$

$$pr_0\left(\overline{([4y_0'],[0],[0])}\right)=[4y_0']_{16}\in 4\mathbb{Z}/16$$
$$\cong\quad [y_0']_4\in\mathbb{Z}/4.$$

So $0\subseteq F_0H_2(E)\subseteq F_1H_2(E)\subseteq F_2H_2(E)=H_2(E)$ gives a filtration with factors

$$\begin{array}{ccc} E^3_{02} & E^3_{11} & E^3_{20}. \\ \| & \| & \| \\ \mathbb{Z}/4 & \mathbb{Z}/2 & \mathbb{Z}/2. \end{array}$$

V 2. Terminology

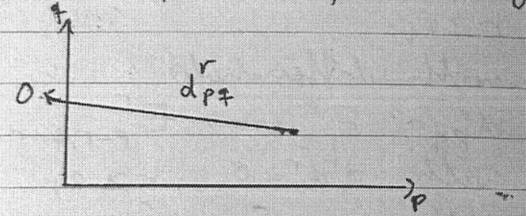
Def 163: A homology spectral sequence (starting on page E^a) in an abelian category $\mathcal{A}$ is
- a family of objects $(E^r_{pq})_{\mathbb{Z}^2}$, $r\geq a$,
- with differentials $d^r_{pq}: E^r_{pq}\longrightarrow E^r_{p-r,q+r-1}$ with $d^rd^r=0$, $r\geq a$,
- with isomorphisms $E^{r+1}_{pq}\simeq H_{pq}(E^r)$, $r\geq a$.

Terminology 164: $n:=p+q$ total degree of E^r_{pq}.

Def 165: A morphism between homology spectral sequences $f: E\longrightarrow E'$ is a family of morphisms in $\mathcal{A}$: $f^r_{pq}: E^r_{pq}\longrightarrow E'^r_{pq}$ which commute with the differentials.

<u>Rk 166:</u> Suppose $E^a_{p,q} = 0 \ \forall_{p,q \in \mathbb{Z}^2}$ s.t. $p<0$ or $q<0$.

Then for $n, r \geq \max\{a, p+1, q+2\}$
$E^r_{p,q} = E^n_{p,q}$
Write $E^\infty_{p,q} := E^r_{p,q}$ for r big enough.



<u>Def 167:</u> (Cohomology spectral sequence)

$(E_r^{p,q})_{p,q}, r \geq a$, $d_r^{pq}: E_r^{pq} \longrightarrow E_r^{p+r, q-r+1}$.

$$E_2^{pq} \xrightarrow{d_2^{pq}} E_2^{p+2, q-1}$$

<u>Def 168:</u> (bounded spectral sequence)
(a) $(E^r_{**})_{r \geq a}$ is called <u>bounded</u> if for every $n \in \mathbb{Z}$ only finitely many $E^a_{p\,n-p}$ are non-zero.

Then for all $p,q \in \mathbb{Z}$ $\exists\, r_0(p,q) \geq a$:

$$E^r_{pq} = E^{r+1}_{p,q} =: E^\infty_{p,q}.$$

(b) We say a bounded $(E^r)_{r \geq a}$ <u>converges to H_*</u>
If $\exists$ family of objects $(H_n)_{n \in \mathbb{Z}}$ in $\mathcal{A}$ and for each $n \in \mathbb{Z}$ a filtration

$$0 = F_s H_n \subseteq F_{s+1} H_n \subseteq \ldots \subseteq F_{t-1} H_n \subseteq F_t H_n = H_n$$

($s = s(n)$, $t = t(n)$, put $F_p H_n = 0$ for $p<s$ and $p>t$)
such that $F_p H_n / F_{p-1} H_n \cong E^\infty_{p\,n-p}$

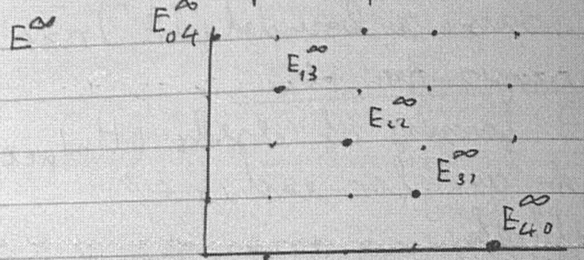
for all $p \in \mathbb{Z}$.
We write

$$E^a_{pq} \Longrightarrow H_{p+q}.$$

(c) A bounded cohomology spectral sequence $(E_r)_{r \geq a}$ converges to H^* if $\forall_{n \in \mathbb{Z}} \exists H_n \in \mathcal{A}$: $\exists$ $0 = F^t H^n \subseteq F^{t-1} H^n \subseteq \ldots \subseteq F^{s+1} H^n \subseteq F^s H^n = H^n$

s.t. $E_\infty^{p,n-p} \simeq F^pH^n / F^{p+1}H^n$

Example 169: Suppose $E^a \Rightarrow H$ and E^a is first quadrant.



Then H_n has a filtration

$0 = F_{-1}H_n \subseteq F_0H_n \subseteq F_1H_n \subseteq \dots \subseteq F_nH_n$

such that

E^∞_{0n} is a subrepresentation of H_n and a quotient of E^a_{0n}

E^∞_{n0} is a quotient of H_n and a subrepr. of E^a_{n0},

more precisely:

$E^a_{0n} \twoheadrightarrow E^\infty_{0n} \simeq F_0H_n \subseteq H_n$

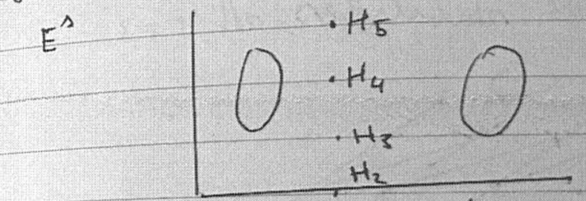
$H_n \twoheadrightarrow \frac{F_nH_n}{F_{n-1}H_n} \simeq E^\infty_{n0} \subseteq E^a_{n0}$

Terminology: $E^a_{0n} \to H_n, H_n \twoheadrightarrow E^a_{n0}$ "edge homomorphisms"

E^∞_{0n}, $n \geq 0$, — fiber terms

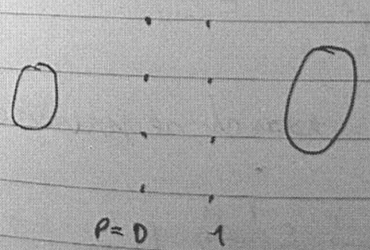
E^∞_{n0}, $n \geq 0$, base terms.

Def 170: A spectral sequence $(E^r)_{r\geq a}$ is said to collapse at E^s if there is at most one non-zero column or at most one non-zero row.



collapsing at the 2^{nd} column.

Examples 171: (a) (2-columns) Suppose $E \Rightarrow H$ s.t. $E^2_{pq} = 0$ unless $p \in \{0,1\}$



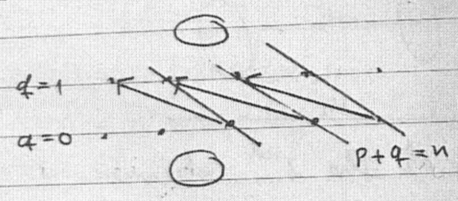
Then $E^2 = E^\infty$ and

$0 \to E^2_{0n} \to H_n \to E^2_{1\,n-1} \to 0$ is exact.

(b) (2 – rows) Suppose $E \Longrightarrow H$
and $E^2_{pq} = 0$, unless $q \in \{0,1\}$.
Then we obtain a long exact sequence

$$\cdots \to H_{n+1} \to E^2_{n+1,0} \xrightarrow{d^2} E^2_{n-1,1} \to H_n \longrightarrow E^2_{n0} \xrightarrow{d^2} E^2_{n-2,1} \longrightarrow H_{n-1} \to \cdots$$

Proof: (b) E stabilizes at $r = 3$.



So we have

$$\begin{tikzcd} E^2_{n-1,1} \arrow[r, two heads] & E^3_{n-1,1} \arrow[r, hook] \arrow[d, equal] & H_n \arrow[r, two heads] & E^3_{n0} \arrow[r, hook] & E^2_{n,0} \arrow[d, "d^2_{n,0}"] \\ \mathrm{im}\, d^2_{n+1,0} \arrow[u, hook] & E^\infty_{n-1,1} & & & E^2_{n-2,1} \end{tikzcd}$$

and we get the exact sequence

$$\cdots \to E^2_{n+1\,0} \xrightarrow{d^2} E^2_{n-1,1} \to H_n \to E^2_{n0} \xrightarrow{d^2} E^2_{n-2\,1} \to \cdots$$

□

We now have to study the unbounded case.

Def 172: (E^∞ terms) Given $(E^r)_{r \geq a}$ we get a nested family of subobjects of E^a_{pq}:

$$0 = B^a_{pq} \subseteq B^{a+1}_{pq} \subseteq B^{a+2}_{pq} \subseteq \cdots \subseteq B^r_{pq} \subseteq \cdots$$

$$\subseteq Z^r_{pq} \subseteq Z^{r-1}_{pq} \subseteq \cdots \subseteq Z^{a+1}_{pq} \subseteq Z^a_{pq} = E^a_{pq}$$

s.t. $E^r_{pq} \simeq Z^r_{pq} / B^r_{pq}$.
Put $Z^\infty_{pq} := \bigcap_{r=a}^{\infty} Z^r_{pq}$, $B^\infty_{pq} := \bigcup_{r=a}^{\infty} B^r_{pq}$

(if they exist) and $E^\infty_{pq} := Z^\infty_{pq} / B^\infty_{pq}$.

Rk 173: (a) If E is bounded then
$\forall_{p,q}\ \exists r_0\ \forall s \geq r_0$: $Z^s_{pq} = Z^{r_0}_{pq}$ and $B^s_{pq} = B^{r_0}_{pq}$.
Thus $\forall_{pq}\ \exists r_0\ \forall s \geq r_0$: $E^s_{pq} = E^\infty_{p,q}$ (Def 172).

So Def 172 and Def 168 (a) are consistent.
(b) Let $f: E \to E'$ be a morphism between homology spectral sequences.

and suppose that E^∞ and E'^∞ exist. Then f induces morphisms
$$f^\infty_{pq}: E^\infty_{pq} \longrightarrow E'^\infty_{pq}$$

<u>Proof:</u>
$$\begin{tikzcd}
Z^\infty_{p,q}/B^\infty_{pq} \arrow[r, dashed, "f^\infty_{pq}"] & Z'^\infty_{pq}/B'^\infty_{pq} \\
Z^\infty_{p,q}/B^a_{pq} \arrow[u, two heads] \arrow[r, dashed] \arrow[ur, phantom, "\circlearrowleft"] & Z'^\infty_{pq}/B'^a_{pq} \arrow[u, two heads] \\
Z^a_{pq}/B^a_{p,q} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, dashed] \arrow[ur, phantom, "\circlearrowleft"] & Z'^a_{pq}/B'^a_{pq} \arrow[u, phantom, sloped, "\subseteq"] \\
E^a_{pq} \arrow[u, phantom, sloped, "\cong"] \arrow[r, "f^a_{pq}"] \arrow[ur, phantom, "\circlearrowleft"] & E'^a_{pq} \arrow[u, phantom, sloped, "\cong"]
\end{tikzcd}$$

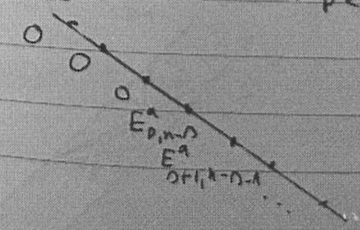
<u>Exercise</u>: Show that all the morphisms $\dashrightarrow$ are well-defined $\square$

<u>Lemma 174</u> (mapping lemma for E^∞)
Let $f: E \longrightarrow E'$ be a morphism of homology spectral sequences.
Suppose $\exists\, r_0 \geq a: \forall_{p,q}: f^{r_0}_{pq}: E^{r_0}_{pq} \longrightarrow E'^{r_0}_{pq}$ is an isomorphism.
Then (i) $\forall_{r \geq r_0} \forall_{pq}: f^r_{pq}: E^r_{pq} \longrightarrow E'^r_{pq}$ is an isomorphism
(ii) $\forall_{pq}: f^\infty_{pq}: E^\infty_{pq} \longrightarrow E'^\infty_{pq}$ is an isomorphism.
(if E^∞ and E'^∞ exist.)

<u>Proof:</u> exercise $\square$

Very useful for convergence (defined later) will be when Z^r_{pq} stabilizes

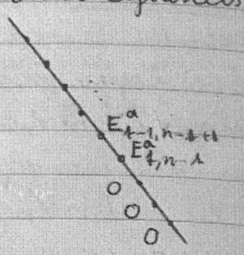
<u>Def 175:</u> (regular spectral sequence)
(a) $(E^r)_{r \geq a}$ is called <u>bounded below</u>, if $\forall_{n \in \mathbb{Z}}\, \exists\, s(n) \in \mathbb{Z}\, \forall_{p < s(n)}: E^a_{p\,n-p} = 0$



Example: 0

For cohomology spectral sequences:

bounded below



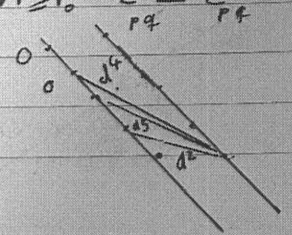
Example:

(b) A spectral sequence E is called regular if $\forall_{p,q}\ \exists r_0\ \forall_{r \geq r_0}$: d^r_{pq} vanishes (d_r^{pq} vanishes)

Rk 176: (i) E is regular if it is bounded below.

(ii) Suppose E is regular. Then $\forall_{p,q}\ \exists r_0 : \forall r \geq r_0\ \ Z^r_{pq} = Z^{r_0}_{pq} = Z^\infty_{pq}$

Proof: (i)



(ii) by definition of regularity □

What should be convergence for homology spectral sequences?

Def 177: Let $(E^r)_{r \geq a}$ be a homology spectral sequence and $H_* = (H_n)_{n \in \mathbb{Z}}$ be a family of objects in $\mathcal{A}$.
We say that $(E^r)_{r \geq a}$ converges weakly to H_* if every H_n has a filtration of objects in $\mathcal{A}$:

$$\ldots \subseteq F_{p-1}H_n \subseteq F_p H_n \subseteq F_{p+1} H_n \subseteq \ldots$$

such that there are isomorphisms

$$E^\infty_{pq} \simeq F_p H_{p+q} / F_{p-1} H_{p+q}.$$

for all $p, q \in \mathbb{Z}$.

Rk 178: E^∞ doesn't see

$$H_n / \bigcup_p F_p H_n$$

and

$$\bigcap_p F_p H_n.$$

Def 179: We say that $(E^r)_{r\geq a}$ *approaches* H_* if it converges weakly to H_* and $\left(\bigcup_p F_p H_n = H_n \text{ and } \bigcap_p F_p H_n = 0\right)$ for all $n \in \mathbb{Z}$.

We say that $(E^r)_{r\geq a}$ converges to H_* if $(E^r)_{r\geq a}$ approaches H_* and $\forall n \in \mathbb{Z}: H_n = \varprojlim_p H_n / F_p H_n$.

Rk 180: If $(E^r)_{r\geq a}$ is bounded below then are equivalent:

1° $(E^r)_{r\geq a}$ converges to H_*

and 2° $(E^r)_{r\geq a}$ approaches H_*.

(exercise)

Convergence Theorem 181: Suppose $(E^r)_{r\geq a}$ converges to H_* and $(E'^r)_{r\geq a}$ converges to H'_*.

Suppose a morphism $h = (h_n)_{\mathbb{Z}} : H_* \to H'_*$ is compatible with a morphism $f : E \to E'$ (in the obvious sense).

Suppose that $\exists r_0 \geq a \; \forall_{p,q} : f^{r_0}_{pq} : E^{r_0}_{pq} \simeq E'^{r_0}_{pq}$.

Then $h : H_* \longrightarrow H'_*$ is an isomorphism, i.e. $\forall n \in \mathbb{Z} : h_n : H_n \xrightarrow{\sim} H'_n$.

Proof: We have exact rows, for $p > s$,

$$\begin{tikzcd}
0 \arrow[r] & F_{p-1}H_n / F_s H_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & F_p H_n / F_s H_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & E^\infty_{p,n-p} \arrow[r] \arrow[d, "\wr", "\text{Lem } 174"'] & 0 \\
0 \arrow[r] & F_{p-1}H'_n / F_s H'_n \arrow[r] & F_p H'_n / F_s H'_n \arrow[r] & E'^\infty_{p,n-p} \arrow[r] & 0
\end{tikzcd}$$

Induction and Snake $\Rightarrow \forall_{p>s} : F_p H_n / F_s H_n \xrightarrow{\sim} F_p H'_n / F_s H'_n$

$$H_n \big/ F_0 H_n = \bigcup_{p>0} F_p H_n \big/ F_0 H_n$$ and same for H'

(This is a filtered colimited)

Universal property for colimits

$\Rightarrow$ h induces an isomorphism

$$H_n \big/ F_0 H_n \xrightarrow{\sim} H'_n \big/ F_0 H'_n$$

Universal property for limits

$\Rightarrow$ $H_n \xrightarrow[\sim]{h} H'_n$ $\square$

IV.3. Leray – Serre spectral sequence

Just knowing the existence of a spectral sequence can have consequences for computing homology groups.

Here we give the example of Leray – Serre spectral sequence for homology group computations for Serre fibrations for example fiber bundles.

Def (Serre fibration) 182:

(a) A pair (X, x_0) with a top. space X and a point $x_0 \in X$ is called based topological space.

(b) A continuous map $E \to B$ between topological spaces is called a Serre fibration if it has the homotopy lifting property w.r.t. any closed disc D^n, $n \in \mathbb{N}_0$, i.e.

For every diagram

$$\begin{tikzcd} X & D^n \arrow[r, "f"] \arrow[d, "\iota_0"] & E \arrow[d, "\pi"] \\ (x,0) & D^n \times [0,1] \arrow[r, "H"] \arrow[ur, phantom, "G"] & B \end{tikzcd}$$

$\exists\ G : D^n \times [0,1] \longrightarrow E$: $\pi \circ G = H$ and $G \circ \iota_0 = f$.

(G is the lift of H via π.)

__Rk 183:__ fiber bundles are Serre fibration.
The idea of Serre fibrations is that one can lift homotopies of cells of CW structures.
(Topology course : next term.)

__Notation 184:__ If B is based: (B, b_0)
Then we write

$$F \xrightarrow{i} E \xrightarrow{\pi} B$$

with $F := \pi^{-1}(b_0)$
(if $b_0 \in \mathrm{im}(\pi)$)

__Terminology:__ B base space
E total space
F Fiber

__Theorem 185 (Serre)__ Let $(F, x_0) \xhookrightarrow{\iota} (E, x_0) \xrightarrow{\pi} (B, b_0)$ be a Serre fibration with $\pi(x_0) = b_0$ and $\pi_0(B) = 0$ (B is pathconnected)
Then there is a long exact sequence

$$\pi_n(F, x_0) \xrightarrow{\iota_*} \pi_n(E, x_0) \xrightarrow{\pi_*} \pi_n(B, b_0)$$

$$\hookrightarrow \pi_{n-1}(F, x_0) \to \pi_{n-1}(E, x_0) \to \pi_{n-1}(B, b_0)$$

$$\hookrightarrow \dots$$

$$\hookrightarrow \pi_0(F, x_0) \to \pi_0(E, x_0) \to 0.$$

Proof: Topology course, next term. $\square$

<u>Thm 186</u> (Leray - Serre spectral sequence)

Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a Serre fibration and $\pi_1(B) = \pi_0(B) = 0$ (B is <u>simply connected</u>)

Then $\exists$ first quadrant spectral sequence $(E^r)_{r \geq 2}$ s.t.

$$E^2_{pq} = H_p(B, H_q(F)) \Longrightarrow H_{p+q}(E)$$

<u>Recall 187:</u> (a) We have the edge maps:

$$\cdot\ H_q(F) = E^2_{0q} \twoheadrightarrow E^\infty_{0q} \hookrightarrow H_q(E)$$

is given by i_*.

$$\cdot\ H_p(E) \twoheadrightarrow E^\infty_{p0} \hookrightarrow E^2_{p0} \cong H_p(B) \otimes_{\mathbb{Z}} H_0(F)$$

So if $\pi_0(F) = 0$, then we get

$H_p(E) \longrightarrow H_p(B)$,

It is given by π_*.

(b) We can use the UCT to compute $H_p(B; H_q(F))$.

$$H_p(B; H_q(F)) \simeq (H_p(B) \otimes_{\mathbb{Z}} H_q(F)) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(B), H_q(F))$$

Eg. <u>p = 0</u>: $H_{p-1}(B) = 0$. So

$$H_0(B; H_q(F)) = H_0(B) \otimes_{\mathbb{Z}} H_q(F) \underset{\substack{\uparrow \\ (\pi_0(B) = 0,\ \text{so } H_0(B) = \mathbb{Z})}}{=} H_q(F)$$

<u>p = 1</u>: $H_1(B) \simeq \pi_1(B) / [\pi_1(B) : \pi_1(B)] \underset{\substack{\uparrow \\ \pi_1(B) = 0}}{=} 0$

$$\Rightarrow H_1(B, H_q(F)) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(H_0(B), H_q(F)) \underset{\substack{\uparrow \\ H_0(B) \text{ is free}}}{\simeq} 0$$

<u>q = 0</u>: Suppose $\pi_0(F) = 0$, i.e. F is path-connected

So $H_0(F) = \mathbb{Z}$.

$$H_p(B; H_q(F)) \simeq H_p(B) \otimes_{\mathbb{Z}} \mathbb{Z} = H_p(B)$$

$\underline{p=2}$ $\quad H_2(B; H_q(F)) \simeq \left(H_2(B)\otimes_{\mathbb{Z}} H_q(F)\right) \oplus T_1^{\mathbb{Z}}(H_1(B), H_q(F))$

$$\underset{\substack{\uparrow \\ H_1(B)=0}}{\simeq} H_2(B)\otimes_{\mathbb{Z}} H_q(F).$$

So for $\pi_1(B)=\pi_0(B)=\pi_0(F)=0$ we get E^2:

$q \backslash p$	0	1	2	3	4
2	$H_2(F)$	0	$*$	$\cdot$	$*$
1	$H_1(F)$	0	$*$	$*$	$*$
0	$\mathbb{Z}$	0	$H_2(B)$	$H_3(B)$	$H_4(B)$

So we get an exact sequence

$$H_2(F)\xrightarrow{i} H_2(E)\xrightarrow{\pi} H_2(B)\xrightarrow{d^2} H_1(F)\to H_1(E)\to 0$$

<u>Exactness</u>: • on $H_1(E)$: $\quad E^3_{01}\simeq H_1(E)$.

and $H_1(F)$: $\quad \| $

$$H_1(F)/\mathrm{im}\, d^2.$$

• on $H_2(B)$: $\ker d^2 = E^3_{02}$ is a quotient of $H_2(E)$.

• on $H_2(E)$: $H_2(F)\to H_2(E)$ has image E^3_{02} and cokernel E^3_{20}.

<u>Example 188</u>: (Loop spaces)

Given (B, b_0) simply connected.

$PB := \{ \gamma : [0,1] \longrightarrow B \mid \gamma$ continuous and $\gamma(0) = b_0 \}$

<u>space of</u> <u>based</u> <u>paths</u> in B.

(with compact open topology)

$\Omega B \subseteq PB$ <u>subspace of</u> <u>based</u> <u>loops</u> (i.e. $\gamma(0) = \gamma(1) = b_0$)

$$\begin{array}{ccccc} \Omega B & \hookrightarrow & PB & \overset{\pi}{\twoheadrightarrow} & B \\ & & \gamma & \longmapsto & \gamma(1) \end{array}$$

is a Hurewicz fibration (also just called fibration), i.e. one can lift every homotopy, see [Hatcher, Prop. 4.64.].
By B being simply connected, we see that ΩB is path connected.

Consider 187(c) for $F = \Omega B$, $E = PB$.
<u>Note</u>: PB is contractible, so $H_n(PB) = 0$ for $n > 0$.

① From 187(c) we get

$$\underset{\substack{\| \\ 0}}{H_2(E)} \longrightarrow H_2(B) \xrightarrow{\sim} H_1(F) \longrightarrow \underset{\substack{\| \\ 0}}{H_1(E)}$$

② Note that $E^3_{21} = E^\infty_{21} = 0$, as $H_3(PB) = 0$.
So

$$H_4(B) \xrightarrow{d^2} H_2(B) \underset{\mathbb{Z}}{\otimes} H_2(B) \xrightarrow{d^2} H_2(\Omega B) \to H_3(B) \to 0$$

is exact.
(For the exactness on $H_2(\Omega B)$ and $H_3(B)$, note that

$$H_3(B) \xrightarrow{\sim} H_2(\Omega B) / \operatorname{im} d^2$$

by the 3rd page.)

<u>Exercise 189:</u> Suppose $n \geq 2$. Then

$$H_p(\Omega S^n) \simeq \begin{cases} \mathbb{Z}, & n-1 \mid p \\ 0, & \text{else} \end{cases}$$

for all $p \in \mathbb{N}_0$.