Example 85: Sheaves(X) := category of sheaves of abelian groups on a top. space X.

Claim: Sheaves(X) has enough injectives

Sketch:
$x \in X$, $\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)$ stalk at x.

Idea: $\forall x \in X$ we choose an injective object of Sheaves(X).
Then take the product of those.

① Skyscraper sheaves:
Take $A \in Ab$ and $x \in X$.

$$(x_* A)(U) := \begin{cases} A, & x \in U \\ 0, & x \notin U \end{cases}$$

For $V \subseteq U$
$$res^U_V : \underset{A}{\underset{\parallel}{(x_* A)(U)}} \xrightarrow{id} \underset{A}{\underset{\parallel}{(x_* A)(V)}}, \text{ if } x \in V$$

and the zero map if $x \notin V$.

② $\mathrm{Hom}_{Ab}(\mathcal{F}_x, A) \simeq \mathrm{Hom}_{Sheaves(X)}(\mathcal{F}, x_* A)$

$$\alpha_x \longleftarrow\!\shortmid\ \alpha$$

$$(\alpha_x : \mathcal{F}_x \longrightarrow (x_* A)_x = A)$$

(exercise).

Thus if A is injective then $x_* A$ is injective in Sheaves(X).

③ $\forall x \in X$ take an injective abelian group I_x and $\mathcal{F}_x \hookrightarrow I_x$

By ② we obtain
$$\mathcal{F} \longrightarrow x_*(I_x)$$

and thus
$$\mathcal{F} \longrightarrow \prod_{x \in X} x_*(I_x).$$

Details: exercise.

II.5. Left derived functors

Goal: General theory behind the construction of the Tor_i functors.

We fix

- a right exact covariant functor $F : \mathcal{A} \to \mathcal{B}$ ($\mathcal{A}$, $\mathcal{B}$ abelian cat.)
- $\forall A \in \mathcal{A}$ a projective resolution $P_\bullet \to A$

(in particular we want $\mathcal{A}$ to have enough projectives)

Ex: $F := M \otimes - : R\text{-mod} \to Ab$.

We define a family of functors $L_i F : \mathcal{A} \to Ab$, $i \geq 0$ as follows

$$L_i F(A) := H_i(F(P_\bullet)), \; i \geq 0$$

On morphisms:

$f : A \to A'$. Use Thm 66.

$\Rightarrow f$ extends to $\tilde{f}_\bullet : P_\bullet \to P'_\bullet$

which gives $F(\tilde{f}) : F(P_\bullet) \to F(P'_\bullet)$ and a family of maps

$$\begin{array}{ccc} H_i(F(\tilde{f})) : H_i(F(P_\bullet)) & \to & H_i(F(P'_\bullet)) \\ \| & & \| \\ L_i F(A) & & L_i F(A') \end{array}$$

Terminology 86: We call $L_i F : \mathcal{A} \to Ab$ the <u>ith left derived functor of F</u>

<u>Example 87:</u> $\mathcal{A} = \mathcal{B} = Ab$

$F : \mathcal{A} \to \mathcal{B}$

$F(A) := \mathbb{Q}/\mathbb{Z} \otimes A$

We compute $L_i F(\mathbb{Z}/2)$

projective resolution

$$0 \to \underset{P_1}{\mathbb{Z}} \xrightarrow{2} \underset{P_0}{\mathbb{Z}} \to \mathbb{Z}/2 \to 0$$

$$\begin{array}{ccccccc} 0 \longrightarrow & F(P_1) & \xrightarrow{2} & F(P_0) & \longrightarrow 0 \\ & \| & & \| & \\ & \mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z} & & \mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z} & \\ & \wr\| & & \wr\| & \\ & \mathbb{Q}/\mathbb{Z} & & \mathbb{Q}/\mathbb{Z} & \end{array}$$

$$\begin{array}{ccc} H_2(F(P_\cdot)) & H_1(F(P_\cdot)) & H_0(F(P_\cdot)) \\ \| & \| & \| \\ 0 & \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} & 0 \\ & \wr\| & \\ & \mathbb{Z}/2\mathbb{Z} & \end{array}$$

<u>Prop. 88:</u> (a) $L_0 F(A) \simeq F(A)$

(b) $L_i F(A)$ doesn't depend on the projective resolution of A up to natural isomorphism.

More precisely: If we choose for all $A \in \mathcal{A}$ a second proj. resolution then we obtain other left derived functors

$$\tilde{L}_i F.$$

The claim in (b) is that $L_i F$ and $\tilde{L}_i F$ are natural isomorphic functors.

Proof: (a) F is right exact.

$\Rightarrow \quad F(P_1) \longrightarrow F(P_0) \longrightarrow F(A) \longrightarrow 0$

is exact $\Rightarrow H_0(F(P_\bullet)) \simeq F(A)$

(b)
$$\begin{tikzcd} A \arrow[r, equal, "id"] & A \\ P_0 \arrow[u] \arrow[r, shift left, "\alpha_0"] & Q_0 \arrow[u] \arrow[l, shift left, "\beta_0"] \\ P_1 \arrow[u] \arrow[r, shift left, "\alpha_1"] & Q_1 \arrow[u] \arrow[l, shift left, "\beta_1"] \\ \arrow[u] & \arrow[u] \end{tikzcd}$$

(Thm 66)

$H_i(F(\beta)) \circ H_i(F(\alpha)) = id_{H_i(F(P_\bullet))}$

$H_i(F(\alpha)) \circ H_i(F(\beta)) = H_i(F(\alpha \circ \beta))$

$\underset{\underset{\alpha \circ \beta \sim id_Q}{\uparrow}}{=} H_i(F(id)) = id_{H_i(F(Q_\bullet))}$

$H_i(F(\alpha)) : L_i A \longrightarrow \tilde{L}_i A$

is a natural isomorphism $\square$

Thm 89: The derived functors

$L_i F$, $i \geq 0$, form a homological δ-functor.

Proof: $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ exact.

$$\begin{tikzcd} A' & A & A'' \\ P'_\bullet \arrow[u] & P_\bullet \arrow[u] & P''_\bullet \arrow[u] \end{tikzcd}$$

Horse shoe $\Rightarrow$ We obtain a projective resolution $\tilde{P}_\bullet \longrightarrow A$ with $\tilde{P}_i := P'_i \oplus P''_i$.

$\Rightarrow$ We get a long exact sequence.

$\overset{\delta}{\hookrightarrow} H_i(F(P'_\bullet)) \longrightarrow H_i(F(\tilde{P}_\bullet)) \longrightarrow H_i(F(P''_\bullet)) \overset{\delta}{\rightarrow}$

$\qquad\qquad\qquad\qquad \wr\wr$

$\qquad\qquad\qquad H_i(F(P_\bullet))$

but we need the exactness of

$$0 \to F(P_i') \to F(\tilde P_i) \to F(P_i'') \to 0$$

<u>Proof</u>: F is right exact.

$$0 \to P_i' \underset{pr_1}{\overset{\iota_1}{\rightleftarrows}} \tilde P_i \xrightarrow{\tilde{pr}_2} P_i'' \to 0 \quad \text{is split exact}$$

$$\Rightarrow \quad F(pr_1) \circ F(\iota_1) = id_{F(P_i')}$$

$$\Rightarrow \quad F(\iota_1) \text{ is a mono.} \qquad \square$$

We need to show that δ is natural.

$$\begin{tikzcd}
& P'_\bullet \arrow[dr, "\epsilon'"] & & \tilde P_\bullet \arrow[dr, "\tilde\epsilon"] & & P''_\bullet \arrow[dr, "\epsilon''"] & & \\
0 \arrow[rr] & & A' \arrow[rr, "i_A"] \arrow[dd, "f'"] & & A \arrow[rr, "\pi_A"] \arrow[dd, "\tilde f"] & & A'' \arrow[r] \arrow[dd, "f''"] & 0 \\
& Q'_\bullet \arrow[dr, "\eta'"] & & \tilde Q_\bullet \arrow[dr, "\tilde\eta"] & & Q''_\bullet \arrow[dr, "\eta''"] & & \\
0 \arrow[rr] & & B' \arrow[rr, "i_B"] & & B \arrow[rr, "\pi_B"] & & B'' \arrow[r] & 0
\end{tikzcd}$$

We extend f' to $\tilde f'$ and f'' to $\tilde f''$.

We need to show that we can extend f to some $\tilde f$ s.t. the whole diagram is commutative.

For the ith layer to be commutative we need $\tilde f_i$ to be of the form

$$\tilde f_i = \begin{pmatrix} \tilde f_i' & \gamma_i \\ 0 & \tilde f_i'' \end{pmatrix} : \begin{matrix} P_i' \\ \oplus \\ P_i'' \end{matrix} \longrightarrow \begin{matrix} Q_i' \\ \oplus \\ Q_i'' \end{matrix}$$

(commutativity with inclusions and the projections)

$\tilde f$ needs to be a chain map.

(next time)

we need to choose the γ_i, $i \geq 0$

The differential $\tilde{d}_n : \tilde{P}_n \longrightarrow \tilde{P}_{n-1}$

have the form

$$\begin{pmatrix} d'_{P,n} & \lambda_{P,n} \\ & d''_{P,n} \end{pmatrix} : \tilde{P}_n \longrightarrow \tilde{P}_{n-1}$$

and $$\begin{pmatrix} d'_{Q,n} & \lambda_{Q,n} \\ & d''_{Q,n} \end{pmatrix} : \tilde{Q}_n \longrightarrow \tilde{Q}_{n-1}$$

$$\begin{pmatrix} d'_{Q,n} & \lambda_{Q,n} \\ & d''_{Q,n} \end{pmatrix} \circ \begin{pmatrix} \tilde{f}'_n & \gamma_n \\ & \tilde{f}''_n \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{f}'_{n-1} & \gamma_{n-1} \\ & \tilde{f}''_{n-1} \end{pmatrix} \circ \begin{pmatrix} d'_{P,n} & \lambda_{P,n} \\ & d''_{P,n} \end{pmatrix}$$

$$\Longleftrightarrow \quad d'_{Q,n} \circ \gamma_n + \lambda_{Q,n} \tilde{f}''_n - \tilde{f}'_{n-1} \lambda_{P,n} - \gamma_{n-1} d''_{P,n} = 0$$

$$\begin{tikzcd}
P_0' \arrow[r, "\nu_1"] \arrow[dr, "\varepsilon'"] \arrow[dd, "\tilde{f}_0'"'] & \tilde{P}_0 \arrow[dr, "\tilde{\varepsilon}"] & & \\
& A' \arrow[r, "i_A"] \arrow[d, "f'"] & A \arrow[r, "\pi_A"] \arrow[d, "f"] & A'' \arrow[d, "f''"] \\
Q_0' \arrow[r, "\eta'"] & B' \arrow[r, "i_B"] & B \arrow[r, "\pi_B"] & B''
\end{tikzcd}$$

Further we need to recall

$$\tilde{\varepsilon} : \begin{matrix} P_0' \\ \oplus \\ P_0'' \end{matrix} \longrightarrow A \quad \text{and}$$

$$\tilde{\eta} : \begin{matrix} Q_0' \\ \oplus \\ Q_0'' \end{matrix} \longrightarrow B.$$

We start to choose γ_i, $i \geq 0$, inductively.

$\underline{\gamma_0}$: $$f \circ \tilde{\varepsilon} = \tilde{\eta} \circ \tilde{f}_0$$

$$\Longleftrightarrow \left(f \circ \varepsilon|_{P_0'},\ f \circ \varepsilon|_{P_0''} \right) = \left(\tilde{\eta}|_{Q_0'},\ \tilde{\eta}|_{Q_0''} \right) \circ \begin{pmatrix} \tilde{f}_0' & \gamma_0 \\ & \tilde{f}_0'' \end{pmatrix}$$

$$f \circ \tilde{\varepsilon}|_{P_0'} = f \circ i_A \circ \varepsilon' = i_B \circ f' \circ \varepsilon'$$
$$= i_B \circ \eta' \circ \tilde{f}_0' = \tilde{\eta}|_{Q_0'} \circ \tilde{f}_0'$$

and $\tilde{\eta}|_{Q'_0} \circ \gamma_0 = f \circ \tilde{\epsilon}|_{P''_0} - \tilde{\eta}|_{Q''_0} \circ \widetilde{f''_0}$

$$\overset{\shortparallel}{i_B \circ \eta' \circ \gamma_0}$$

$$\begin{tikzcd} Q'_0 \arrow[d, "\eta'"] & & P''_0 \arrow[dl] \\ B' \arrow[r, "i_B"'] & B \arrow[r, "\pi_B"'] & B'' \end{tikzcd}$$

$\pi_B \circ (f \circ \tilde{\epsilon}|_{P''_0} - \tilde{\eta}|_{Q''_0} \circ \widetilde{f''_0})$

$= f'' \circ \pi_A \circ \tilde{\epsilon}|_{P''_0} - \eta'' \circ \widetilde{f''_0}$

$= f'' \circ \epsilon'' - \eta'' \circ \widetilde{f''_0} = 0.$

P''_0 is projective $\Rightarrow \exists\ \gamma_0 : P''_0 \longrightarrow Q'_0$

s.t. $i_B \circ \eta' \circ \gamma_0 = f\, \tilde{\epsilon}|_{P''_0} - \tilde{\eta}|_{Q''_0} \circ \widetilde{f''_0}.$

$\underline{\gamma_n,\ n > 0:}$ Assume we have chosen

$\gamma_0, \dots, \gamma_{n-1}$.

<u>Claim:</u>
The maps (*) $\lambda_{Q,n} \widetilde{f''_n} - \widetilde{f'_{n-1}} \lambda_{P,n} - \gamma_{n-1} d''_{P,n}$

$$: P''_n \longrightarrow Q'_{n-1}$$

has its image in

- $\ker \eta'$, if $n = 1$
- $\ker d'_{Q,n-1}$, if $n > 1$

So we can use projectivity of P''_n to obtain

$$\begin{tikzcd} & P''_n \arrow[dl, dashed, "\exists \gamma_n"'] \arrow[d] \\ Q'_n \arrow[r] & Q'_{n-1} \end{tikzcd}$$

<u>Proof of the claim:</u>

<u>$n=1$:</u> Apply $i_B \circ \eta'$ on $(*)$

- $i_B \circ \eta' \circ \tilde{f}'_0 \lambda_{P,1} = i_B \circ f'_0 \circ \epsilon' \circ \lambda_{P,1}$

$$= f \circ i_A \circ \epsilon' \circ \lambda_{P,1}$$

$$= -f \circ \tilde{\epsilon}|_{P''_0} \circ d''_{P,1}$$

- $i_B \eta' \gamma_0 \, d''_{P,1} = f \, \tilde{\epsilon}|_{P''_0} \circ d''_{P,1} - \tilde{\eta}|_{Q''_0} \circ d''_{Q,1} \circ \tilde{f}''_1$

So $\quad i_B \eta' \circ (*) = 0.$

<u>$n>1$:</u> Apply $d'_{Q,n-1}$

- $d'_{Q,n-1} \circ \tilde{f}'_{n-1} \circ \lambda_{P,n} = \tilde{f}'_{n-2} \circ d'_{P,n-1} \circ \lambda_{P,n}$

$$= -\tilde{f}'_{n-2} \circ \lambda_{P,n-1} \circ d''_{P,n}$$

- $d'_{Q,n-1} \circ \gamma_{n-1} \circ d''_{P,n} =$

$$\left(\tilde{f}'_{n-2} \lambda_{P,n-1} + \gamma_{n-2} \, d''_{P,n-1} - \lambda_{Q,n-1} \tilde{f}''_{n-1} \right) \circ d''_{P,n}$$

$$= \tilde{f}'_{n-2} \lambda_{P,n-1} d''_{P,n} - \lambda_{Q,n-1} \tilde{f}''_{n-1} \circ d''_{P,n}$$

- $d'_{Q,n-1} \, \lambda_{Q,n} \, \tilde{f}''_n = -\lambda_{Q,n-1} \circ d''_{Q,n} \tilde{f}''_n$

$$= -\lambda_{Q,n-1} \circ f''_{n-1} \circ d''_{P,n}.$$

$\square$

<u>Rk 90:</u> In the proof of the last theorem all the maps for the half box have been computed and fixed before extending f.