

Chapter 1

Chain complexes

I.1. 1st definitions and examples.

Let R be any ring (maybe non-commutative)

$R\text{-mod}$ category of left R -modules
 $\text{mod-}R$ " " right " "

Def 1: (chain complex)

A chain complex of R -modules is a family of R -modules $(C_n)_{n \in \mathbb{Z}}$ together with a fam. of R -module homomorphisms

$$d_n : C_n \longrightarrow C_{n-1}, \quad n \in \mathbb{Z},$$

st. $d_n \circ d_{n+1} = 0 \quad \forall n \in \mathbb{Z}.$

Terminology:

- d_n "differentials"
- $z_n = \ker(d_n) = \text{cycles}$ "n-chain"
- $B_n = \text{Im}(d_n) = \text{boundaries}$ "module of n-boundaries"
- $H_n(C) := \frac{Z_n}{B_n}$ "n-th homology module of C"
- We also just write d for d_n .

Example 2: $R = \mathbb{Z}/8\mathbb{Z}$

$$C_n := \mathbb{Z}/8\mathbb{Z} \in R\text{-mod.}$$

$$d_n: C_n \rightarrow C_{n-1} \quad d_n([2]_8) = [4]_8$$

$$Z_n = \ker(d_n) = \mathbb{Z}/8\mathbb{Z}$$

$$B_n = \text{Im}(d_{n+1}) = \mathbb{Z}/8\mathbb{Z}$$

$$H_n = \frac{Z_n}{B_n} = \frac{\mathbb{Z}/8\mathbb{Z}}{\mathbb{Z}/8\mathbb{Z}} \cong \mathbb{Z}/8\mathbb{Z}$$

Def 3: (category of chain complexes)

Chl(mod-R):

objects: chain complexes of right R-modules

morphisms: chain maps

$$\text{i.e. } u = (u_n): C \rightarrow D$$

$$\text{where } u_n: C_n \rightarrow D_n \text{ R-module}$$

$$\text{a.t. } d_n^{(D)} \circ u_n = u_{n-1} \circ d_n^{(C)} \quad \forall n \in \mathbb{Z}$$

$$\begin{array}{ccccccc} C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \xrightarrow{d_{n-1}} & C_{n-2} \\ \downarrow u_{n+1} & & \downarrow u_n & & \downarrow u_{n-1} & & \downarrow u_{n-2} \\ D_{n+1} & \xrightarrow{d_{n+1}} & D_n & \xrightarrow{d_n} & D_{n-1} & \xrightarrow{d_{n-1}} & D_{n-2} \end{array}$$

Remark 4: Let $u: C \rightarrow D$ be a morphism of chain complexes. Then

$$u_n(Z_n(C)) \subseteq Z_n(D) \\ u_n(B_n(C)) \subseteq B_n(D)$$

for all $n \in \mathbb{Z}$.

Hence u induces for maps

$$\bar{u}_n: H_n(C) \rightarrow H_n(D)$$

Proof:

$$\pi \in C_n \text{ a.k. } d_n(\pi) = 0 \\ d_n^{(D)}(u_n(\pi)) = u_{n+1}(d_n^{(C)}(\pi)) = u_{n+1}(0) = 0$$

etc.

□

Then $H_n: Ch(mod-R) \rightarrow mod-R$ is a covariant functor.

Def 5: A morphism $u: C \rightarrow D$ is called a quasi-isomorphism if $\forall n \in \mathbb{Z}: \bar{u}_n: H_n(C) \rightarrow H_n(D)$

is an isomorphism of R -modules.

Exercise 6: The following are equivalent:

1° C is exact, i.e.

$$\forall n \in \mathbb{Z}: Z_n(C) = B_n(C)$$

2° C is acyclic, i.e. $\forall n \in \mathbb{Z}: H_n(C) = 0$

3° $0 \rightarrow C$ is a quasi-isom.

"Zero complex $\rightarrow 0 \rightarrow \dots$ "

Def 7: (certain complex)

Given a chain complex C we can change the numbering

$$C^n = C_{-n}, \quad d^{(n)} = d_{-n}$$

(ascending numbering)

$$\rightarrow C^2 \xrightarrow{d^{(2)}} C^3 \xrightarrow{d^{(3)}} C^4 \xrightarrow{d^{(4)}} \dots$$

A chain complex C in a family $(C^n)_{n \in \mathbb{Z}}$ of R -modules together with R -module homomorphisms $d^n: C^n \rightarrow C^{n-1}$, $n \in \mathbb{Z}$, s.t. $d^n \circ d^{n-1} = 0$, $\forall n \in \mathbb{Z}$.

$B^n(C) := \text{im}(d^{n-1})$
"module of n -coboundaries"

$Z^n(C) := \ker(d^n)$
"module of n -cocycles"

$H^n(C) := Z^n(C) / B^n(C)$
 n th cohomology module.
homomorphisms and quasi-isomorphisms are similarly defined.

Def 8: A chain complex C is called bounded if $C_n = 0$ for all $n \leq k$ and all $n > l$.

"Chain amplitude in $[0, l]$ "

Similar: "bounded above"
"bounded below"
"non-negative"

Notation:

$Ch_+ :=$ subcategory of bounded chain complexes

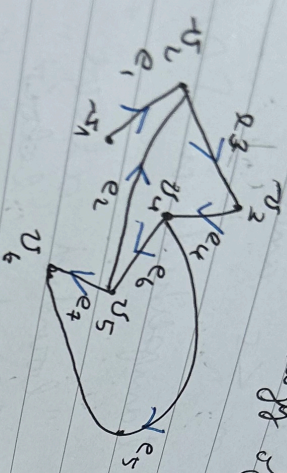
$Ch_+ :=$ subcategory of bounded below chain complexes

$Ch_- :=$ " " " " bounded above " "

$Ch_{\geq 0}, Ch_{\leq 0}$

Similar

Example 9: (Homology of a graph)



$V := \{v_1, \dots, v_6\}$ vertices
 $E := \{e_1, \dots, e_6\}$ edges

We have put orientations on the edges

incidence matrix $V \times E$ matrix

$$A = \begin{pmatrix} -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ -1 & 0 & 1 & -1 & -1 & -1 \end{pmatrix}$$

$$C_1 = \bigoplus_{j=1}^7 R e_j \quad C_0 = \bigoplus_{i=1}^6 R v_i$$

$$C_n = 0, \quad n \neq 0, 1.$$

$$d_1: C_1 \rightarrow C_0 \quad [d_1] := A$$

$$d_1(e_5) = v_6 - v_4$$

$$d_n: C_n \rightarrow C_{n-1} \quad d_n = 0, \quad n \neq 1.$$

$$B_0 = \bigoplus_{j=2}^6 R(v_j - v_1)$$

$$Z_0 = \bigoplus_{i=1}^6 R v_i = B_0 \oplus R v_1$$

$$\Rightarrow H_0(C) \simeq R v_1 \simeq R$$

$$Z_1 = \ker(d_1) = R(e_2 + e_3 + e_4 + e_5) \oplus R(e_5 - e_7 - e_6)$$

(Why?)

$$B_1 = 0. \quad \text{So } H_1(C) \simeq R^2$$

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In general if Γ is connected then

$$H_0(C) \subseteq R$$

$$H_1(C) \subseteq R^{|\mathcal{E}| - |\mathcal{V}| + 1}$$

Example 10:

(singular homology)

$$X \text{ Top. space}$$

$$S_k := \bigoplus R^k$$

$$\text{fec}(\Delta_k, X)$$

$(\Delta_k \text{ standard } k\text{-simplex in } \mathbb{R}^k)$

The set of vertices is ordered.

$$v_0, v_1, v_2, v_3, \dots$$

$$\Delta_3 = [v_0, v_1, v_2, v_3]$$

So the set of $k-1$ faces of Δ_k is ordered. ($0 \leq i \leq k$) index for the faces.

$$\text{All faces: } [v_{i_1}, v_{i_2}, \dots, v_{i_{k-1}}]$$

$$\partial_i: S_k \rightarrow S_{k-1} \text{ restriction on the } i\text{th face.}$$

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$$d: S_k \rightarrow S_{k-1}$$

$$d = \sum_{i=0}^k (-1)^i \partial_i$$

$$\text{Then } d \circ d = 0.$$

$H_n(S)$ "the singular homology of X with coeff. in R ."

denoted by $H_n(X, R)$.

If X is a simplicial complex then $C(X) \rightarrow S(X)$ is a quasi-isomorphism.

Introduction to Abelian Categories

Def 11: Let $\mathcal{A}$ be a category.
($\mathcal{A}$ has objects, morphisms and satisfies the category axioms.)

Ab-category: (1) $\text{Hom}_{\mathcal{A}}(A, B)$ is an abelian group $\forall A, B \in \text{Obj}(\mathcal{A})$.
(2) (distributivity)

$$\forall A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

$$h \circ (g \circ f) \circ i = (h \circ g \circ i) + (h \circ f \circ i)$$

A functor $F: \mathcal{A} \rightarrow \mathcal{B}$ between Ab-categories is called additive if $F: \text{Hom}_{\mathcal{A}}(A, B) \rightarrow \text{Hom}_{\mathcal{B}}(F(A), F(B))$ is a group homomorphism $\forall A, B \in \text{Obj}(\mathcal{A})$.

Zero object: An object Z of $\mathcal{A}$ is called initial (terminal) if for all $B \in \text{Obj}(\mathcal{A})$:
 $|\text{Hom}_{\mathcal{A}}(A, B)| = 1$
 $|\text{Hom}_{\mathcal{A}}(B, A)| = 1$.

An object of $\mathcal{A}$ which is initial and terminal is called a zero-object of $\mathcal{A}$.

additive category: An Ab-category

$\mathcal{A}$ is called (3) $\mathcal{A}$ has a zero object and (4) $\forall A, B \in \text{Obj}(\mathcal{A})$: $\exists$ product $A \times B$ in $\text{Obj}(\mathcal{A})$.

Product: This is an object C with morph. $C \xrightarrow{p_A} A, C \xrightarrow{p_B} B$

$$\begin{array}{c} \text{p.h. } q_A \searrow \\ \mathcal{A} \xrightarrow{q_A} A \\ \text{p.h. } q_B \searrow \\ \mathcal{B} \xrightarrow{q_B} B \end{array} \quad \exists! q: D \rightarrow C$$

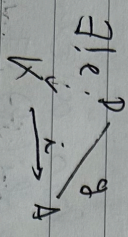
$$q_A = p_A \circ q, \quad q_B = p_B \circ q$$

kernel and cokernel of a morphism:

Suppose that $\mathcal{A}$ is an additive category and $f \in \text{Hom}_{\mathcal{A}}(A, B)$.

A kernel (cokernel) of f is a morphism $\lambda: K \rightarrow A$ ($j: B \rightarrow C$) s.t. $f \circ \lambda = 0$ ($j \circ f = 0$) and it is universal with this property.

[For $\lambda: K \rightarrow A \xrightarrow{f} B$ with $f \circ \lambda = 0$, $\exists! e: D \rightarrow K: g = \lambda \circ e$.]



Abelian category: An additive category $\mathcal{A}$.

- (5) Every morphism has a kernel and a cokernel and
- (6) Every mono is the kernel of any of its cokernels.
- (7) Every epi is the cokernel of any of its kernels.

(mono and epi are defined right and left cancellation properties, respectively)

Examples 12:

- (1) R -mod is an abelian category. (What are the kernels and the cokernels?)

- (2) $\text{Ch}(R\text{-mod})$ is an abelian category.

Def: objects = chain complexes of R -modules
morphisms = chain maps

Ab: (1) $f = (f_n)_{n \in \mathbb{Z}}, g = (g_n)_{n \in \mathbb{Z}}: A \rightarrow B$

chain maps.

Then $f + g := (f_n + g_n)_{n \in \mathbb{Z}}: A \rightarrow B$

is a chain map

($\text{Hom}_{\text{Ch}}(A, B), +$) is an abelian group.

(2) distributivity holds from
— " — for R -module
homomorphisms.

additive category:-

- (3) zero object in the
complex $\rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow$
(4) existence of product:

$$A, B \in \text{Obj}(\mathcal{C}h)$$

$A \times B$ is the complex

$$\rightarrow A_{-1} \times B_{-1} \rightarrow A_0 \times B_0 \xrightarrow{(d_0^A, d_0^B)} A_1 \times B_1 \rightarrow A_2 \times B_2 \rightarrow \dots$$

(check the universal
property.)

abelian category:-

$$(5) f = (f_n)_{n \in \mathbb{Z}} : A \rightarrow B \text{ a chain map}$$

$\ker(f)$ is the complex

$$\dots \xrightarrow{d_{n-2}^{(A)}} \ker(f_{n-1}) \xrightarrow{d_{n-1}^{(A)}} \ker(f_n) \xrightarrow{d_n^{(A)}} \ker(f_{n+1}) \xrightarrow{d_{n+1}^{(A)}} \dots$$

(in the categorical sense:

the kernel of f is given by
 $\text{ind. } \ker(f) \rightarrow A$

$$\text{ind. } \ker(f_n) \hookrightarrow A_n \text{ } \Bigg|$$

$\text{coker}(f)$ is the complex

$$\dots \rightarrow \text{coker}(f_{-1}) \rightarrow \text{coker}(f_0) \rightarrow \dots$$

$$\begin{array}{ccc} B_{-1} & \xrightarrow{\parallel} & B_0 \\ \swarrow \text{im}(f_{-1}) & & \swarrow \text{im}(f_0) \end{array}$$

(6) For R -mod: $\text{monic} = \text{injective}$
 $\text{epi} = \text{surjective}$.

$$f = (f_n)_{n \in \mathbb{Z}} : A \rightarrow B \text{ monic}$$

Then $\forall n$: f_n is injective.

So f_n is the kernel of im

$\text{coker}(f_n)$.

So f is the kernel of im

$\text{coker}(f)$.

(7) Similar.

□

Def 13: (exact sequence)

Let $\mathcal{A}$ be an abelian category.

1) The image $\text{im}(f)$ of $f \in \text{Hom}_{\mathcal{A}}(A, B)$ is defined to be $\ker(\text{coker}(f))$.

2) A sequence

$A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$ is called exact if $\text{im}(f)$ is a kernel of g .

Remark 14:

(a) In R -mod

$\{f(a) \mid a \in A\} \hookrightarrow B$
inclusion

is an image of f .
We forget about the inclusion and just say that

$\{f(a) \mid a \in A\}$ is the image of f .

(b) Let $\mathcal{A}$ be an abelian category and

$A \xrightarrow{f} B \xrightarrow{g} C$ a sequence in $\mathcal{A}$ and $I \xrightarrow{h} B$ in $\text{im}(f)$ and $K \xrightarrow{e} B$ a $\ker(g)$.

$I \xrightarrow{a} K \xrightarrow{1} B \xrightarrow{g} C \xrightarrow{f} 0$

$2^\circ \exists ! \eta: I \rightarrow K$ s.t.

$e \circ \eta = h$.

$3^\circ \exists ! \eta: I \rightarrow K$ s.t.

$e \circ \eta = h$.

Proof: $2^\circ \Leftrightarrow 3^\circ$ because e is monic.

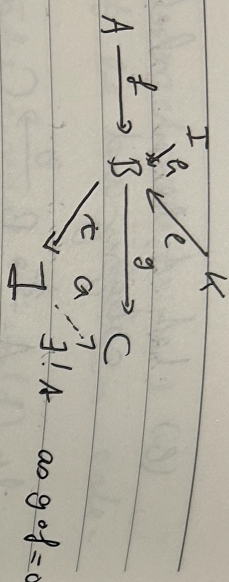
$2^\circ \rightarrow 1^\circ$ h is a $\ker(\text{coker}(f))$.

Then

$\exists ! \eta, \eta \circ f = h$
 $A \xrightarrow{f} B$

$\Rightarrow g \circ f = g \circ h \circ f = g \circ e \circ \eta \circ f = 0 \circ \eta \circ f = 0$

$$1^0 \Rightarrow 2^0$$



$$\pi: B \rightarrow J \text{ a cokernel of } f.$$

$$\text{Thus } g \circ f = f \circ \pi \text{ and } \pi \circ f = f \circ 0 = 0.$$

So as e is a $\ker(g)$, we get 2^0 .

□

I.2. Double complexes.

We have defined $Ch(X)$

The category of chain complexes in $\mathcal{A}$ for an abelian category $\mathcal{A}$. $Ch(\mathcal{A})$ is also an abelian category.

So we can consider

$$Ch(Ch(\mathcal{A}))$$

The category of chain complexes in $Ch(\mathcal{A})$.

obj: chain complexes of

chain complexes in $\mathcal{A}$

maps:

chain maps between objects of $Ch(Ch(\mathcal{A}))$.

A way to study objects of $Ch(Ch(\mathcal{A}))$ is to study double complexes.

Def 15: (double complexes)

Let $\mathcal{C}$ be an abelian category
A family of objects $(C_{p,q})_{\mathbb{Z} \times \mathbb{Z}}$

in $\mathcal{C}$ together with maps

$$d_{p,q}^h : C_{p,q} \rightarrow C_{p-1,q} \quad \text{and}$$

$$d_{p,q}^v : C_{p,q} \rightarrow C_{p,q-1}, \quad p, q \in \mathbb{Z},$$

in $\mathcal{C}$ is called a double complex (or bicomplex)

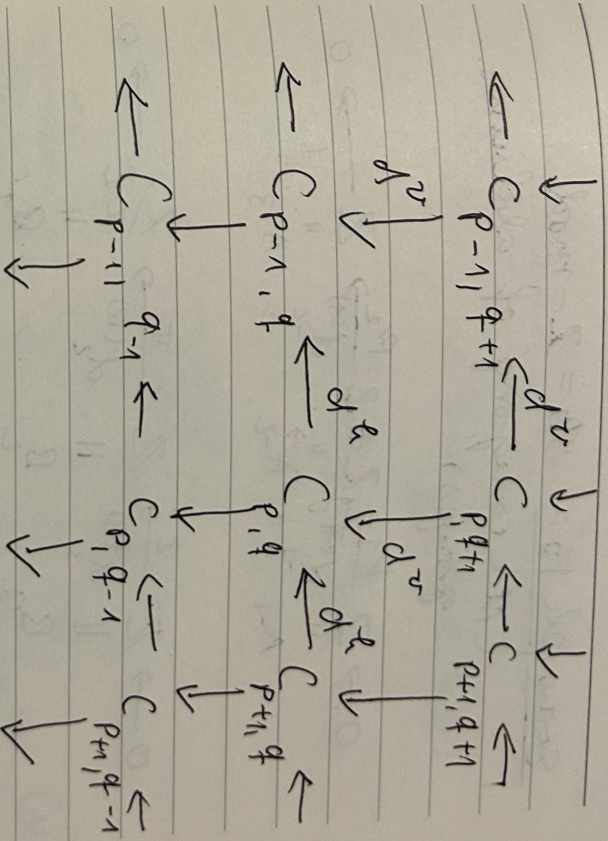
if $\forall p, q \in \mathbb{Z} : d_{p-1,q}^h \circ d_{p,q}^h = 0$

$$d_{p,q-1}^v \circ d_{p,q}^v = 0$$

$$\text{and } d_{p-1,q}^h \circ d_{p,q}^v + d_{p,q-1}^v \circ d_{p,q}^h = 0$$

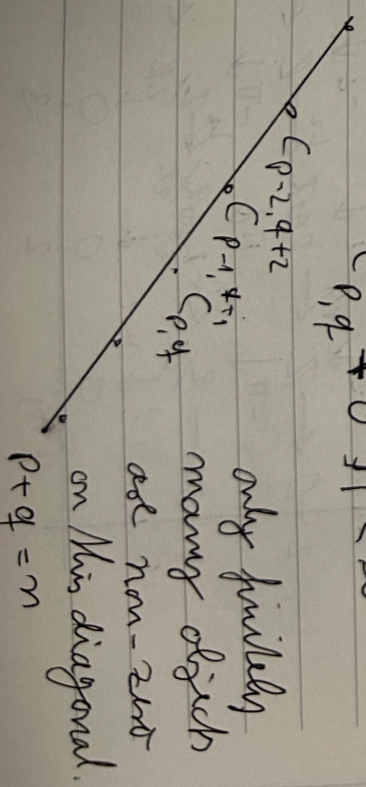
$$\text{or short } d_{p,q}^h \circ d_{p,q}^v = d_{p,q-1}^v \circ d_{p,q}^h = 0$$

Attn: of notation of "0"



A double complex $\mathcal{C}$ is called bounded if

$$\forall n \in \mathbb{Z} : \exists (p,q) \in \mathbb{Z} \times \mathbb{Z} \mid p+q=n \text{ and } C_{p,q} \neq 0 \mid p+q < \infty$$



only finitely many objects are non-zero on this diagonal.

Example 16: $\mathcal{A} = \mathbb{Z}\text{-mod.}$

(the category of abelian groups)

$$0 \rightarrow \mathbb{Z} \xrightarrow{d_1^{(A)}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d_2^{(B)}} \mathbb{Z} \rightarrow 0$$

$\parallel d_1^{(A)} \quad \parallel \quad \parallel d_2^{(B)}$
 $A_{-1} \quad A_{-2} \quad A_{-3}$

$$0 \rightarrow \mathbb{Z} \xrightarrow{d_1^{(B)}} \mathbb{Z} \xrightarrow{d_2^{(B)}} \mathbb{Z} \rightarrow 0$$

$\parallel d_1^{(B)} \quad \parallel d_2^{(B)}$
 $B_3 \quad B_2 \quad B_1$

(a) $C_{p,q} := \text{Hom } \mathbb{Z}(A_{-p}, B_q)$

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 4 & \leftarrow 0 & \xleftarrow{p_1} & 0 & \xleftarrow{h_2} & 0 & \xleftarrow{-2} 0 \\
 & & & & \downarrow & & \downarrow \\
 3 & \leftarrow 0 & \xleftarrow{p_1} & \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{h_2} & \mathbb{Z} & \xleftarrow{-2} 0 \\
 & & & & \downarrow & & \downarrow \\
 2 & \leftarrow 0 & \xleftarrow{p_1} & \mathbb{Z} & \xleftarrow{h_2} & \mathbb{Z} & \xleftarrow{-\pi} 0 \\
 & & & & \downarrow & & \downarrow \\
 1 & \leftarrow 0 & \xleftarrow{p_1} & \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{h_2} & \mathbb{Z} & \xleftarrow{-\pi} 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

e.g. $C_{2,1} = \text{Hom}_{\mathbb{Z}}(A_{-2}, B_1)$

$= \text{Hom}_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z})$

$\cong \mathbb{Z} \oplus \mathbb{Z}$

• $C_{p,q} \xleftarrow{(d_p^{(A)})^*} C_{p+1,q}$

$(d_q^{(B)})^* \downarrow \quad \downarrow d_{q+1}^{(B)} \quad (*)$

$C_{p,q-1} \xleftarrow{(d_p^{(A)})^*} C_{p+1,q-1}$

• sign trick: $d_{p+1,q}^h := (d_p^{(A)})^*$

$d_{p,q}^v := (-1)^p (d_q^{(B)})^*$

$(C_{p,q}, d^h, d^v)$ is a double complex.

$$\text{as } (d_{p-1}^{(A)})^* \circ (d_p^{(A)})^* = \underbrace{(d_p^{(A)} \circ d_{p-1}^{(A)})^*}_0$$

$$= 0$$

$$\text{and } (d_{q-1}^{(B)})^* \circ (d_q^{(B)})^* = \underbrace{(d_q^{(B)} \circ d_{q-1}^{(B)})^*}_0$$

$$\text{and } d^v \circ d^h + d^h \circ d^v = 0 \text{ as } (x)$$

is a commutative diagram.

(t) We also get an object of $\text{Ch}(\text{Ch}(R\text{-mod}))$

$$\text{Hom}_Z(A_{*+1}, B_q)$$

$$(d_q^{(B)})^* \downarrow$$

$$\text{Hom}_Z(A_{*+1}, B_{q-1})$$

$$(d_{q-1}^{(B)})^* \downarrow$$

$$\text{Hom}_Z(A_{*+1}, B_{q-2})$$

Remark 17: $\text{Ch}(\text{Ch}(A))$ can be identified with the category of double complexes in $\mathcal{A}$ via the sign trick.

Def 18: $\mathcal{A} = R\text{-mod}$.

Let $(C_{p,q})_{p,q \in \mathbb{Z}}$ be a double complex in $\mathcal{A}$. We define two total complexes associated to C .

Total Product Complex $\text{Tot}^\Pi(C) := \text{Tot}(C)$

$$\text{Tot}^\Pi(C)_n := \prod_{p+q=n} C_{p,q}$$

Total direct complex $\text{Tot}^\oplus(C)$

$$\text{Tot}^\oplus(C)_n := \bigoplus_{p+q=n} C_{p,q}$$

The differential is given by

$$d = d^1 + d^0$$

Example 1.9: Consider C of

C is bounded, no $\text{Tot}^{\Pi}(C)$
 $= \text{Tot}^{\oplus}(C)$.

$\text{Tot}^{\oplus}(C)$:

n :	7	6	5	4	3	2	1
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$$\begin{array}{ccccccc}
 0 & \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} & 0 & 0 & 0 \\
 \oplus \rightarrow \oplus & \rightarrow \mathbb{Z} & \oplus \rightarrow \mathbb{Z} \oplus \mathbb{Z} & \rightarrow \mathbb{Z} & \rightarrow 0 & \rightarrow 0 & \rightarrow 0 \\
 0 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
 \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
 0 & 0 & 0 & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} & 0
 \end{array}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -\pi \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \pi \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \pi \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -\pi \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \pi \end{pmatrix}$$

Check: Complex property: $d \circ d = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ -2 & \pi & 0 \\ 0 & -\pi & 1 \end{pmatrix} \cdot \begin{pmatrix} \pi & 0 & 0 \\ 2 & 1 & 0 \\ 0 & \pi & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ -2\pi + 2\pi & \pi & 0 \\ -2\pi & -\pi & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

etc.

homology groups of $\text{Tot}^{\oplus}(C)$:

Claim: They are 0. (Exercise)

e.g. at $n=5$:

$$\text{If } \begin{pmatrix} \pi & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} z_1, z_2 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then $0 = z_1$ and $(2z_1, 2z_2) = -(0, z)$.

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$$\text{or } \begin{pmatrix} (z_1, z_2) \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} (0, z_2) \\ -2z_2 \\ 0 \end{pmatrix}$$

$$= d_{\mathbb{C}} \left(\begin{pmatrix} z_2 \\ 0 \\ 0 \end{pmatrix} \right)$$