HOMOLOGICAL ALGEBRA, FALL 2025
PROBLEM SHEET 3

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Problem 1 (10, truncation)**.** Define the good and brutal truncations for cochain complexes.

Problem 2 (20, exact sequence of chain complexes)**.**

(i) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of chain complexes in an abelian category $\mathcal{A}$ such that $g \circ f$ is zero. Prove in categorical terms that this sequence is exact in $\mathbf{Ch}(\mathcal{A})$ if and only if for all integers n the sequence $A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$ is exact in $\mathcal{A}$.

(ii) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of chain complexes in an abelian category. Show that if two of the chain complexes are acyclic, then the remaining one is acyclic.

Problem 3 (40, category **K**)**.** Solve Exercise 1.4.5 (1.,2.,3., and 4.) in [Wei94]. It is about the category of chain complexes of R-modules with chain homotopy classes of chain maps and the question if this category is abelian.

Problem 4 (10, Snake lemma)**.** Prove the Snake lemma for abelian categories using categorical arguments, i.e. without using the embedding theorem.

Problem 5 (10*, homotopy classes of chain maps)**.** Compute all the chain homotopy classes from C to C for the chain complex C of abelian groups, given by $C_n = \mathbb{Z}/8$ and d_n is the multiplication with 4.

References

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

Date: Please hand in before the lecture on Friday, **October 10th$^{\text{th}}$ 2025**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference.