III 6. Universal Coefficient Theorem.

Question 143: Let $P_\bullet \in Ch(\text{mod-}R)$
$M \in R\text{-mod}$, $N \in \text{mod-}R$.
We have two canonical maps

$$\Phi_n : H_n(P) \otimes_R M \longrightarrow H_n(P \otimes_R M)$$
$$\Phi_n([x] \otimes_R m) := [x \otimes_R m]$$

$$\Psi_n : H^n(\mathrm{Hom}_R(P_n, N)) \to \mathrm{Hom}_R(H_n(P_\bullet), N)$$
$$\Psi_n([f]) := \overline{f}\big|_{Z_n}$$

We will see, that under a mild conditions on $d(P_n) = B_{n-1}(P)$
Φ_n is injective and Ψ_n is surjective.

What are $\mathrm{coker}(\Phi_n)$ and $\ker(\Psi_n)$?

Prop. 160: Let R be right noetherian. Then

(i) $\forall A \in \text{mod-}R$ f.g. $fd(A) = pd(A)$

(ii) Tor-dim $(R) = $ r.gl.dim(R)

Proof: (ii) follows from (i) as we can use R/I to compute Tor-dim and r.gl.dim.

(i) We have $fd(A) \leq pd(A)$.

"$\geq$": Suppose $d := fd(A) < \infty$.
To show $pd(A) \leq d$.

Take a resolution

$$0 \to M_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

with P_i free and f.g.

159 $\Rightarrow$ M_d is flat

R right noetherian $\Rightarrow$ M_d is f.p.

126 $\Rightarrow$ M_d is projective

155 $\Rightarrow$ $pd(A) \leq d$. $\square$

<u>Step 6:</u> ker ∂_n :

$$0 \to d(P_n) \to Z_n \to H_{n-1}(P) \to 0$$

is a flat resolution of $H_{n-1}(P)$.
Thus $\mathrm{Tor}_1^R(H_{n-1}(P), M)$ is the first homology of

$$0 \to d(P_n) \otimes_R M \xrightarrow{\partial_n} Z_{n-1} \otimes_R M \to 0,$$

i.e. it is ker ∂_n □

<u>UCT for homology 145:</u> Suppose $R = \mathbb{Z}$ and all P_n are free.
Then the sequence in Thm 144 is (non-canonically) split.

<u>Proof:</u> $0 \to Z_n \to P_n \xrightarrow{d} d(P_n) \to 0$ is split as $d(P_n)$ is projective.
Thus $Z_n \otimes M$ is a direct summand of $P_n \otimes_R M$.
Thus $H_n \otimes_R M$ is a direct summand of $H_n(P \otimes_R M)$. □

<u>Thm 146:</u> (Künneth formula for complexes)
Let $Q_\bullet \in Ch(R\text{-mod})$ and suppose that all P_n and $d(P_n)$ are flat. Then there is an exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(P) \otimes_R H_q(Q) \to H_n(P \otimes_R Q) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(P), H_q(Q)) \to 0$$

If $R = \mathbb{Z}$ and all P_n are free then the sequence is non-canonically split.

<u>Proof:</u> exercise □

<u>Example 147:</u> (i) X top. space, $S(X)$ the singular chain complex. $R = \mathbb{Z}$.

$$H_n(X, M) := H_n(S(X) \otimes_{\mathbb{Z}} M)$$

"nth homology of X with coeff. in M."

Write $H_n(X) := H_n(X;\mathbb{Z})$

All $S_n(X)$ are free

$\underset{UCT}{\Longrightarrow}$ $H_n(X,M) \simeq (H_n(X)\otimes M) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X),M)$

UCT in topology.

(ii) $H_n(X\times Y) \simeq \left(\bigoplus_{p+q=n} \left(H_p(X)\otimes H_q(Y)\right)\right) \oplus \left(\bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(X),H_q(Y))\right)$

III 6.2. UCT for Hom

Thm 148 (UCT for cohomology)

Suppose all P_n and $d(P_n)$ are projective. Then for all n $\exists$ (non-canonically) split exact sequence

$$0 \longrightarrow \mathrm{Ext}_R^1(H_{n-1}(P),N) \to H^n(\mathrm{Hom}_R(P,N)) \xrightarrow{\Phi} \mathrm{Hom}_R(H_n,N) \to 0$$

Proof: $P_n \simeq Z_n \oplus d(P_n)$ and

$$0 \longrightarrow \mathrm{Hom}(d(P_n),N) \to \mathrm{Hom}(P_n,N) \to \mathrm{Hom}(Z_n,N) \to 0$$

is split exact.

The proof is similar to the proofs of 144 and 145 $\square$

Example 149: X top. space. $R=\mathbb{Z}$

$H^*(X,N) := H^*(\mathrm{Hom}_{\mathbb{Z}}(S(X),N))$

cohomology of X with coeff. in N.

Example 158: $\mathrm{Ext}^n_{\mathbb{Z}/p^2}(\mathbb{Z}/p, \mathbb{Z}/p) \simeq \mathbb{Z}/p$

for all $n \geq 0$.

$\Rightarrow \mathrm{id}_{\mathbb{Z}/p^2}(\mathbb{Z}/p) = \infty$

But $\mathrm{id}_{\mathbb{Z}/p}(\mathbb{Z}/p) = 0$

$\mathrm{id}_{\mathbb{Z}}(\mathbb{Z}/p) = 1$

Lm 159 (fd lemma)

Let $A \in \text{mod-}R$. T.a.e.

1° $\mathrm{fd}(A) \leq d$

2° $\forall_{B \in R\text{-mod}} \forall_{n > d}$: $\mathrm{Tor}^R_n(A, B) = 0$

3° $\forall_{B \in R\text{-mod}}$: $\mathrm{Tor}^R_{d+1}(A, B) = 0$

4° If $0 \to M_d \to F_{d-1} \to \dots \to F_0 \to A \to 0$ is a resolution of A with all F_i flat then M_d is flat.

Pf: Similar to 155 □

Proof of global dim thm: (Thm 151)

$1' \Leftrightarrow 2' \Leftrightarrow 4'$ by 155 and 157.

$2' \geq 3$. So assume $2. > 3$. Then $1 > 3$.

Take $B \in \text{mod-}R$ s.t. $\mathrm{id}(B) > \sup\{\mathrm{pd}(R/I) \mid I \text{ right ideal of } R\}$

$\overset{\shortparallel}{d}$

Take a resolution

$0 \to B \to I^0 \to I^1 \to \dots \to I^{d-1} \to M^d \to 0$

with I^j injective.

Then for all right ideals J we have

$$\mathrm{Ext}^1_R(R/J, M^d) = \mathrm{Ext}^{d+1}_R(R/J, B) = 0$$

Bear $\Rightarrow M^d$ is injective

$\Rightarrow \mathrm{id}(B) \leq d$ ↯ □

The proof of Thm 153 is similar using Prop 122 ($1^\circ \Leftrightarrow 4^\circ$) instead of Bear's criterion.

III 6.1. UCT for $\otimes$

Thm 144: Suppose P_n and $d(P_n)$ are flat for all $n \in \mathbb{Z}$. Then $\forall_{n\in\mathbb{Z}} \exists$ exact sequence

$$0 \to H_n(P) \otimes_R M \xrightarrow{\Phi} H_n(P \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(P), M) \to 0$$

(Künneth formula).

Proof: Step 1: Z_n is flat

Pf: Consider the long exact $\mathrm{Tor}_*(-, M)$ sequence for

$$0 \to Z_n \to P_n \xrightarrow{d} d(P_n) \to 0$$

P_n and $d(P_n)$ are flat

$\Rightarrow \mathrm{Tor}_1^R(Z_n, M) = 0$

M arbitrary $\Rightarrow Z_n$ is flat. □

Step 2: $0 \to Z_n \otimes M \to P_n \otimes M \to d(P_n) \otimes M \to 0$ is exact.

Pf: Follows, because $\mathrm{Tor}_1^R(d(P_n), M) = 0$ □

Step 3: Establishing the exact sequence with $H_n(P \otimes_R M)$.

$$0 \to Z \otimes_R M \to P \otimes_R M \to d(P) \otimes M \to 0$$

is an exact sequence of chain complexes. $Z \otimes_R M$ and $d(P) \otimes_R M$ have zero-differentials. Take long exact sequence in homology:

$$\begin{array}{ccccccccc} H_{n+1}(dP \otimes M) & \xrightarrow{\partial_{n+1}} & H_n(Z \otimes M) & \to & H_n(P \otimes M) & \to & H_n(dP \otimes M) & \xrightarrow{\partial_n} & H_{n-1}(Z \otimes M) \to \\ \wr\!\!\mid & & \wr\!\!\mid & & & & \wr\!\!\mid & & \wr\!\!\mid \\ d(P_{n+1}) \otimes M & & Z_n \otimes M & & & & d(P_n) \otimes M & & Z_{n-1} \otimes M \end{array}$$

$$\Rightarrow 0 \to \operatorname{coker} \partial_{n+1} \to H_n(P \otimes M) \to \ker \partial_n \to 0$$

Step 4: $\partial_n = \mathrm{incl}_{d(P_n) \subseteq Z_{n-1}} \otimes \mathrm{id}_M$.

Pf: Easy exercise □

Step 5: $\operatorname{coker}(\partial_{n+1}) \cong H_n(P) \otimes M$ by right exactness of $- \otimes_R M$.

$$H^n(X,N) \cong \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X),N) \oplus \mathrm{Hom}(H_n(X),N)$$

IV Homological dimension

Def 150: Let $A \in$ mod-R

(a) $pd(A) := \inf\{n \in \mathbb{N}_0 \mid \exists$ projective resolution $0 \to P_n \to P_{n-1} \to \dots \to P_1 \to P_0 \to A \to 0\}$

projective dimension of A

(b) $f.d.(A) := \inf\{n \in \mathbb{N}_0 \mid \exists$ flat resol. $0 \to F_n \to \dots \to F_0 \to A \to 0\}$

flat dim of A

(c) $i.d.(A) := \inf\{n \in \mathbb{N}_0 \mid \exists$ injective resol. $0 \to A \to I^0 \to \dots \to I^n \to 0\}$

injective dim of A.

Thm 151: (Global dim theorem)
The following numbers (in $\mathbb{N}_0 \cup \{\infty\}$) are the same:

1. $\sup\{id(B) \mid B \in$ mod-$R\}$
2. $\sup\{pd(A) \mid A \in$ mod-$R\}$
3. $\sup\{pd(R/I) \mid I$ right ideal of $R\}$
4. $\sup\{d \in \mathbb{N}_0 \mid \exists_{A,B \in \text{mod-}R}: \mathrm{Ext}^d_R(A,B) \neq 0\}$

It is called right global dim of R. r.gl.dim(R)

Lemma 155: (Pd lemma)

Let $A \in \text{mod-}R$. T.a.e.

1° $\mathrm{pd}(A) \leq d$

2° $\forall_{B \in \text{mod-}R}\, \forall n > d$: $\mathrm{Ext}^n_R(A,B) = 0$

3° $\forall_{B \in \text{mod-}R}$: $\mathrm{Ext}^{d+1}_R(A,B) = 0$

4° If $0 \to M_d \to P_{d-1} \to \dots \to P_0 \to A \to 0$ is exact with all P_i projective then the syzygy M_d is projective.

Proof: $1° \Rightarrow 2° \Rightarrow 3° \Rightarrow 4°$ ✓
$\Uparrow$ 4°, dim shifting (for $3° \Rightarrow 4°$). □

Example 156: $R = \mathbb{Z}/p^2$, $A = \mathbb{Z}/p \in \text{mod-}R$

A is not projective as $\mathrm{Ext}^1_R(A,A) \neq 0$, because

$$\begin{array}{c} [z] \mapsto [pz] \quad [z] \longmapsto [z] \\ 0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \to 0 \end{array}$$

is not split, because $\mathbb{Z}/p^2$ has only one subgroup of order p.

$$\dots \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

is a projective (in fact a free) resolution of $\mathbb{Z}/p$.

All syzygys are $p\mathbb{Z}/p^2 \simeq \mathbb{Z}/p$

Lem 155 $\Rightarrow$ $\mathrm{pd}_R(\mathbb{Z}/p) = \infty$

$\Rightarrow$ $\mathrm{gl.dim}(R) = \infty$.

Lem 157: (id lemma) Let $B \in \text{mod-}R$. T.a.e.

1° $\mathrm{id}(B) \leq d$

2° $\forall_A\, \forall n > d$: $\mathrm{Ext}^n_R(A,B) = 0$

3° $\forall_A$: $\mathrm{Ext}^{d+1}_R(A,B) = 0$

4° If $0 \to B \to I^0 \to \dots \to I^{d-1} \to M^d \to 0$ is exact s.t. all I^i are injective. Then M^d is injective.

Proof: Similar to 155. □

<u>Rk 152.</u> (a) A field has global dimension 0

(b) $\mathbb{Z}$ has gl dim 1

(c) $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ & \mathbb{Q} \end{pmatrix} =: R$

Then $r.gl.dim(R) = 1$

$l.gl\ dim(R) = 2$.

(Exercise, hint: Use 151 3.)

Note: R is right noetherian, but not left noetherian.

(d) Question: What are left and right global dim of $M_2(\mathbb{R})$ and $M_2(\mathbb{Z})$?

(e) We will see later that $r.gl.dim(R) = l.gl.dim(R)$ if R is right and left noetherian.

<u>Thm 153:</u> (Tor dim thm).
The following number are equal.

1. $\sup\{fd(A) \mid A \in \text{mod-}R\}$
2. $\sup\{fd(R/I) \mid I \text{ right ideal of } R\}$
3. $\sup\{fd(B) \mid B \in R\text{-mod}\}$
4. $\sup\{fd(R/I) \mid I \text{ left ideal of } R\}$
5. $\sup\{d \mid Tor_d^R(A,B) \neq 0$ for some $A \in \text{mod-}R$ and $B \in R\text{-mod}\}$

<u>Tor dimension of R</u> (<u>weak dimension of R</u>)

<u>Example 154:</u> (a) $R = \mathbb{Z}$: $fd(\mathbb{Q}) = 0$, $pd(\mathbb{Q}) = 1$.

(b) We always have
$fd(A) \leq pd(A)$
and
$\text{Tor-dim}(R) \leq r.gl.dim(R)$ and $l.gl.dim(R)$.

For the proof of 151 and 153 we need three lemmas (pd, id, fd).

Lemma 155: (Pd lemma)
Let $A \in \text{mod-}R$. T.a.e.:
1° $pd(A) \leq d$
2° $\forall_{B \in \text{mod-}R} \forall n > d$: $\operatorname{Ext}^n_R(A,B) = 0$

3° $\forall_{B \in \text{mod-}R}$: $\operatorname{Ext}^{d+1}_R(A,B) = 0$

4° If $0 \to M_d \to P_{d-1} \to \dots \to P_0 \to A \to 0$
is exact with all P_i projective
then the syzygy M_d is
projective.

Proof: $1^\circ \Rightarrow 2^\circ \Rightarrow 3^\circ \Rightarrow 4^\circ$ ✓
$\Uparrow$ dim shifting.
4° □

Example 156: $R = \mathbb{Z}/p^2$, $A = \mathbb{Z}/p \in \text{mod-}R$

A is not projective as
$\operatorname{Ext}^1_R(A,A) \neq 0$, because

$$0 \to \mathbb{Z}/p \xrightarrow{[z] \mapsto [pz]} \mathbb{Z}/p^2 \xrightarrow{[z] \mapsto [z]} \mathbb{Z}/p \to 0$$

is not split, because $\mathbb{Z}/p^2$

has only one subgroup of order p.

$$\dots \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

is a projective (in fact a free) resolution of $\mathbb{Z}/p$.
All syzygys are $p\mathbb{Z}/p^2 \cong \mathbb{Z}/p$

Lem 155 $\Rightarrow$ $pd_R(\mathbb{Z}/p) = \infty$

$\Rightarrow$ $gl.dim(R) = \infty$.

Lem 157: (id lemma) Let $B \in \text{mod-}R$
T.a.e.
1° $id(B) \leq d$
2° $\forall_A. \forall n > d$: $\operatorname{Ext}^n_R(A,B) = 0$
3° $\forall_A$: $\operatorname{Ext}^{d+1}_R(A,B) = 0$

4° If $0 \to B \to I^0 \to \dots \to I^{d-1} \to M^d \to 0$
is exact s.t. all I^i are injective. Then M^d is injective.

Proof: Similar to 155. □

Example 158: $\mathrm{Ext}^n_{\mathbb{Z}/p^2}(\mathbb{Z}/p, \mathbb{Z}/p) \simeq \mathbb{Z}/p$

for all $n \geq 0$.

$\Rightarrow \mathrm{id}_{\mathbb{Z}/p^2}(\mathbb{Z}/p) = \infty$

But $\mathrm{i.d.}_{\mathbb{Z}/p}(\mathbb{Z}/p) = 0$

$\mathrm{id}_{\mathbb{Z}}(\mathbb{Z}/p) = 1$

Lem 159 (fd lemma):

Let $A \in$ mod-R. T.a.e.

1° $\mathrm{fd}(A) \leq d$

2° $\forall_{B \in R\text{-mod}} \forall_{n > d}$: $\mathrm{Tor}^R_n(A, B) = 0$

3° $\forall_{B \in R\text{-mod}}$: $\mathrm{Tor}^R_{d+1}(A, B) = 0$

4° If $0 \to M_d \to F_{d-1} \to \dots \to F_0 \to A \to 0$ is a resolution of A with all F_i flat then M_d is flat.

Pf: Similar to 155 □

Proof of global dim thm: (Thm 151)

$1' \Leftrightarrow 2' \Leftrightarrow 4'$ by 155 and 157.

$2' \geq 3'$. So assume $2. > 3.$ Then $1. > 3.$

Take $B \in$ mod-R s.t. $\mathrm{id}(B) > \sup\{\mathrm{pd}(R/I) \mid I \text{ right ideal of } R\}$

$\overset{\shortparallel}{d}$

Take a resolution

$0 \to B \to I^0 \to I^1 \to \dots \to I^{d-1} \to M^d \to 0$

with I^j injective.

Then for all right ideals J we have

$$\mathrm{Ext}^1_R(R/J, M^d) = \mathrm{Ext}^{d+1}_R(R/J, B) = 0$$

Baer $\Rightarrow M^d$ is injective

$\Rightarrow \mathrm{id}(B) \leq d$ ↯ □

The proof of Thm 153 is similar using Prop 122 ($1^\circ \Leftrightarrow 4^\circ$) instead of Baer's criterion.

Prop. 160: Let R be right noetherian. Then

(i) $\forall A \in$ mod-R f.g.: $fd(A) = pd(A)$

(ii) Tor-dim(R) = r.gl.dim(R).

Proof: (ii) follows from (i) as we can use R/I to compute Tor-dim and r.gl.dim.

(i) We have $fd(A) \leq pd(A)$.

"$\geq$": Suppose $d := fd(A) < \infty$.
To show $pd(A) \leq d$.

Take a resolution

$$0 \to M_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

with P_i free and f.g.

159 $\Rightarrow$ M_d is flat

R right noetherian $\Rightarrow$ M_d is f.p.

126 $\Rightarrow$ M_d is projective

155 $\Rightarrow$ $pd(A) \leq d$. □