V Spectral sequences

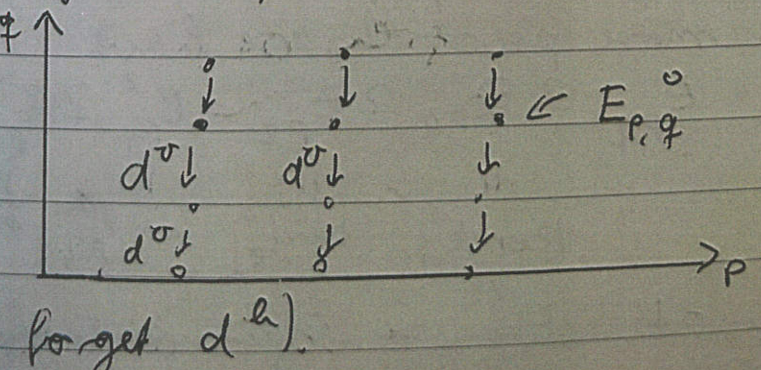
Given a double complex, we want to compute / approximate the homology of the $\oplus$-total complex.

Idea: (Leray) We compute a sequence of subquotients of $H_n^{\oplus}$ using spectral sequences.

V1. Idea of a spectral sequence of a complex.

Let E be a double complex.
For $r \in \mathbb{N}_0$ we construct a diagram E^r supported on parallel lines in $\mathbb{Z}^2$ of slope $\frac{1-r}{r}$.

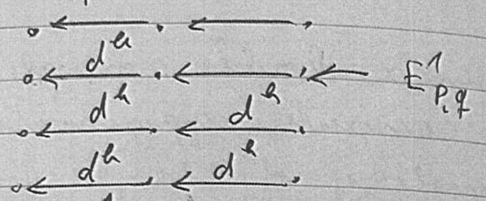
E^0:



(we forget d^h).

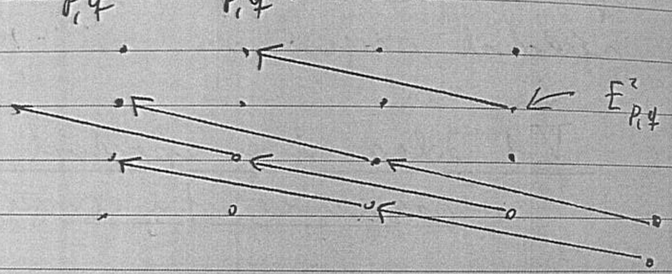
$d^v =: d^0_{p,q}$

E^1: $E^1_{p,q} := H_q(E^0_{p*}) =: H_{p,q}(E^0)$



d^h defines $d^1_{p,q}$

E^2: $E^2_{p,q} := H_{p,q}(E^1)$



What is $d^2_{p,q}$?

<u>Note</u>: $E^2_{p,q}$ and $E^1_{p,q}$ are subquotients of $E_{p,q}$, more precisely we have

$$\begin{array}{ccccccccc} E_{p,q} & \supseteq & Z^1_{p,q} & \supseteq & Z^2_{p,q} & \supseteq & B^2_{p,q} & \supseteq & B^1_{p,q} \supseteq \{0\} \\ & & \| & & \| & & \| & & \| \\ & & \ker d^1_{p,q} & & \ker d^2_{p,q} & & \operatorname{im} d^2_{p+1,q} & & \operatorname{im} d^1_{p,q+1} \end{array}$$

o.t. $E^2_{p,q} = Z^2/B^2$ and $E^1_{p,q} = Z^1/B^1$

<u>Construct $d^2_{p,q}$</u>: $[a]_{B^2} \in E^2_{p,q}$

$\Rightarrow$ $d^h(a) \in \operatorname{im} d^v$, as $d^1([a]_{B^1}) = 0$.

$\Rightarrow$ $\exists\, b \in E_{p-1,q+1}$: $-d^v(b) = d^h(a)$

$$\begin{tikzcd} \bullet & -b \arrow[l, "d^h"'] \arrow[d, "d^v"] & \\ & \bullet & a \arrow[l, "d^h"] \end{tikzcd}$$

Put $d^2_{p,q}([a]_{B^2}) := [d^h(-b)]_{B^2}$.

<u>Example 161</u>

E:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 & \overset{(0,2)}{\mathbb{Z}/32} \arrow[l] \arrow[d, "8"'] & \overset{(1,2)}{\mathbb{Z}/32} \arrow[l, "16"'] \arrow[d, "8"'] & \overset{(2,2)}{\mathbb{Z}/32} \arrow[l, "2"'] \arrow[d, "8"'] & 0 \arrow[l] \\ 0 & \overset{(0,1)}{\mathbb{Z}/32} \arrow[l] \arrow[d, "8"'] & \overset{(1,1)}{\mathbb{Z}/32} \arrow[l, "16"'] \arrow[d, "8"'] & \overset{(2,1)}{\mathbb{Z}/32} \arrow[l, "2"'] \arrow[d, "8"'] & 0 \arrow[l] \\ 0 & \overset{(0,0)}{\mathbb{Z}/32} \arrow[l] \arrow[d] & \overset{(1,0)}{\mathbb{Z}/32} \arrow[l, "16"'] \arrow[d] & \overset{(2,0)}{\mathbb{Z}/32} \arrow[l, "2"'] \arrow[d] & 0 \arrow[l] \\ & 0 & 0 & 0 & \end{tikzcd}$$

Note: This is a double complex as $d^v \circ d^h + d^h \circ d^v = 0$.

E^0: $A := \mathbb{Z}/32$

$$\begin{array}{ccc} 0 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ A & A & A \\ 8\downarrow & 8\downarrow & 8\downarrow \\ A & A & A \\ 8\downarrow & 8\downarrow & 8\downarrow \\ A & A & A \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}$$

E^1: (Note: $\dfrac{4\mathbb{Z}/32\mathbb{Z}}{8\mathbb{Z}/32\mathbb{Z}} \simeq \dfrac{4\mathbb{Z}}{8\mathbb{Z}} \simeq \mathbb{Z}/2\mathbb{Z}$ $\quad [4z]_8 \longmapsto [z]_2$)

$$\mathbb{Z}/8\mathbb{Z} \xleftarrow{0} \mathbb{Z}/8\mathbb{Z} \xleftarrow{2} \mathbb{Z}/8\mathbb{Z}$$

$$\mathbb{Z}/2\mathbb{Z} \xleftarrow{0} \mathbb{Z}/2\mathbb{Z} \xleftarrow{0} \mathbb{Z}/2\mathbb{Z}$$

$$\mathbb{Z}/8\mathbb{Z} \xleftarrow{0} \mathbb{Z}/8\mathbb{Z} \xleftarrow{2} \mathbb{Z}/8\mathbb{Z}$$

E^2:

$$\begin{tikzcd} {[4z]_8}\ \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ {[0]_2}\ \mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2\ {[z]_2} \arrow[llu] \\ \mathbb{Z}/8 & \mathbb{Z}/2 & \mathbb{Z}/2\ {[z]_2} \arrow[llu] \end{tikzcd}$$

$d^2_{2,0}$: $\mathbb{Z}/2 = E^2_{2,0} \ni [z]_2 \leftarrow\!\shortmid [4z]_8 \leftarrow\!\shortmid [4z]_{32}$

under subquotients

$$\begin{array}{ccc} [-16z]_{32}\ A & \xleftarrow{16} & A\ [z]_{32} \\ & & 8\downarrow \\ & & A \xleftarrow{2} A \ni [4z]_{32} \\ & & \Psi \\ & & [8z]_{32} \end{array}$$

$$[0]_2 \longleftarrow\!\shortmid [4z]_2 \longleftarrow\!\shortmid [-16z]_{32}$$
$$[4z]_2 = [0]_2$$

$d^2_{2,1}$: $\mathbb{Z}/2 = E^2_{2,1} \ni [z]_2 \longleftarrow\!\shortmid [z]_2 \longleftarrow\!\shortmid [4z]_{32}$

$$\begin{array}{ccc} [16z]_{32}\ A & \xleftarrow{16} & A\ [z]_{32} \\ & & 8\downarrow \\ & & A \xleftarrow{2} A \ni [4z]_{32} \\ & & [8z]_{32} \end{array}$$

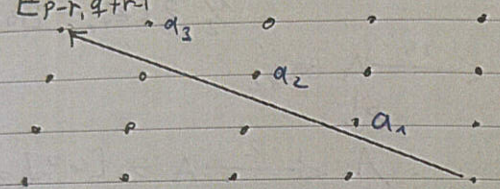
$$[4z]_8 \longleftarrow\!\shortmid [4z]_8 \longleftarrow\!\shortmid [-16z]_{32}$$

E^3:

$$\begin{array}{ccc} \mathbb{Z}/4 & \mathbb{Z}/2 & \mathbb{Z}/2 \\ \mathbb{Z}/2 & \mathbb{Z}/2 & 0 \\ \mathbb{Z}/8 & \mathbb{Z}/2 & \mathbb{Z}/2 \end{array}$$

How to get the differentials for E^r with $r \geq 3$?

$r=4$: $E^r_{p-r,q+r-1}$



$E^r_{p,q} \ni [a]_{B^r}$

Put $a_0 := a \in E_{p,q}$

Claim: $\exists\, a_1 \in E_{p-1,q+1}, a_2 \in E_{p-2,q+2}, \dots, a_{r-1} \in E_{p-r+1,q+r-1}$ s.t.

For all $1 \leq i \leq r-1$ we have
$d^h(a_{i-1}) + d^v(a_i) = 0$.

Then put $d^r_{p,q}([a]_{B^r}) := [d^h(a_{r-1})]_{B^r}$

We call $a_0, a_1, \dots, a_{r-1}$ a defining sequence

Proof of the claim:
By induction on r:
$r=1$: ✓
$r \geq 2$: $d^{r-1}_{p,q}([a]_{B^{r-1}}) = [0]_{B^{r-1}}$

Thus taking a defining sequence $a_0, a_1, \dots, a_{r-2}$ for $d^{r-1}_{p,q}$

we obtain $d^h(a_{r-2}) \in B^{r-1}$

$\Rightarrow \exists$ defining sequence $b_1, b_2, \dots, b_{r-2}$ s.t. $[d^h(b_{r-2})]_{B^{r-2}} = [d^h(a_{r-2})]_{B^{r-2}}$

Substract $(b_1, \dots, b_{r-2})$ from $(a_1, \dots, a_{r-2})$
$\Rightarrow$ W.l.o.g. $d^h(a_{r-2}) \in B^{r-2}$
By induction
$\rightarrow$ W.l.o.g. $d^h(a_{r-2}) \in B^1 = \operatorname{im} d^v$
Take $a_{r-1} \in E_{p-r,q+r-1}$ s.t.
$d^v(a_{r-1}) = -d^h(a_{r-2})$.

Then $a_0, \dots, a_{r-1}$ is a defining sequence for $d^r_{p,q}$ □

Exercise 162: Prove that $d^r_{p,q}$ is well-defined and $d^r \circ d^r = 0$.

We want to study the relation of E^3 to the total $\oplus$ complex in Example 161.

$$0 \leftarrow \mathrm{Tot}^{\oplus}(E)_0 = \mathbb{Z}/_{32} \xleftarrow{d} \underset{\underset{\mathbb{Z}/_{32}\oplus\mathbb{Z}/_{32}}{\cong}}{\mathrm{Tot}^{\oplus}(E)_1} \xleftarrow{d} \underset{\underset{\mathbb{Z}/_{32}\oplus\mathbb{Z}/_{32}\oplus\mathbb{Z}/_{32}}{\cong}}{\mathrm{Tot}^{\oplus}(E)_2} \xleftarrow{d} \underset{\underset{\mathbb{Z}/_{32}\oplus\mathbb{Z}/_{32}}{\cong}}{\mathrm{Tot}^{\oplus}(E)_3} \xleftarrow{d} \underset{\underset{\mathbb{Z}/_{32}}{\cong}}{\mathrm{Tot}^{\oplus}(E)_4} \xleftarrow{d} 0$$

$$d = d^v + d^h.$$

We compute $H_n(E) := H_n(\mathrm{Tot}^{\oplus}(E))$

<u>$n=0$:</u> We write $H_n(E) = Z^{\oplus}_n / B^{\oplus}_n$.

$$Z^{\oplus}_0 = \mathbb{Z}/_{32}$$

$$B^{\oplus}_0 = \{ [8z_0 + 16 z_1]_{32} \mid z_0, z_1 \in \mathbb{Z} \} = 8\mathbb{Z}/_{32}.$$

So $H_0(E) \simeq \mathbb{Z}/_{8\mathbb{Z}}$.

<u>$n=1$:</u> $\mathrm{Tot}^{\oplus}(E)_1 \xrightarrow{d_1} \mathrm{Tot}^{\oplus}(E)_0$

$$([z_0]_{32}, [z_1]_{32}) \longmapsto [8z_0 + 16 z_1]_{32}$$

$$\mathrm{Tot}^{\oplus}(E)_2 \xrightarrow{d_2} \mathrm{Tot}^{\oplus}(E)_1$$

$$([y_0]_{32}, [y_1]_{32}, [y_2]_{32}) \longmapsto ([8y_0 + 16 y_1]_{32}, [8y_1 + 2y_2]_{32})$$

$$Z^{\oplus}_1 = \{ ([z_0]_{32}, [z_1]_{32}) \mid z_0 \equiv_4 -2z_1 \}$$

$$= \{ ([2z_1 + 4k]_{32}, [z_1]) \mid z_1, k \in \mathbb{Z} \}$$

$$B^{\oplus}_1 = \{ ([8y_0 + 16y_1]_{32}, [2y_2]_{32}) \mid y_0, y_1, y_2 \in \mathbb{Z} \}$$

$$\begin{array}{ccccc} \mathbb{Z}/_2 & \hookrightarrow & Z^{\oplus}_1/B^{\oplus}_1 & \twoheadrightarrow & \mathbb{Z}/_2 \\ \cong & & \cong & & \cong \\ E^3_{01} & & H_1(E) & & E^3_{10} \end{array} \quad \text{exact.}$$

$$[k]_2 \longmapsto [([4k]_{32}, [0]_{32})]_{B^{\oplus}_1}$$

$$[([2z + 4k]_{32}, [z]_{32})]_{B^{\oplus}_1} \longmapsto [z]_2$$

<u>$n=2$:</u> $\mathrm{Tot}^{\oplus}(E)_3 \xrightarrow{d_3} \mathrm{Tot}^{\oplus}(E)_2$

$([v_1]_{32}, [v_2]_{32}) \longmapsto ([16v_1]_{32}, [8v_1+2v_2], [8v_2])$

$Z_2^{\oplus} = \ker(d_2) = \{([y_0],[y_1],[y_2]) \mid y_0 \equiv_4 -2y_1 \text{ and } y_2 \equiv_{16} -4y_1\}$

$= \{([2y_1+4k]_{32}, [y_1]_{32}, [-4y_1+16l]_{32} \mid y_1, k, l \in \mathbb{Z}\}$

$B_2^{\oplus} = \{([16v_1], [8v_1+2v_2], [8v_2]) \mid v_1, v_2 \in \mathbb{Z}\}$

For $p = 0,1,2$ we put

$F_p Z_2^{\oplus} = Z_2^{\oplus} \cap \{c_p = c_{p+1} = \dots = 0\}$

coord. c_k for $k \geq p$ are 0

- $F_2 Z_2^{\oplus} = Z_2^{\oplus}$, $F_2 B_2^{\oplus} = B_2^{\oplus}$, $F_2 Z_2^{\oplus} / F_2 B_2^{\oplus} \simeq H_2(E)$
- $F_1 Z_2^{\oplus} = \{([4y_0'], [4y_1'], [0]) \mid y_0', y_1' \in \mathbb{Z}\}$

  $F_1 B_2^{\oplus} = \{([16v_1], [8v_2'], [0]) \mid v_1, v_2' \in \mathbb{Z}\}$

  $F_1 Z_2^{\oplus} / F_1 B_1^{\oplus} \simeq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

- $F_0 Z_2^{\oplus} = \{([4y_0'], [0], [0]) \mid y_0' \in \mathbb{Z}\}$

  $F_0 B_2^{\oplus} = \{([16v_1], [0], [0]) \mid v_1 \in \mathbb{Z}\}$

  $F_0 Z_2^{\oplus} / F_0 B_2^{\oplus} \simeq \mathbb{Z}/4\mathbb{Z}$.

Then we get a description of $H_2(E)$ by a filtration in $H_2(E)$.

$$0 \longrightarrow F_1 Z_2^{\oplus}/F_1 B_2^{\oplus} \longrightarrow F_2 Z_2^{\oplus}/F_2 B_2^{\oplus} \xrightarrow{pr_2} \mathbb{Z}/2\mathbb{Z} \to 0 \quad \text{exact.}$$

$F_2 Z_2^{\oplus}/F_2 B_2^{\oplus} \cong H_2(E)$, $\quad \mathbb{Z}/2\mathbb{Z} \cong E^3_{2,0}$

$$0 \longrightarrow F_0 Z_2^{\oplus}/F_0 B_2^{\oplus} \longrightarrow F_1 Z_2^{\oplus}/F_1 B_2^{\oplus} \xrightarrow{pr_1} \mathbb{Z}/2\mathbb{Z} \to 0$$

$\mathbb{Z}/2\mathbb{Z} \cong E^3_{1,1}$

$$0 \longrightarrow F_0 Z_2^{\oplus}/F_0 B_2^{\oplus} \xrightarrow{pr_0} \mathbb{Z}/4\mathbb{Z} \to 0$$

$\mathbb{Z}/4\mathbb{Z} \cong E^3_{0,2}$

$pr_2(\overline{([2y_1+4k],[y_1],[-4y_1+16l])}) := [-4y_1+16l]_8 \in 4\mathbb{Z}/8\mathbb{Z}$

$\cong [y_1]_2 \in \mathbb{Z}/2\mathbb{Z}$.

$$pr_1\left(\overline{([4y_0']_{32},[4y_1']_{32},[0]_{32})}\right) = [4y_1']_8 \in 4\mathbb{Z}/8\mathbb{Z}$$
$$\cong$$
$$[y_1']_2 \in \mathbb{Z}/2$$

$$pr_0\left(\overline{([4y_0'],[0],[0])}\right) = [4y_0']_{16} \in 4\mathbb{Z}/16$$
$$\cong$$
$$[y_0']_4 \in \mathbb{Z}/4.$$

So $0 \subseteq F_0 H_2(E) \subseteq F_1 H_2(E) \subseteq F_2 H_2(E) = H_2(E)$ gives a filtration with factors

$$\begin{array}{ccc} E^3_{02} & E^3_{11} & E^3_{20}. \\ \| & \| & \| \\ \mathbb{Z}/4 & \mathbb{Z}/2 & \mathbb{Z}/2. \end{array}$$