II 6. Right derived functors

$F : \mathcal{A} \to \mathcal{B}$ left exact additive functor between abelian categories $\mathcal{A}, \mathcal{B}$.

Suppose $\mathcal{A}$ has enough injectives.

($i \geq 0$) ith right derived functor of F:

$A \in \mathcal{A}$. $A \to I^{\bullet}$ injective resolution.

We get

$$0 \to F(A) \to F(I^0) \to F(I^1) \to \ldots$$

Remove $F(A)$

$$0 \to F(I^0) \to F(I^1) \to F(I^2) \to \ldots$$

Take cohomologies

$$R^i F(A) := H^i(F(I^{\bullet}))$$

Transfer to the left-derived case:

$F^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$ is right exact and $\mathcal{A}^{op}$ has enough projectives. Thus we have

$$L_i(F^{op}) \quad , i \geq 0.$$

Claim: $R^i F(A) \cong (L_i(F^{op}))^{op}(A)$.

(exercise.)

So we apply all the results of II.5.

Def: $A \in \mathcal{A}$ is called F-acyclic if $R^i F(A) = 0 \ \forall i \geq 1$.

Def 92: (Ext functors)

$A \in R\text{-mod}$

$\mathrm{Hom}(A, -) : R\text{-mod} \to Ab$ is left exact.

$i \geq 0$, $\mathrm{Ext}^i_R(A, B) := R^i \mathrm{Hom}_R(A, -).(B)$.

Prop. 93. Let $B \in R\text{-mod}$. T.a.e.:

1° B is injective

2° $\mathrm{Ext}^1_R(A,B) = 0$ $\forall A \in R\text{-mod}$

3° $\mathrm{Ext}^i_R(A,B) = 0$ $\forall A \in R\text{-mod}$, $\forall i \geq 1$

Proof: $3^\circ \Rightarrow 2^\circ$ ✓

$1^\circ \Rightarrow 3^\circ$: $\mathrm{Hom}_R(-,B)$ is exact $\Rightarrow 3^\circ$.

$2^\circ \Rightarrow 1^\circ$: Take a short exact sequence

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

$$\Rightarrow 0 \to \mathrm{Hom}_R(A_3,B) \to \mathrm{Hom}_R(A_2,B) \to \mathrm{Hom}_R(A_1,B) \to \underset{\overset{\parallel}{0}}{\mathrm{Ext}^1(A_3,B)}$$

is exact

So B is injective. $\square$

Prop 94: Let $A \in R\text{-mod}$. T.a.e.

1° A is projective

2° $\mathrm{Ext}^1_R(A,B) = 0$ $\forall B \in R\text{-mod}$

3° $\mathrm{Ext}^i_R(A,B) = 0$ $\forall i \geq 1$ $\forall B \in R\text{-mod}$.

(exercise)

Example 95: $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z})$.

Take $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, a short exact sequence.

$$0 \to \overset{=0}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z})} \to \overset{=0}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q})} \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$$

$$\to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \to \overset{=0}{\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q})} \to \overset{=0}{\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})}$$

$$\to \mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \to \overset{=0}{\mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q})} \to \overset{=0}{\mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})}$$

long exact sequence.

$$\Rightarrow \operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \simeq \begin{cases} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}), & i=1 \\ 0, & i \neq 1 \end{cases} \quad (i \geq 0)$$

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \simeq \hat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n\mathbb{Z}$$

$$\frac{1}{n}\mathbb{Z}/\mathbb{Z} \quad 0$$

$$\operatorname{Hom}\left(\frac{\frac{1}{n}\mathbb{Z}}{\mathbb{Z}}, \frac{\frac{1}{n}\mathbb{Z}}{\mathbb{Z}}\right) \simeq \mathbb{Z}/n\mathbb{Z}$$

<u>Ref 96</u>: (R^iG for a contravariant functor)
Let $G: \mathcal{A} \to \mathcal{B}$ be contravariant and left exact.
(i.e. $G: \mathcal{A}^{op} \to \mathcal{B}$ is covariant and left exact.)

For $A \in \mathcal{A}$ take a projective resolution $P_\bullet \to A$

we get $0 \to G(A) \to G(P_0) \to G(P_1) \to \cdots$

and $0 \to G(P_0) \to G(P_1) \to \cdots$
$R^iG(A) := H^i(G(P_\bullet))$.

<u>ith right derived functor of G.</u>

<u>Example 97</u>: (a) $B \in R$-mod.

$\operatorname{Hom}_R(-, B): R\text{-mod} \to Ab$

is left exact and contravariant.

<u>Example</u> $R = \mathbb{Z}$, $B = \mathbb{Z}$

$$R^i\operatorname{Hom}_{\mathbb{Z}}(-, B)(\mathbb{Q}/\mathbb{Z}) = ?$$

Take a projective resolution of $\mathbb{Q}/\mathbb{Z}$

$$0 \to \ker(\pi) \to \bigoplus_{i=1}^{\infty} \mathbb{Z} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0$$

$$(a_i)_{\mathbb{Z}} \mapsto \left[\sum_{i=1}^{\infty} a_i\right]_{\mathbb{Z}}$$

We obtain after applying $Hom(-,\mathbb{Z})$:

$$0 \to Hom(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \to Hom(\bigoplus_{\mathbb{N}} \mathbb{Z}, \mathbb{Z}) \to Hom(\ker(\pi), \mathbb{Z}) \to 0.$$

Remove $Hom(\mathbb{Q}/\mathbb{Z}, \mathbb{Z})$.

$$\begin{array}{ccc} 0 \to Hom(\bigoplus \mathbb{Z}, \mathbb{Z}) & \xrightarrow{res} & Hom(\ker(\pi), \mathbb{Z}) \to 0 \\ \| & & \| \\ H^0(G(P_\cdot)) & & H^1(G(P_\cdot)) \end{array}$$

res is injective $\Rightarrow$ $H^0(G(P_\cdot)) = 0$

$$\| \\ Hom(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}).$$

What is $R^1 Hom(-,\mathbb{Z})(\mathbb{Q}/\mathbb{Z})$?

$i > 1$: $R^i Hom(-,\mathbb{Z})(\mathbb{Q}/\mathbb{Z}) = 0$.

(b) $Sheaves(X) \xrightarrow{\Gamma} Ab$

$\mathcal{F} \longmapsto \Gamma(\mathcal{F}) := \mathcal{F}(X)$

Γ is left exact.

$$H^i(X, \mathcal{F}) := R^i\Gamma(\mathcal{F})$$

is called the ith sheaf cohomology of $\mathcal{F}$.

II.7. Balancing Tor and Ext.

Theorem 98: Let $A \in \text{mod-}R$, $B \in R\text{-mod}$.

Then $L_n(- \otimes_R B)(A) \simeq L_n(A \otimes_R -)(B)$,

for all $n \in \mathbb{N}_0$.

Proof: We choose projective resolutions

$$P_\bullet \xrightarrow{\epsilon} A \quad \text{in mod-}R$$
$$Q_\bullet \xrightarrow{\eta} B \quad \text{in } R\text{-mod}$$

We consider the following double complex

$$\begin{tikzcd}
 \arrow[d] & \arrow[d] & \arrow[d] & & \\
A \otimes Q_2 \arrow[d] & P_0 \otimes Q_2 \arrow[d, "d"] & P_1 \otimes Q_2 \arrow[l, "d"'] \arrow[d, "-d"] & P_2 \otimes Q_2 \arrow[l, "d"'] \arrow[d, "d"] & \arrow[l] \\
A \otimes Q_1 \arrow[d] & P_0 \otimes Q_1 \arrow[d, "d"] & P_1 \otimes Q_1 \arrow[l, "d"'] \arrow[d, "-d"] & P_2 \otimes Q_1 \arrow[l, "d"'] \arrow[d, "d"] & \arrow[l] \\
A \otimes Q_0 & P_0 \otimes Q_0 & P_1 \otimes Q_0 \arrow[l, "d"'] & P_2 \otimes Q_0 \arrow[l, "d"'] & \arrow[l] \\
& P_0 \otimes B & P_1 \otimes B \arrow[l] & P_2 \otimes B \arrow[l] & \arrow[l]
\end{tikzcd}$$

We consider $A \otimes Q_\bullet$ as a double complex concentrated in the 0th column.

Recall $\text{Tot}(P_\bullet \otimes Q_\bullet)_i := \bigoplus_{k+l=i} (P_k \otimes Q_l)$.

Then $\text{Tot}(A \otimes Q)_i = A \otimes Q_i$.

So $A \otimes Q = \text{Tot}(A \otimes Q) \xleftarrow{\epsilon \otimes id_Q} \text{Tot}(P \otimes Q)$

and $P \otimes B \xleftarrow{id_P \otimes \eta} \text{Tot}(P \otimes Q)$.

Claim: $\epsilon \otimes id_Q$ and $id_P \otimes \eta$ are quasi-isomorphisms

Pf: We will show that the mapping cones of $\epsilon \otimes id_Q$ and $id_P \otimes \eta$ are acyclic.

(1) mapping cone of $\epsilon \otimes id_Q$:

We put $A \otimes Q_\bullet$ with $-d$ in column for $p=-1$.

$$\underset{\text{with } -d}{A \otimes Q_\bullet} \xleftarrow{\epsilon} P_\bullet \otimes Q_\bullet$$

p −1 0 1 2 3

We call this double complex C.
$Tot(C)[1]$ is the cone of $\epsilon \otimes id_Q$.

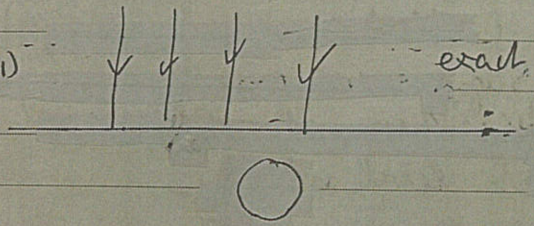
Thus we need to show that $Tot(C)$ is acyclic.

This is done by the "Acyclic Assembly Lemma".

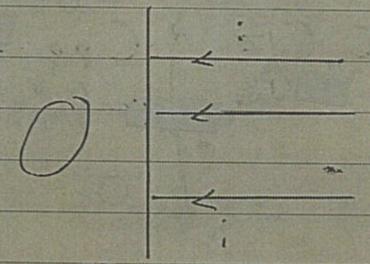
(2) mapping cone of $id_{P_\bullet} \otimes \eta$:
Similar to (1) □

□

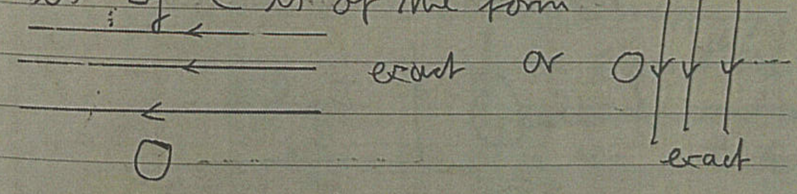
<u>Lemma 99:</u> (Acyclic Assembly Lemma)
Let C be a double complex of R-mod.
(a) If C is concentrated on an upper half plan with exact columns

(a1)


or if

(a2)


then $Tot^{\Pi}(C)$ is acyclic

(b) If C is of the form


then $Tot^{\oplus}(C)$ is acyclic.

Proof: Step 1: We reduce to Case (a1).

- Interchanging rows and columns, we obtain (a1) $\Leftrightarrow$ (a2) and (b1) $\Leftrightarrow$ (b2) (reflection on $p-q=0$)
- Suppose we are in Case (b2). We truncate C: each column at level n.

$$(\tau_n C)_{pq} = \begin{cases} C_{p,q}, & q > n \\ \ker(d^v_{p,n} : C_{p,n} \to C_{p,n-1}), & q = n \\ 0, & q < n. \end{cases}$$

$I_n C$ satisfies (a1) and
$\mathrm{Tot}^{\oplus}(I_n C) = \mathrm{Tot}^{\Pi}(I_n C)$.

$$\begin{array}{c|l} 0 & I_n C \\ \hline 0 & 0 \end{array}$$

If (a1) is proven then
$\mathrm{Tot}^{\oplus}(I_n C) = \mathrm{Tot}^{\Pi}(I_n C)$ is acyclic.
Thus $\mathrm{Tot}^{\oplus}(C)$ is acyclic.

Step 2: Proof of (a1):

We only prove the exactness of $\mathrm{Tot}^{\Pi}(C)$ at 0.
Take a 0-cycle of $\mathrm{Tot}^{\Pi}(C)$:

$$c = (c_{p,q})_{p+q=0}.$$

W.l.o.g. $c_{p,q} = 0 \ \forall q < 0$.

$$\begin{tikzcd} \cdots & b_{-2,3} \arrow[l, mapsto] \arrow[d, mapsto] & & \\ & c_{-2,2} & b_{-1,2} \arrow[l, mapsto] \arrow[d, mapsto] & \\ & & c_{-1,1} & b_{0,1} \arrow[l, mapsto] \arrow[d, mapsto] \\ & & & c_{00} \end{tikzcd}$$

$b_{0,1}$ exists by exactness.
$b_{-1,2}$ mapping to $c_{-1,1} - d^h(b_{0,1})$ exists by exactness of the -1^{th} column.
etc.

□