HOMOLOGICAL ALGEBRA, FALL 2025
PROBLEM SHEET 5

PROF. DANIEL SKODLERACK

Problem 1 (10, delta-functor)**.** Let P be a projective R-left module and consider the following family of functors from $Ch_{\geqslant 0}(R\text{-mod})$ to mod-R.

$$T_n(M_*) := H_n(Hom_R(P, M_*)),\ n \geqslant 0.$$

Here $Hom_R(P, M_*)$ is the conically defined complex of right R-modules $Hom_R(P, M_m)$. The differentials are the push-forward maps from the differentials of M_*. Prove that $(T, \partial \circ Hom_R(P, -))$ is a universal δ-functor. ($\partial_n, n \geqslant 1$, are the connecting morphisms for H_*.)

Problem 2 (10, projectives in $Ch(\mathcal{A})$)**.** Prove Example 61(b), see [Wei94, Exercise 2.2.1].

Problem 3 (10, injective module)**.** Consider the ring $R = \mathbb{Z}/m$ and $d|m$. Consider $M := \mathbb{Z}/d$ as a quotient module of R by $d\mathbb{Z}/m\mathbb{Z}$. Find the conditions on d for which M is an injective R-module and prove your found conjecture.

Problem 4 (10, divisible module)**.**
(i) Let R be an integral domain. Prove that an R-module M is injective if and only if it is R-divisible, i.e. for every $x \in M$ and every nonzero $r \in R$ there is an element $y \in M$ such that $ry = x$.
(ii) Take $R = \mathbb{Z}$. Find an injective resolution for $\mathbb{Z}$ in the category of abelian groups.

Problem 5 (10**, projective module)**.** Determine if the Taylor series ring $\mathbb{Z}[[X]]$ is a projective $\mathbb{Z}[X]$-module.

References

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

Date: Please hand in before the lecture on Friday, **October 24th$^{\text{th}}$ 2025**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference.