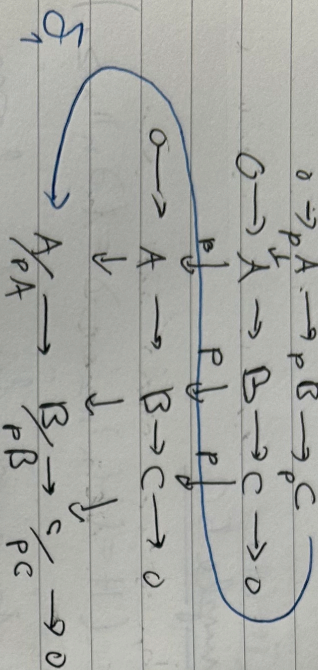


$T_n := 0$ for $n \neq 0, 1$.

$\delta_1 :=$ Snake Lemma on



$$\delta_n := 0 \quad \forall n \neq 1.$$

Then (T, δ) is a δ -functor from $\mathcal{A}$ to $\mathcal{B}$.

Re: We also have the concept of cohomological δ -functor (generalizing the idea of cohomology groups)

Then $\delta^n: T^n(C) \rightarrow T^{n+1}(A)$.

Def 5.1: A morphism of δ -functors $\delta: T \rightarrow T'$ is a family of natural transformations $f_n: T_n \rightarrow T'_n$ ($f^n: T^n \rightarrow T'^n$) which commute with δ .

In other words, for every exact sequence in $\mathcal{A}$ we get a commutative ladder in $\mathcal{B}$.

homological

Def 5.2: (a) A δ -functor $(T, \delta): \mathcal{A} \rightarrow \mathcal{B}$ is called universal, if for all δ -functors $S: \mathcal{A} \rightarrow \mathcal{B}$ and all natural transformations $f_0: S_0 \rightarrow T_0$ $\exists!$ morphism $f: S \rightarrow T$ which extends f_0 .

(b) A cohomological δ -functor $(T, \delta): \mathcal{A} \rightarrow \mathcal{B}$ is called universal if for all coh. δ -functors $S: \mathcal{A} \rightarrow \mathcal{B}$

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and all natural transformations

$$f^0: T^0 \rightarrow S^0$$

$$\exists! f: T \rightarrow S \text{ extending } f^0$$

Remark 53: $H: Ch_{\geq 0}(A) \rightarrow A$
and $H^*: Ch_{\geq 0}^2(A) \rightarrow A$
are universal $\mathcal{S}$ -functors
(Proof maybe later.)

Example 54: (a) Take $\ell \geq 1$
a positive integer.

Reminder: For $A \in Ch_{\geq 0}(A)$
we have
 $\mathcal{O}_{\geq \ell}(A) := \cdots \rightarrow A_{\ell+2} \rightarrow A_{\ell+1} \rightarrow A_{\ell} \rightarrow 0 \rightarrow 0 \rightarrow \cdots$

$S := H_0 \mathcal{O}_{\geq 0}$ is a homological $\mathcal{S}$ -
functor: $Ch_{\geq 0}(A) \rightarrow A$.

Claim:

S is not universal

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Proof: We show it for $\ell=1$.
($\ell > 1$ is similar)

Consider H and $\mathcal{O}: H_0 \rightarrow S_0$.

$$\begin{array}{ccccccc} \rightarrow H_0(A) & \rightarrow H_0(K) & \rightarrow H_0(B) & \rightarrow H_0(C) & \rightarrow 0 \\ \downarrow 1 & \downarrow 0 & \downarrow 0 & \downarrow 0 & \\ \rightarrow C_1 & \rightarrow 0 & \rightarrow 0 & \rightarrow 0 & \rightarrow 0 \\ \downarrow \beta_1(C) & & & & \end{array}$$

inclusion, for $n \geq 1$.

So we have at least two ex-
tensions of $\mathcal{O}: H_0 \rightarrow S_0$.

① $\mathcal{O}: H \rightarrow S$

②

$$f(A) = \begin{cases} \mathcal{O}, & n=0 \\ \text{incl}: H_1(A) \hookrightarrow A_1 / \beta_1(A), & \text{if } n=1 \end{cases}$$

$$\left[\begin{array}{l} \text{id } H_n(A), \text{ if } n > 1. \end{array} \right]$$

□

(b) $\exists! T$ is universal when $T \oplus T$
is universal (consider projections).

II.2. Projective Resolution

Def 55: Let R be an algebra over k .
 $P \in \text{Obj}(\mathcal{A})$ is called projective if

$$\forall \text{ epi } g: M \twoheadrightarrow N \quad \exists f: P \rightarrow M$$

$$\text{ s.t. } g \circ f = \text{id}_N$$

$$\left(\begin{array}{ccc} & P & \\ \swarrow g & \downarrow f & \\ M & \xrightarrow{g} & N \end{array} \right)$$

Pl 56: P projective

$$\Leftrightarrow \forall M \xrightarrow{g} N: \text{Hom}(P, M) \xrightarrow{g_*} \text{Hom}(P, N)$$

$$\Leftrightarrow \text{Hom}(P, -): \mathcal{A} \rightarrow \text{Ab}$$

is exact.

□

$T \otimes T$ mod $\text{ann}(M \otimes N)$ is isomorphic to $\text{ann}(M \otimes N)$

Proof: The second " $\Leftarrow$ " is an exercise.

(Note that for

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \text{ exact}$$

the sequence

$$0 \rightarrow \text{Hom}(P, A) \rightarrow \text{Hom}(P, B) \rightarrow \text{Hom}(P, C)$$

is exact, even if P is not projective)

" $\Leftarrow$ " : This is by definition. □

In a module category we say: projective module.

Example 57: $\mathcal{A} = R\text{-mod}$.

Free R -modules are projective.

$$P_1 := F := \bigoplus_{i \in \mathbb{Z}} R e_i$$

$$M \xrightarrow{g} N$$

$$\text{Take } m_i \in M \text{ s.t. } g(m_i) = f(e_i).$$

Put $\mathcal{Q}(v_i) := m_i$ and define $\mathcal{Q}$ by linear extension. $\square$

lem 58: $A = R\text{-mod}$. $P \in \mathcal{Q}_A(X)$

Then P is projection iff P is isomorphic to a direct summand of a free module.

Proof: "only if": Take a subset

$S := \{v_i \mid i \in I\}$ of P of rank $(S) = P$.

$$F := \bigoplus_{i \in I} R v_i \xrightarrow{\pi} P$$

$\downarrow \cong$
 $\bigoplus_{i \in I} P$

has a splitting Δ .

$$\Rightarrow \underbrace{\text{im}(\Delta)}_{\cong P} \oplus \ker(\pi) = F$$

"if": Suppose $\exists F$ free of rank n such that $F = P \oplus Q$ for some Q .

Consider $P \oplus Q = F$

$$\begin{array}{ccc} \exists \tilde{\mathcal{Q}}: & P & \downarrow \text{pr}_P \\ & & P \\ M & \xrightarrow{f} & N \end{array}$$

$$f := 1 \circ \text{pr}_P$$

Example 57 $\Rightarrow \exists \tilde{\mathcal{Q}}: F \rightarrow M$

$$\text{a.t. } g \circ \tilde{\mathcal{Q}} = f$$

$$\text{But } \mathcal{Q} := \tilde{\mathcal{Q}}|_P$$

$$\begin{aligned} \text{Then } g \circ \mathcal{Q}(P) &= g(\tilde{\mathcal{Q}}(P, 0)) \\ &= \tilde{f}(P, 0) \\ &= f(P). \end{aligned} \quad \square$$

Example 59: ① R a field

Then all submodules of R -mod are projective and free.

② $R = \mathbb{Z}$ All submodules

of a free R -module are free, because $\mathbb{Z}$ is a PID. Therefore by Lem 58 all projectives are free.

③ $Ab_f :=$ category of all finite abelian groups has no projectives except 0.

P4: Assume $A \in Obj(Ab_f)$ is a projective.

A is a finite abelian group $\neq 0$
 $\Rightarrow \exists \exists p_1, \dots, p_r$ primes $e_1 \geq 1, \dots, e_r \geq 1$

$$A = \mathbb{Z}_{p_1^{e_1}} \oplus \dots \oplus \mathbb{Z}_{p_r^{e_r}}$$

As in Lem 58 it follows $\mathbb{Z}_{p_i^{e_i}}$ is projective.

$$\text{But } \mathbb{Z}_{p_1^{e_1+1}} \xrightarrow{\pi} \mathbb{Z}_{p_1^{e_1}}$$

p_1 : Assume it has a splitting Δ .

$$\Rightarrow \Delta([1]) = [1 + p_1^{e_1} z]_{p_1^{e_1+1}}$$

for some $z \in \mathbb{Z}$.

$$\Rightarrow 0 = \Delta([p_1^{e_1+1}]) = [p_1^{e_1+1}]_{p_1^{e_1+1}} \neq [0]$$

Def 60: An abelian category $\mathcal{A}$ is said to have enough projectives if

$\forall A \in Obj(\mathcal{A}) \exists P \in Obj(\mathcal{A})$ projective $f \in Hom_{\mathcal{A}}(P, A) : f$ epi.

Example 61:

(a) R -mod has enough projectives.

(b) Assume $\mathcal{A}$ has enough projectives. Then

$\text{Ch}(\mathcal{A})$ has enough projectives.

The projectives in $\text{Ch}(\mathcal{A})$ are exactly the split exact complexes

where P_n is projective in $\mathcal{A}$.

(Exercise)

Def 62: $M \in \text{Obj}(\mathcal{A})$. A left resolution of M is a complex

$\mathcal{P}$ with $P_i = 0$, $i \leq 0$, and a map $\varepsilon: P_0 \rightarrow M$ in $\mathcal{A}$.

$$\rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \rightarrow 0$$

is exact.

It is called projective resolution if all P_i are projective.

Example 63: (a) If M is projective

then $\rightarrow 0 \rightarrow M \rightarrow 0$ with

$M \xrightarrow{\varepsilon = \text{id}} M$ is a projective resolution of M .

augmented complex:

$$\begin{array}{ccccccc} 0 & 0 & & M & & & \\ \rightarrow & P_2 & \rightarrow & P_1 & \rightarrow & P_0 & \xrightarrow{\text{id}} M \rightarrow 0 \end{array}$$

(62) $(2, X) = 2Z[X] + RZ[X] = M$
is a $R[X]$ -module.

$\cong R$

$$P \mapsto (-X^i P, 2P) \quad (P, 0) \mapsto 2P + 2Q$$

$$0 \rightarrow R \xrightarrow{\epsilon} R \oplus R \xrightarrow{\epsilon} M \rightarrow 0$$

is exact, and

with ϵ

$$0 \rightarrow R \xrightarrow{\epsilon} R \oplus R \rightarrow 0 \text{ is a}$$

projective resolution of $(2, X)$.

Q: $\Rightarrow (2, X)$ projective in
 R -mod?

lem 64: Every R -module
has a projective resolution.

Proof: We will use that
 R -mod has enough projectives.

$M \in \text{Obj}(R\text{-mod})$

By applying:

$$\text{Ext}(E) \hookrightarrow P_1 \xrightarrow{\epsilon_1} \text{Ext}(E) \hookrightarrow P_0 \xrightarrow{\epsilon_0} M$$

(enough)

etc.

□

In fact we see that we
can replace R -mod by
any abelian cat. $\mathcal{A}$ with
enough projectives in lem 64.

Prob 65: Given a projective resolution $P_\bullet$
of M with $\epsilon: P_0 \rightarrow M$

we can extend ϵ to
a chain map

$$P_\bullet \xrightarrow{\epsilon} M$$

where M is interpreted as

$$\rightarrow 0 \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow 0$$

degree

II.3. Injective resolution

The dual concept of projective object is injective object, in fact injectives in $\mathcal{A}$ are the projectives in $\mathcal{A}^{op}$.

Def 68: An object $I \in \text{Obj}(\mathcal{A})$ is called injective if diagram

$$\begin{array}{ccc} 0 \rightarrow A \xrightarrow{t} B \\ \downarrow \alpha \quad \downarrow \beta \\ I \xrightarrow{f} g \end{array}$$

$$\exists g: B \rightarrow I : g \circ f = \alpha.$$

Baer's criterion 69:

An R -module M is injective iff $\forall$ left ideal J and $\chi: J \rightarrow M$ (R -mod-homomorphism)

$$\exists g \in \text{Hom}_R(R, M) : \chi = g \circ \text{incl}_{J \subseteq R}$$

$$\begin{array}{ccc} J \subseteq R \\ \downarrow \alpha \quad \downarrow \beta \\ M \xrightarrow{f} g \end{array}$$

Proof: "only if" follows from Def 68.
"if" Given $0 \rightarrow A \xrightarrow{t} B$

$$\downarrow \alpha$$

$$M = \{ (A', \alpha') \mid A \subseteq A' \subseteq B, \alpha': A' \rightarrow M \}$$

$$\alpha' \downarrow_A = \alpha$$

Zorn's Lemma $\Rightarrow \exists (A, \alpha) \in \mathcal{M}$

max w.r.t. $\leq$

$$\text{Assume } A \neq R.$$

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Take $a \in R \setminus A$

$J := \{r \in R \mid ra \in A\}$ left ideal.

By the criterion $J \xrightarrow{\iota} \hat{A} \xrightarrow{\hat{\alpha}} M$
 extends to R .

$$\begin{array}{ccc} R & \xrightarrow{\iota} & \hat{A} \\ & \searrow \hat{\alpha} & \\ & & M \end{array}$$

Put $A' := \hat{A} + Ra$

$\alpha': A' \rightarrow M$

$\alpha'(a + ra) := \hat{\alpha}(a) + r\hat{\alpha}(a)$

well-defined? (exercise)

Then $(A', \alpha') \cong (A, \hat{\alpha}) \hookrightarrow M$ $\square$

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Examples 70:

(a) If R is a field. Then every
 R -vector space is injective,
 as the only ideals in R are
 $\{0\}$ and R .

(b) $\mathbb{Z}$ is not an injective
 abelian group

$$\begin{array}{ccc} 2\mathbb{Z} & \subseteq & \mathbb{Z} \\ \frac{1}{2} \downarrow & & \\ \mathbb{Z} & \subsetneq & \mathbb{Q} \end{array}$$

Pr: Assume $\exists g \in \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$:

$g|_{2\mathbb{Z}} = \text{mult. with } \frac{1}{2}$

$$\Rightarrow 2g(1) = g(2) = \frac{1}{2} \cdot 2 = 1$$

$\Rightarrow 2$ has a multiplicative inverse in $\mathbb{Z}$ $\hookrightarrow \square$

(c) $R = \mathbb{Z}/m$. Then R is a projective R -module.

Proof: $\mathcal{I} \subseteq R$ ideal, i.e.
 $\mathcal{I} = d\mathbb{Z}/m\mathbb{Z}$ with $d|m$.
 $(A := \frac{m}{d})$

Consider $\mathcal{I} \subseteq R$
 $\downarrow \quad \exists g$
 R

$$\text{Then } [A]_m \alpha ([d]) = \alpha ([m]_m) = [0]_m$$

Thus $\alpha([d]) = [a]_m$
 with $m \nmid a$, i.e. $d \nmid a$.
 But $b := \frac{a}{d}$.

Thus $\alpha([z]_m) = [bz]_m$
 and g exists: $g([z]_m) = [bz]_m$. $\square$