

Note: $\mathcal{A}$ is abelian iff $\mathcal{A}^{\text{op}}$ is abelian.

lem 71: let I be an object in an abelian cat: $\mathcal{A}$. T.a.e.:

- 1° I is injective in $\mathcal{A}$
- 2° I is projective in $\mathcal{A}^{\text{op}}$
- 3° The contravariant functor

$$\text{Hom}_{\mathcal{A}}(-, I) : \mathcal{A} \rightarrow \text{Ab}$$

is exact. (i.e. takes short exact sequences to short exact sequences)

Proof: $1^\circ \Leftrightarrow 2^\circ$ ✓ just reverse arrows.

$2^\circ \Leftrightarrow 3^\circ$ I proj. in $\mathcal{A}^{\text{op}}$

$\Leftrightarrow \text{Hom}_{\mathcal{A}^{\text{op}}}(I, -)$ is exact.
 $\mathcal{A}^{\text{op}} \rightarrow \text{Ab}.$

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We compare $\text{Hom}_{A^{\text{op}}} (I, -)$ with $\text{Hom}_A (-, I)$.

Take $1 \in \text{Hom}_A (M, N) = \text{Hom}_{A^{\text{op}}} (N, M)$

Wegert $g_1 \xrightarrow{\quad} f_{\text{op}g} = g \circ f$
 $\text{Hom}_{A^{\text{op}}} (I, N) \xrightarrow{f_{\text{op}g}} \text{Hom}_{A^{\text{op}}} (I, M)$

$\parallel$
 $\text{Hom}_A (N, I) \xrightarrow{f^*} \text{Hom}_A (M, I)$
 $\parallel$

So $\text{Hom}_{A^{\text{op}}} (I, -)$ is exact iff $\text{Hom}_A (-, I)$ is exact.

□

Example 72: We know already

that $\mathbb{Z}$ is not injective in $\mathbb{Z}$ -mod, as it is not $\mathbb{Z}$ -divisible.

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So $\text{Hom}_{\mathbb{Z}} (-, \mathbb{Z})$ should not be exact.

Indeed. $0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 0$

exact in $A = \mathbb{Z}$ -mod.

$0 \rightarrow \text{Hom}_{\mathbb{Z}} (\mathbb{Z}_2, \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}} (\mathbb{Z}, \mathbb{Z}) \xrightarrow{2} \text{Hom}_{\mathbb{Z}} (\mathbb{Z}, \mathbb{Z}) \rightarrow 0$
 $\parallel$
 $0 \rightarrow 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow 0$
 $\parallel$

not exact here.

Def 73: $M \in \mathcal{A}$. A contains complex

$0 \rightarrow I^{-2} \rightarrow I^{-1} \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots$
 $\parallel$
 $0 \quad \parallel \quad 0$

($I \in \text{Ch}^{\geq 0}(A)$)

Agrees with a morphism $M \hookrightarrow I^0$

is called a right resolution of M if

$$(*) \quad 0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

is exact.

We also call $(*)$ a right resolution of M and also see it as a morphism

$$M \rightarrow I^\bullet$$

$$\text{in } \text{Ch}^{\geq 0}(\mathcal{A})$$

A right resolution of M is called injective if all $I^i, i \geq 0$, are injective in $\mathcal{A}$.

Def 74: An abelian category $\mathcal{A}$ is said to have enough injectives if $\forall M \in \text{Obj}(\mathcal{A})$ $\exists I \in \text{Obj}(\mathcal{A})$ injective and $\exists M \hookrightarrow I$ in $\mathcal{A}$.

Lemma 75: If $\mathcal{A}$ has enough injectives then every object in $\mathcal{A}$ has an injective resolution.

Proof: We do $\mathcal{A}^{\text{op}}$ and take a projective resolution. $\square$

Comparison Theorem 76:

Given in $\mathcal{A}$ the following diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \rightarrow & E^0 & \rightarrow & E^1 \rightarrow E^2 \rightarrow \dots \\ & & \downarrow f & & & & \\ 0 & \rightarrow & N & \rightarrow & I^0 & \rightarrow & I^1 \rightarrow I^2 \rightarrow \dots \end{array}$$

I^i are injective

Then $\exists f: E^\bullet \rightarrow I^\bullet$ chain morphism a.t. f extends f' .

f is unique upto isomorphism.

Proof: Thm. 6.6. $\square$

Lemma 7.1: R -mod has enough injectives.

Proof: $M \in \mathbb{Z}$ -mod.

$N := \text{Hom}_{\mathbb{Z}}(R, M)$
is a left R -module via

$$(r \cdot f)(s) := f(sr).$$

$$\left((r_1 r_2) f \right)(s) = f(sr_1 r_2) \\ (r_1 (r_2 f))(s) = (r_2 f)(sr_1) = f(sr_1 r_2)$$

We prove 2 facts now:

- ① N is injective if M is injective in $\mathbb{Z}$ -mod.
- ② $\mathbb{Z}$ -mod has enough injectives.

Using ①, ② we finish the proof:

Take $A \in R$ -mod. Then

$$A \in \mathbb{Z}\text{-mod.}$$

- ② $\Rightarrow \exists I \in \mathbb{Z}$ -mod injective

$$\text{or } \exists A \hookrightarrow I. \text{ in } \mathbb{Z}\text{-mod.}$$

Then $\text{Hom}_{\mathbb{Z}}(R, I)$ is injective

in R -mod by ① and

$$A \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, I)$$

$$a \mapsto q_a \\ (q_a(r) := f(ar))$$

Exercise: This is an R -module isomorphism.

$$\text{Monom: } q_a = 0 \Rightarrow 0 = q_a(1) = f(a)$$

$$\Rightarrow a = 0. \\ \uparrow \\ f \text{ injective} \quad \square$$

lem 78: R -mod has enough injectives

Proof: $A \in \mathcal{Q}$ -mod.

$$I := \prod \mathbb{Q}/\mathbb{Z}$$

$$f \in \text{Hom}_{\mathbb{Q}}(A, \mathbb{Q}/\mathbb{Z})$$

$$A \xrightarrow{\quad} I$$

$$a \mapsto (f(a))$$

Claim: This map is injective

Pr: Let $a \in A \setminus \{0\}$.

$$\text{Then } \langle a \rangle \simeq \begin{cases} \mathbb{Z} \\ \mathbb{Z}/m\mathbb{Z} \end{cases}$$

$$\Rightarrow \exists \langle a \rangle \xrightarrow{f \neq 0} \mathbb{Q}/\mathbb{Z}$$

$$\cap \mathbb{Q} \neq \emptyset$$

$$A \ni f \text{ as } \mathbb{Q}/\mathbb{Z}$$

is injective $\square$

lem 79: $\forall M \in R\text{-mod } \forall A \in \mathcal{A}_R$

The map

$$\text{Hom}_{\mathcal{A}_R}(M, A) \xrightarrow{\quad \zeta \quad} \text{Hom}_{\mathcal{A}_R}(M, \text{Hom}_{\mathcal{A}_R}(R, A))$$

$$(\zeta(f))(m)(r) := f(rm)$$

is bijective.

Proof: We guess the inverse.

$$(\tau(q))(m) := (q(m))(1)$$

$$(((\zeta \circ \tau)(q))(m))(r) = \tau(q)(rm)$$

$$= (q(rm))(1)$$

$$= (f \circ q(m))(1)$$

$$= (q(m))(1 \cdot r)$$

$$\text{So } (\zeta \circ \tau)(q) = q$$

$$((\tau \circ \zeta)(f))(m) = ((\zeta(f))(m))(1)$$

$$= f(m)$$

$\square$

So if we plug in an exact sequence of R -modules on the right side of τ then we get an exact sequence of abelian groups on the left side if A is injective.

This shows (1).

II.4. Adjoint functors.

Def 80: A pair of functors

$$L: \mathcal{A} \rightarrow \mathcal{B}, \quad R: \mathcal{B} \rightarrow \mathcal{A}$$

are called adjoint if there is a natural pairing $(\tau_{A,B})$

$$\begin{matrix} A \in \mathcal{A} \\ B \in \mathcal{B} \end{matrix}$$

of isomorphisms

$$\text{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{\tau_{A,B}} \text{Hom}_{\mathcal{A}}(A, R(B))$$

L is called the left adjoint
 R is called the right adjoint.

natural means

$$\begin{array}{ccc} \text{Hom}_{\mathcal{B}}(L(A), B) & \xrightarrow{\tau_{A,B}} & \text{Hom}_{\mathcal{A}}(A, R(B)) \\ \downarrow \tau_{A,B} & & \downarrow \tau_{A,B} \\ \text{Hom}_{\mathcal{B}}(L(A), B) & \xrightarrow{\tau_{A,B}} & \text{Hom}_{\mathcal{A}}(A, R(B)) \end{array}$$

and $\forall g \in \text{Hom}_A(A_1, A_2)$

$$\text{Hom}_B(L(A_1), B) \xrightarrow{\tau_{A_1, B}} \text{Hom}_A(A_1, R(B))$$

$$L(B)^* \uparrow \quad \cap \quad \uparrow g^*$$

$$\text{Hom}_B(L(A_2), B) \xrightarrow{\tau_{A_2, B}} \text{Hom}_A(A_2, R(B))$$

Example 81:

$L: R\text{-mod} \longrightarrow \mathbb{Z}\text{-mod}$
 $L = \text{forgetful functor}$
 (forgets R -module structure)

$$R: \mathbb{Z}\text{-mod} \longrightarrow R\text{-mod}$$

$$R(A) := \text{Hom}_{\mathbb{Z}}(R, A)$$

Then (L, R) are adjoint
 by lemma 7.9.

A, B abelian cat.

Def 82: A functor $R: B \rightarrow A$
preserves injectives if

$\forall I$ injective in $B: R(I)$ in-
 jective in A .

A functor $L: A \rightarrow B$ preserves
projectives if

$\forall P$ projective in $A: L(P)$ projec-
 tive in B .

Theorem 83:

(a) Let R be an additive functor
 $R: B \rightarrow A$ right adjoint to
 an exact functor
 Then R preserves injectives.

(b) Let $L: A \rightarrow B$ be an additive
 functor, left adjoint to an
 exact functor. Then L preserves
 projectives.

Proof:

(a) Let $L: A \rightarrow B$ be an exact left adjoint of R

and I be injective in B .

Then $\text{Hom}_B(L(\cdot), I)$

is exact. Thus $\text{Hom}_A(-, R(I))$

is exact, i.e. $R(I)$ is injective.

(b) Exercise. $\square$

Exercise 84: Let (L, R) be

an adjoint pair of additive functors

Then (i) R is left exact

(ii) L is right exact.

4 covariant additive

Def: $F: A \rightarrow B$ right exact

if $\forall 0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ exact.

$F(A_1) \rightarrow F(A_2) \rightarrow F(A_3) \rightarrow 0$ exact.

A contravariant functor

$F: A \rightarrow B$ is called right

exact if $\forall 0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ exact:

$F(A_3) \rightarrow F(A_2) \rightarrow F(A_1) \rightarrow 0$

is exact.

In Exercise 84 and Def 80

L and R are supposed to be contravariant.