

Proof of Thm 25.

Consider

$$\begin{array}{ccccccc}
 A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \xrightarrow{h_n} & 0 \\
 \downarrow d_{A_{n+1}} & & \downarrow d_{B_{n+1}} & & \downarrow d_{C_{n+1}} & & \\
 0 \longrightarrow Z_{n+1}(A) & \xrightarrow{f_{n+1}} & Z_{n+1}(B) & \xrightarrow{g_{n+1}} & Z_{n+1}(C) & &
 \end{array}$$

Snake Lemma

$$\Rightarrow \exists \text{ natural } \partial_n : H_n(C) \longrightarrow H_{n-1}(A)$$

$$0, 1. \quad H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{h} H_{n-1}(A)$$

$$\xrightarrow{f} H_{n-1}(B) \xrightarrow{g} H_{n-1}(C)$$

is exact. $\square$

Pl 28:

in terms of R-mod.

$$\partial_n([c_n]) = [a_{n-1}]$$

such that

$$f_{n-1}(a_{n-1}) = d(e_n)$$

for some $b_n \in B_n$ s.t. $g(e_n) = c_n$.

I4 Chain Homotopies

Motivation 19: Let K be a field.

K -mod = category of K -vector spaces
(also called K -vec.)

Consider a short complex in K -mod

$$0 \rightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \rightarrow 0$$

$\parallel \quad \quad \parallel \quad \quad \parallel$
 $\mathbb{Z}_0 \quad \quad \mathbb{Z}_0 \quad \quad \mathbb{Z}_0$
 $\oplus \quad \quad \oplus \quad \quad \oplus$
 $B_1 \quad \quad B_0 \quad \quad B_{-1}$

$\swarrow \quad \quad \swarrow \quad \quad \swarrow$
 $\exists \lambda \quad \quad \exists \lambda \quad \quad \exists \lambda$
 $d_0 \circ d_1 = \text{id}$

$\mathbb{Z}_0 = H_0 \oplus B_0$ by taking a direct complement of B_0 in $\mathbb{Z}_0$.

Max conditions d_{n-1}

$$\begin{array}{ccccccc} & & \xrightarrow{\quad} C_{n+1} & \xrightarrow{\quad} C_n & \xrightarrow{\quad} C_{n-1} & \xrightarrow{\quad} & \\ & \parallel & & \parallel & & \parallel & \\ B_{n+1} & \oplus & B_n & \oplus & B_{n-1} & \oplus & B_{n-2} \\ \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\ H_{n+1} & \oplus & H_n & \oplus & H_{n-1} & \oplus & H_{n-2} \\ \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\ B_{n+1}' & \oplus & B_n' & \oplus & B_{n-1}' & \oplus & B_{n-2}' \end{array}$$

$\swarrow \quad \quad \swarrow \quad \quad \swarrow$
 $\exists \lambda \quad \quad \exists \lambda \quad \quad \exists \lambda$
 $d_{n-1} \circ d_n = \text{id}$

$$d_{n-1} := d_n \circ d_{n-1} = \begin{cases} (d_n)' & \text{on } B_{n-1} \\ 0 & \text{on } H_{n-1}' \oplus B_{n-1}' \end{cases}$$

and additive.

Note: C_n is the direct sum of the relevant objects and we obtain a splitting $D = (D_n) \mathbb{Z}$.

D satisfies: $d_n \circ d_{n-1} = \text{pr}_{B_{n-1}}$

and $\Delta_{n-2} \circ d_{n-1} = \text{pr}_{B_{n-1}^1}$

W.T.F. $C_{n-1} = B_{n-1} \oplus H_{n-1}' \oplus B_{n-1}^1$

So $d_{n+1} d_n + d_{n-1} d_n = \text{pr}_{B_n + B_n^1}$

Proposition: $\text{ker}(d_0 + \text{id}) \subseteq \text{Coker}(d_0 + \text{id})$
 $\subseteq H(C)$.

$\cdot$ $d \circ d = d$

$\cdot$ $H(C) \xrightarrow{\sim} H' \subseteq C$

and

$C = B^1 \oplus H^1 \oplus B \xrightarrow{\text{pr}_{H^1}} H' \xrightarrow{\sim} H(C)$

are quasi-isomorphisms.

Proof: (Exercise).

Def 30: A chain complex C in $\mathcal{A}$ is called split if $\exists D_n: C_n \rightarrow C_{n+1}$

such that $d = d \circ d$.

An acyclic C is called

split exact if it is split.

The maps D_n are called splitting maps.

Example 31: In $\mathbb{Z}$ -mod.

$\rightarrow \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \rightarrow$

is acyclic but not split.

Proof: Assume it has a splitting

$D = (D_n)_{\mathbb{Z}}$.

$d(\mathbb{Z}) = d_n D_{n-1} d_n (\mathbb{Z} \mathbb{Z}_4) = d_n D_{n-1} (\mathbb{Z} \mathbb{Z}_4)$

$\stackrel{=}{=} d_n (2 D_{n-1} (\mathbb{Z} \mathbb{Z}_4)) = 4 D_{n-1} (\mathbb{Z}_4) = [0]_4$

D_{n-1} additive. Take $\mathbb{Z} = 1$. $\square$

$$\pi_n^{(D)} \circ (f_n|_Z) \circ d_{n+1}^{(C)}$$

$$= (\pi_n^{(D)} \circ d_{n+1}^{(D)}) \circ f_{n+1}$$

$$= 0 \circ f_{n+1} = 0.$$

Thus $\exists ! H_n(f) : H_n(C) \rightarrow H_n(D)$
such that

$$H_n(f) \circ \pi_n^{(C)} = \pi_n^{(D)} \circ f_n|_Z$$

Now consider $d^{(D)} \circ d^{(C)}$
instead of f .

$$\pi_n^{(D)} \circ (d^{(D)} \circ d^{(C)})|_{Z_n(C)}$$

$$= \pi_n^{(D)} \circ (d^{(D)} \circ d)|_{Z_n(C)} + 0$$

$$= 0 \circ d|_{Z_n(C)} = 0.$$

Thus $H_n(d^{(D)} \circ d^{(C)}) = 0$,
by uniqueness.

This shows $H_n(f) = H_n(g)$
for all $n \in \mathbb{Z}$ $\square$

Exercise 34: Prove Prop. 33 directly
for $\mathcal{A} = R\text{-mod}$.

Prop. 35: Let C be a chain complex
in $\mathcal{A}$. Then are equivalent:

1° $C \xrightarrow{0} C$ is a homotopy
equivalence

2° C is split exact.

Proof: $1^\circ \Rightarrow 2^\circ$ $1^\circ \Rightarrow \exists g: C \rightarrow C$ s.t. $gg = 0$
and $g \circ 0 \sim \text{id}_C$,
i.e. $0 \sim \text{id}_C$.

Thus $\exists d = (d_n)_n : \text{id}_C = d \circ d$.
 $\Rightarrow d = d \circ d \Rightarrow C$ is split.

By Prop 33 $0 = H_n(0) = H_n(\text{id}_C) = \text{id}_{H_n(C)}$
for all $n \in \mathbb{Z}$.
 $\Rightarrow H_n(C) = 0 \quad \forall n \in \mathbb{Z}$.

$2 \Rightarrow 1^0$ C is split

$$\Rightarrow \exists d = (d_n)_{n \in \mathbb{Z}} : C_n \xrightarrow{d} C_{n+1}$$

$$\text{o.t. } d \circ d = d.$$

$$\text{Put } \tilde{d} := d - d \circ d$$

$$\text{Then } d \tilde{d} = d \circ d = d$$

$$\text{and } (d \tilde{d})(\tilde{d} d) = (d \circ d \circ d \circ d)(d d) = 0.$$

$$\text{Thus w.l.o.g. } (d \circ d)(d d) = 0.$$

$$\text{Put } f_n := \text{id}_{C_n} - (d \circ d + d d)_n$$

$$\text{Then } f_n^2 = \text{id}_n - 2(d \circ d + d d)_n + (d \circ d + d d)_n^2$$

$$= f_n.$$

$$\text{Claim: } f_n = 0.$$

Proof:

(a) (for $d = R\text{-mod}$)

$$(d \circ d)^2 = d \circ d \quad \text{and} \quad (d d)^2 = d d.$$

$$\Rightarrow \text{ker}(d \circ d) \oplus \text{im}(d \circ d)_n = C_n$$

$$\text{and } \text{im}(d \circ d)_n \oplus \text{ker}(d d)_n = C_n.$$

$$\text{Now } \text{im}(d_{n+1} \circ d_n) = \text{im}(d_{n+1}) \\ = \text{ker}(d_n) = \text{ker}(d_{n+1} \circ d_n).$$

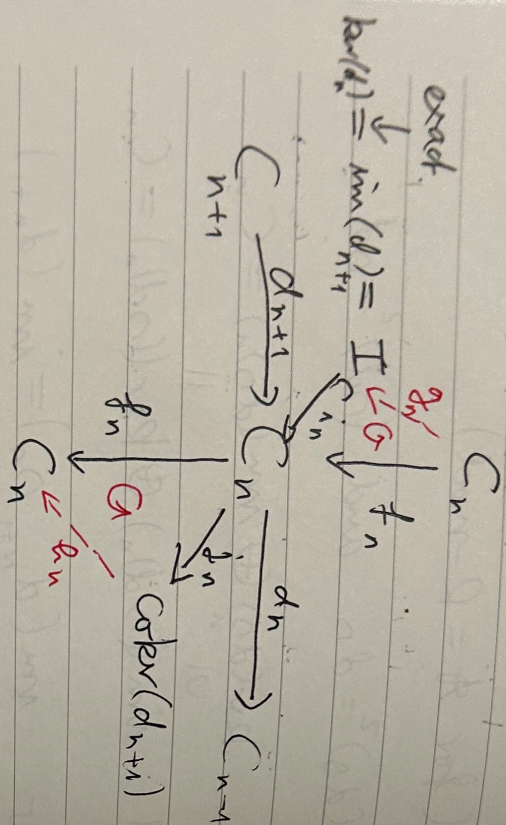
$$\text{And } \text{ker}(d_{n+1} \circ d_n) \supseteq \text{im}(d_n \circ d_n)$$

$$\text{Thus } \text{ker}(d \circ d)_n \text{ and } \text{im}(d d)_n$$

$$\text{im}(d \circ d)_n \quad \text{ker}(d \circ d)_n$$

$$\text{Thus } (d \circ d + d d)_n = \text{id}_n. \quad \square$$

(e) (for general Δ)



$$d_n \circ f_n = d_n (\text{id}_n - d_{n-1} \circ d_{n-1} \circ \dots \circ d_1 \circ d_0)$$

$$= d_n - d_n \circ d_{n-1} \circ \dots \circ d_1 \circ d_0 = 0$$

given the existence of f_n .

$f_n \circ d_{n+1} = 0$ give the existence of f_{n+1} .

$$\text{Thus } f_n = f_n^2 = f_n \circ \underbrace{d_0 \circ \dots \circ d_{n-1}}_0 \circ f_n = 0. \quad \square$$

Example 36: How many

homotopy equivalence classes are in

$$\text{Hom}_{\mathcal{A}(\mathbb{Z}\text{-mod})}(C, C)$$

for $C: \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \dots$?

Note: $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/4, \mathbb{Z}/4) \cong \mathbb{Z}/4$.

$$\text{Given } \Delta = (D_n)_{n \geq 1} \quad D_n: \mathbb{Z}/4 \rightarrow \mathbb{Z}/4$$

$$\text{Then } d_{n+1} \circ D_n + D_{n+1} \circ d_n$$

$$= 2(D_n + D_{n+1})$$

and for any family $(f_n)_{n \geq 1}$ of $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/4, \mathbb{Z}/4)$ we find Δ s.t.

Ans: $f_n = D_n + D_{n+1}$.
In particular $2f$ is null $\{ \otimes \}$ homotopic.

Suppose $g \in \text{Hom}_{\text{ch}}(C, C)$ and

$$g_0([1]_4) \in \{[1]_4, [3]_4\}$$

Then $g_n([1]_4) \in \{[1]_4, [3]_4\}$ (why?)

and $\text{id}_n - g_n$ is null homotopic, because of (*).

id_n is not null homotopic, because C is not split exact.

$$\text{Thus } \text{Hom}_{\text{ch}}(C, C) \cong \{[0]_n, [\text{id}]_n\}$$