$$\partial\sigma\begin{pmatrix}(z_1,z_2)\\ z\\ 0\end{pmatrix} = \begin{pmatrix}(0,z_2)\\ -2z_2\\ 0\end{pmatrix}$$

$$= d_6\left(\begin{pmatrix}z_2\\0\\0\end{pmatrix}\right).$$

At the end of I.2. we introduce two further operations on chain complexes.

<u>Def 20:</u> (truncations)

Let C be a chain complex, $n \in \mathbb{Z}$. We define two kind of truncations of C.

<u>(good) truncation of C below n:</u>

$$(\tau_{\geq n}C)_j := \begin{cases} 0, & j<n \\ Z_n, & j=n \\ C_j, & j>n \end{cases}$$

<u>above n:</u>

$$(\tau_{<n}C)_j = C_j/(\tau_{\geq n}C)_j$$

<u>brutal truncation of C above n.</u>

$$(\sigma_{<n}C)_j := \begin{cases} C_j, & j<n \\ 0, & j\geq n \end{cases}$$

<u>below n:</u>

$$\sigma_{\geq n}C := C/\sigma_{<n}C.$$

<u>Remark 21:</u>

$$H_m(\tau_{\geq n}C) = \begin{cases} 0, & m<n \\ H_m(C), & m\geq n \end{cases}$$

$$H_m(\tau_{<n}C) = \begin{cases} H_m(C), & m<n \\ 0, & m\geq n \end{cases}$$

$$H_m(\sigma_{<n}C) = \begin{cases} H_m(C), & m<n-1 \\ Z_{n-1}, & m=n-1 \\ 0, & m\geq n \end{cases}$$

$$H_m(\sigma_{\geq n}C) = \begin{cases} 0, & m<n \\ C_n/B_n, & m=n \\ H_m(C), & m>n. \end{cases}$$

Example 22:

$$C_k = \begin{cases} \mathbb{Z}/4\mathbb{Z}, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$\begin{array}{ccccccccccccc} & C_3 & & C_2 & & C_1 & & C_0 & & C_{-1} & & \\ \cdots \to & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{0} & 0 & \to & 0 \end{array}$$

$$H_n(C) = \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}/2\mathbb{Z}, & n = 0 \end{cases}$$

$\underline{\tau_{\geq 2} C}$:

$$\begin{array}{ccccccccccc} & 3 & & 2 & & 1 & & 0 & & & \\ \to & \mathbb{Z}/4\mathbb{Z} & \to & 2\mathbb{Z}/4\mathbb{Z} & \to & 0 & \to & 0 & \to & 0 \end{array}$$

$\underline{\tau_{<2} C}$:

$$\begin{array}{ccccccccccccc} & 3 & & 2 & & 1 & & 0 & & -1 & & -2 \\ \to & 0 & \to & 0 & \to & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/4\mathbb{Z} & \to & 0 & \to & 0 \end{array}$$

$\underline{\sigma_{<2} C}$:

$$\begin{array}{ccccccccccc} & 3 & & 2 & & 1 & & 0 & & & \\ \to & 0 & \to & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/4\mathbb{Z} & \to & 0 & \to 0 \end{array}$$

$$H_2(\sigma_{<2} C) = 2\mathbb{Z}/4\mathbb{Z}$$

$\underline{\sigma_{\geq 2} C}$:

$$\begin{array}{ccccccccccccc} & 3 & & 2 & & 1 & & 0 & & -1 & & -2 \\ \to & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/4\mathbb{Z} & \to & 0 & \to & 0 & \to & 0 & \to & 0 \end{array}$$

$$H_2(\sigma_{\geq 2} C) \simeq \mathbb{Z}/2\mathbb{Z}.$$

Def 23: (translation)

Let C be a chain complex and D be a cochain complex $p \in \mathbb{Z}$.

$C[p]_n := C_{n+p}$ — p^{th} translate of C

$D[p]^n := D^{n-p}$

(move the complex in direction of the arrows.)

$$C: \leftarrow C_{-2} \leftarrow C_{-1} \leftarrow C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow$$

$$C[2]: \leftarrow C_0 \leftarrow C_1 \leftarrow \underset{\underset{C[2]_0}{\shortparallel}}{C_2} \leftarrow C_3 \leftarrow C_4 \leftarrow$$

(C_p is on degree 0 of $C[p]$)

$$D: \to D_{-1} \to D_0 \to D_1 \to D_2 \to D_3 \to$$

$$D[2]: \to D_{-3} \to D_{-2} \to D_{-1} \to D_0 \to D_1 \to \ldots$$

The differential of $C[p]$ and $D[p]$ is $(-1)^p d$.

Similarly we translate chain maps.

For $f: C \longrightarrow C'$, $f[p]: C[p] \longrightarrow C'[p]$ is defined by $f[p]_n = f_{n+p}$

and for cochain maps $g: D \to D'$ we put $g[p]^n := g^{n-p}$.

<u>Example 24:</u> Let C be a chain complex. Consider $H(C)$ as a chain complex with 0-differentials.

$$0 \to H(C) \to C/B(C) \xrightarrow{d} Z(C)[-1] \to H(C)[-1] \to 0$$

is an exact sequence of chain complexes.

<u>Note:</u>

$$C_n/B_n(C) \xrightarrow{d} Z_{n-1}(C) = Z(C)[-1]_n$$

I3. Long exact sequences.

<u>Theorem 25:</u> Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of chain complexes (in an abelian category)

Then $\exists$ natural $\partial_n : H_n(C) \to H_{n-1}(A)$ such that

$$\begin{aligned} \cdots \longrightarrow\ & H_{n+1}(C) \\ \xrightarrow{\partial_{n+1}} H_n(A) \longrightarrow H_n(B) \longrightarrow\ & H_n(C) \\ \xrightarrow{\partial_n} H_{n-1}(A) \longrightarrow H_{n-1}(B) \longrightarrow\ & H_{n-1}(C) \\ \xrightarrow{\partial_{n-1}} H_{n-2}(A) \longrightarrow \cdots & \end{aligned}$$

is exact.

Here <u>natural</u> means:

Given a morphism m between exact sequences of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "m_A"] & B \arrow[r] \arrow[d, "m_B"] & C \arrow[r] \arrow[d, "m_C"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

$m \downarrow$ in $Ch(Ch(\mathcal{A}))$

Then

$$\begin{tikzcd} H_n(C) \arrow[r, "\partial_n"] \arrow[d, "m_{C,n}"'] \arrow[dr, phantom, "\circlearrowleft"] & H_{n-1}(A) \arrow[d, "m_{A,n-1}"] \\ H_n(C') \arrow[r, "\partial_n'"'] & H_{n-1}(A') \end{tikzcd}$$

Example 26:

degree

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 3 & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\iota_1"] \arrow[d, "6"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "pr_2"] \arrow[d, "12\oplus 6"] & \mathbb{Z} \arrow[r] \arrow[d, "6\iota_2"] & 0 \\ 2 & 0 \arrow[r] & \mathbb{Z}/24\mathbb{Z} \arrow[r, "(2,0)"] \arrow[d, "6"] & \mathbb{Z}/48\mathbb{Z}\oplus\mathbb{Z} \arrow[r, "1\oplus 1"] \arrow[d, "3\oplus 1"] & \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[d, "\overline{pr_2}"] & 0 \\ 1 & 0 \arrow[r] & \mathbb{Z}/12\mathbb{Z} \arrow[r, "\iota_1"] \arrow[d] & \mathbb{Z}/12\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z} \arrow[r, "pr_2"] \arrow[d] & \mathbb{Z}/3\mathbb{Z} \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & 0 & \\ & & A & B & C & \end{tikzcd}$$

$$\begin{tikzcd} & H_3(A) & H_3(B) & H_3(C) \\ \partial_4 = 0 \arrow[r] & \mathbb{Z} \arrow[r, "1"] & \mathbb{Z} \arrow[r, "0"] & 0 \\ & H_2(A) & H_2(B) & H_2(C) \\ \partial_3 = 0 \arrow[r] & \mathbb{Z}/3\mathbb{Z} \arrow[r, "(1,0)"] & \mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} \arrow[r, "0\oplus 1"] & \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} \\ & H_1(A) & H_1(B) & H_1(C) \\ \partial_2 \arrow[r] & \mathbb{Z}/6\mathbb{Z} \arrow[r, "1"] & \mathbb{Z}/3\mathbb{Z} \arrow[r, "0"] & 0 \\ \partial_1 = 0 \arrow[r] & 0 & & \end{tikzcd}$$

$\operatorname{im} \partial_2 = 3\mathbb{Z}/6\mathbb{Z}$, $\ker \partial_2 = 0 \oplus \mathbb{Z}/2\mathbb{Z}$

$\Rightarrow \quad \partial_2([a]_2, [b]_2) = [3a]_6$

Terminology: ∂_n is called connecting homomorphism.

The baby case of Thm 25 is the Snake lemma.

<u>Lemma 27</u>: (Snake lemma)

Consider the following diagram in an abelian category $\mathcal{A}$.

$$\begin{tikzcd} & A_1 \arrow[r] \arrow[d, "f"] & B_1 \arrow[r] \arrow[d, "g"] & C_1 \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A_0 \arrow[r] & B_0 \arrow[r] & C_0 & \end{tikzcd}$$

Suppose it is commutative with exact rows.

Then $\exists$ a natural $\partial : \ker(h) \to \operatorname{coker}(f)$ s.t.

$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h)$$

is exact.

<u>Proof</u>: We only prove it for the case $\mathcal{A} = R\text{-mod}$.
(This is sufficient, but the general case can also be tried by the reader as an exercise, not using the embedding theorem)

<u>Construction of ∂</u>:

$c_1 \in \ker(h)$
Take $b_1 \in B_1$ s.t. $b_1 \mapsto c_1$
Put $b_0 := g(b_1)$. Then $b_0 \mapsto 0$.
Thus $\exists\, a_0 \in A_0 : a_0 \mapsto b_0$

Define $\partial(c_1) := [a_0]_{\operatorname{im}(f)}$.

Where do we need that $A_0 \to B_0$ is injective?

Well-definedness and exactness are left as an exercise $\square$