

I 5. Mapping cone

We consider here $\mathcal{A} = R\text{-mod}$,
i.e. we argue as $\mathcal{A}$ would be
 $R\text{-mod}$, but the theory is true
for $\mathcal{A}$ abelian in general.

$$\text{Ch} = \text{Ch}(\mathcal{A}) = \text{Ch}(R\text{-mod.})$$

Def 37: Let $f: B \rightarrow C$ be a
chain map. We call the
following chain complex
 $\text{cone}(f)$ the mapping cone of f .

$$\text{cone}(f)_n := B_{n-1} \oplus C_n$$

$$d: \text{cone}(f)_n \rightarrow \text{cone}(f)_{n-1}$$

$$d(b, c) := (-d_B(b), -f(b) + d_C(c))$$

$$\left(\begin{array}{cc} \text{In matrix form} \\ \begin{pmatrix} -d_B & 0 \\ -f & d_C \end{pmatrix} & \begin{matrix} B_{n-1} \\ \oplus \\ C_n \end{matrix} \end{array} \right) \rightarrow \begin{matrix} B_{n-2} \\ \oplus \\ C_{n-1} \end{matrix}$$

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Rh 38:

This is a complex as

$$\begin{pmatrix} -d_B & \\ -f & d_C \end{pmatrix} \circ \begin{pmatrix} -d_B & \\ -f & d_C \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \\ f d_B - d_C f & 0 \end{pmatrix} = 0.$$

Using $\text{cone}(f)$ we can fit $H_n(f)$ into a long exact sequence.

Prop. 39: The exact sequence

$$0 \rightarrow C \xrightarrow{f_2} \text{cone}(f) \xrightarrow{-f_1} B[-1] \rightarrow 0$$

generates a long exact sequence in homology where the connecting morphism is $H_n(f), n \in \mathbb{Z}$.

Proof:

$$\begin{array}{ccccc}
 & & & H_{n-1}(B) & \\
 & & & \parallel & \\
 C \hookrightarrow H_n(C) & \xrightarrow{H(L_2)} & H_n(\text{Cone}(f)) & \xrightarrow{-H(p_1)} & H_n(B[-1]) \\
 & & & \parallel & \\
 & & & H_{n-1}(B) & \\
 \hookrightarrow H_{n-1}(C) & & & \partial_n &
 \end{array}$$

$$\partial_n([e]) = [-f(e)] = H(f)([e]).$$

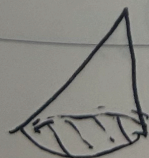
□

Cor 40: f is a quasi-isomorphism
iff $\text{Cone}(f)$ is acyclic.

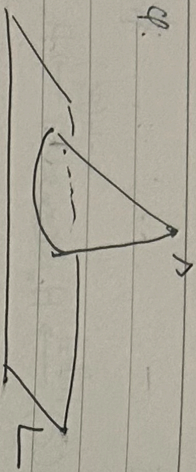
Proof: Consider the long exact sequence in Prop 39. □

Remark 41: Let $\varphi: K \rightarrow L$ a cellular map between cell complexes. (CW complexes)

$$CK: K \times [0,1] \xrightarrow{\varphi} L \times [0,1] \xrightarrow{\varphi} L \times [0,1] \xrightarrow{\varphi} L \times [0,1]$$



C^q :



$$\frac{CK \sqcup L}{(K, 0) \sim q(K)}$$

The the cell complex of C^q , relative to $\sim$ in the mapping cone of $f := q_*$.

$$f: C_{\text{cell}}(K) \rightarrow C_{\text{cell}}(L)$$

$$\text{cone}(f)_n = C_{\text{cell}}(K)_{n-1} \oplus C_{\text{cell}}(L)_n$$

freely generated by $n-1$ cells of K by n -cells of L .
 $\text{cone}(f)_n = C_{\text{cell}}(Cf)_n$

$$\text{cone}(f)_0 = C_{\text{cell}}(L)_0 = C_{\text{cell}}(Cf)_0 \sim \mathbb{Z} \Delta$$

Remark 4.2: $\text{cone}(f)$ is an abelian for any something about f being a quasi-isomorphism or not.

The next construction: the cylinder $\text{cyl}(f)$, will describe f completely in its differential up to homotopy equivalence.

Def 4.3: C cylinder of f
 $f: B \rightarrow C$ chain map.

$$\text{cyl}(f)_n := B_n \oplus B_{n-1} \oplus C_n$$

$$d_n(b, c, s) = (d(b) + b', -d(c'), d(c) - f(c'))$$

On matrix form:

$$\begin{pmatrix} d_B & id_B \\ -d_B & -f \\ -f & d_C \end{pmatrix} : \begin{matrix} B_n \\ \oplus B_{n-1} \\ \oplus C_n \end{matrix} \longrightarrow \begin{matrix} B_{n-1} \\ \oplus B_{n-2} \\ \oplus C_{n-1} \end{matrix}$$

$cyl(f)$ is called mapping cylinder of f .

Notation 4.4:

$$\begin{aligned} cone(C) &:= cone(id_C) \\ cyl(C) &:= cyl(id_C) \end{aligned}$$

Remark 4.5: (topological cylinder)

$$\begin{aligned} cyl(Q) & \quad K \times [0, 1] \simeq K \\ & \quad \xrightarrow{\quad} L \\ & \quad K \times \{0\} \text{ identified with } Q(K) \end{aligned}$$

$$C_{cell}(cyl(Q))_n = C_{cell}(K)_n \oplus C_{cell}(K)_{n-1} \oplus C_{cell}(L)_n$$

Here $Q: K \rightarrow L$ is a cellular map.

Prop 4.6: Two chain maps

$f, g: C \rightarrow D$ are chain homotopic iff there exists a chain map $\Phi: cyl(C) \rightarrow D$ s.t.

$$\begin{array}{ccc} & C & \\ \uparrow i_1 & \searrow f & \\ cyl(C) & \xrightarrow{\Phi} & D \end{array}$$

Proof: exercise $\square$

Lemma 4.7: $C \xrightarrow{\alpha} cyl(f)$,

$d(C) := (0, 0, c)$, is a quasi-isomorphism. Even more: α is a chain homotopy equivalence.

Proof:

$$\beta: \text{cyl}(f) \longrightarrow C$$

$$\beta(a, a', c) := f(a) + c.$$

$$\cdot (d_C \circ \beta)(a, a', c) = f(d_B(a) + d_C(c))$$

$$\beta(d_{\text{cyl}}(a, a', c)) = f(d_B(a + a') - f(a') + d_C(c))$$

$$= f(d_B(a) + d_C(c)).$$

So β is a chain map.

$$\cdot \beta \circ \alpha = \text{id}_C$$

$$\cdot (d_C \circ \beta)(a, a', c) = (0, 0, f(a) + c)$$

Consider $\Delta_n: \text{cyl}(f)_n \longrightarrow \text{cyl}(f)_{n+1}$

$$\Delta_n(a, a', c) := (0, a, 0).$$

Then

$$\Delta_{n-1} d_n(a, a', c) + d_{n+1} \Delta_n(a, a', c)$$

$$= (0, d_B(a) + a', 0) + (a, -d_B(a), -f(a))$$

$$= (a, a', -f(a)) = (\text{id}_{\text{cyl}(f)} - \alpha\beta)(a, a', c).$$

Diagram 4.8:

$$\begin{array}{ccccc} 0 & \xrightarrow{\quad} & B & \xrightarrow{\quad} & C \\ & & \searrow f & \nearrow \beta & \\ & & \text{cyl}(f) & & \\ 0 & \xrightarrow{\quad} & B & \xrightarrow{\quad} & \text{cyl}(f) & \xrightarrow{\quad} & C \\ & & \downarrow \alpha & \uparrow \beta & \downarrow \beta \alpha & \uparrow \alpha & \\ 0 & \xrightarrow{\quad} & C & \xrightarrow{\quad} & \text{cyl}(f) & \xrightarrow{\quad} & C \end{array}$$

(1) β is a chain homotopy equivalence.

So up to n —

f is equal to $\alpha\beta$ up to $\text{cyl}(f)$.

Note: α is just an inclusion, or

The information about f is decoded in the differential of $\text{cyl}(f)$.

(iv) Both exact rows give isomorphic long exact sequences.

$$\begin{array}{ccccccc}
 \xrightarrow{-\partial} & H_n(B) & \xrightarrow{H(u_1)} & H_n(\text{cyl}(U)) & \xrightarrow{H(\text{pr}_{(2,3)})} & H_n(\text{cone}(U)) & \xrightarrow{-\partial} H_{n-1}(B) \\
 \parallel & & & \uparrow \text{is } \cong & \parallel & & \parallel \\
 \rightarrow & H_n(B \cup \{1\}) & \xrightarrow{H(u_2)} & H_n(C) & \xrightarrow{H(v_2)} & H_n(\text{cone}(U)) & \xrightarrow{-H(\text{pr}_1)} H_{n-1}(B \cup \{1\})
 \end{array}$$

Proof : $[C, c] \in H_n(\text{cone}(U))$

$$\partial([C, c]) = [C, c] \in H_{n-1}(B)$$

$$H(\text{pr}_1)([C, c]) = [C]$$

□

II Derived functor

Def. 4.9 (δ -functor)

Let $\mathcal{A}, \mathcal{B}$ be two abelian categories.

A family $(T = (T_n)_{n \geq 0}, S = (S_n)_{n \geq 0})$

where

- $T_n : \mathcal{A} \rightarrow \mathcal{B}$ is an additive functor. Put $T_n := 0$ for $n < 0$

- S_n attaches to an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ a morphism $\delta_n : T_n(C) \rightarrow T_{n-1}(A)$

Put $S_n := 0$ for $n < 0$.

is called a δ -functor,

if (i) for each exact sequence in $\mathcal{A}$: $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ the sequence

$$\dots \rightarrow T_{n+1}(C) \xrightarrow{T_{n+1}} T_n(A) \rightarrow T_n(B) \rightarrow T_n(C) \xrightarrow{\delta_n} T_n(A) \rightarrow \dots$$

is exact,

and

(ii) for all morphisms between exact sequences, i.e. $\forall$ commutative diagrams in $\mathcal{A}$ with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A' & \rightarrow & B' & \rightarrow & C' \rightarrow 0 \end{array}$$

The diagrams

$$\begin{array}{ccc} T_{n+1}(C) & \rightarrow & T_n(A) \\ \downarrow & & \downarrow \\ T_{n+1}(C') & \rightarrow & T_n(A') \end{array}$$

($n \geq 0$) are commutative.

, i.e. the ladder formed by the long exact sequences from (i) is commutative.

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$$(a) \quad (H = (H_n)_{n \geq 0}, \partial = (\partial_n)_{n \geq 1})$$

is a δ -functor from

$$\text{Ch}(\mathcal{A})_{\geq 0} \text{ to } \mathcal{A}.$$

(b) $\mathcal{A} = \text{Ab}$ category of abelian groups
 $= \mathbb{Z}$ -mod.

Fix prime p .

$$\text{But } T_0(A) = A/pA$$

$$T_1(A) = {}_p A := \{a \in A \mid pa = 0\}$$

(subgroup of p -torsion of A)

$T_n := 0$, for $n \notin \{0, 1\}$.

$\delta_1 :=$ Snake lemma on

$$\begin{array}{ccccccc}
 0 & \rightarrow & pA & \rightarrow & pB & \rightarrow & pC \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \delta_1 & \rightarrow & A/pA & \rightarrow & B/pB & \rightarrow & C/pC \rightarrow 0
 \end{array}$$

$\delta_n := 0 \quad \forall n \neq 1$.

Then (T, δ) is a δ -functor from $\mathcal{A}$ to $\mathcal{A}'$.