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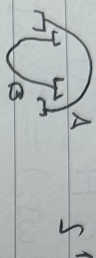
Example 185: (a) $M = S^2 \times S^1$

$$H^k_{dR}(M) \simeq \begin{cases} \mathbb{R}, & k \in \{0, 3\} \\ 0, & k \in \{1, 2\} \end{cases}$$

Note: By Poincaré duality

$$H^k_{dR}(M) \simeq H^{3-k}_{dR}(M).$$

$$S^1 = A \cup B$$



$$U := S^2 \times A, V := S^2 \times B$$

$$dR-M-V: 0 \rightarrow \mathbb{R} \rightarrow \mathbb{R} \oplus \mathbb{R} \xrightarrow{\mathbb{R}^2} H^0(U \cup V) \rightarrow 0$$

$$\hookrightarrow H^1_{dR}(M) \rightarrow 0 \text{ because } U \cap V \simeq S^2.$$

$$\text{So } H^1_{dR}(M) \simeq \mathbb{R} \simeq H^2_{dR}(M).$$

(b) Question: Given $h_1 \in \mathcal{M}_0$ orientable

$\exists$ closed connected 3-mf. M s.t.

$$\dim H^1_{dR}(M) = h_1, 2$$

$$\underline{h_1=0}: M = S^3 \text{ and } M = \mathbb{R}P^3$$

$$\underline{h_1=1}: M = S^2 \times S^1$$

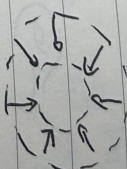
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Def 186: (connected sum)

$$(a) \quad \text{on } \mathbb{R}^m: B_{1/2}(0) = \{x \in \mathbb{R}^m \mid \frac{1}{2} < |x|_2 < 1\}$$

$$\text{But } gl: B_{1/2}(0) \xrightarrow{\varphi} B_{1/2}(0)$$

$$gl(x) := \left(\frac{3}{2} - |x|_2\right) \frac{x}{|x|_2}$$



(c) Let M and N be connected

$m \geq 1$ dim. mf. and

$(U, \varphi), (V, \psi)$ be charts on M and

N , resp., s.t. $\varphi(U) = \varphi(V) = B_1(0)$.

$$\text{Put } U_{1/2} := \varphi^{-1}(B_{1/2}(0))$$

$$V_{1/2} := \psi^{-1}(B_{1/2}(0))$$

$$M \# N := (M \setminus \varphi^{-1}(B_{1/2})) \cup (N \setminus \psi^{-1}(B_{1/2}))$$

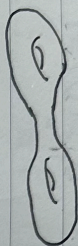
$$\{ (p, q) \in U_{1/2} \times V_{1/2} \mid gl(\varphi(p)) = \varphi(q) \}$$

Connected sum of M and N

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Example 187:

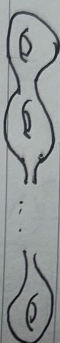
(a) $T^2 \# T^2$



(c) $T^2 \simeq T^2 \# S^2$



(c) $\underbrace{T^2 \# \dots \# T^2}_g$



(d) $K^2 \# T^2$



Lemma 188: Let M and N be orientable connected mf. of dim $m \geq 1$. Then $M \# N$ is orientable.

Proof: Take orientations on M and N such that $\varphi^{-1} \circ \psi \circ \psi^{-1} \rightarrow U$. (see Def 186 (c)) does not preserve the orientation. Then $\varphi^{-1} \circ \psi \circ \psi$ preserves the orientation. $\square$

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Prop. 189: Let M, N be closed connected mf. of dim $m \geq 2$.

Then

(i) $H^k(M \# N) \simeq \begin{cases} \mathbb{R}, & k=0 \\ H^k(M) \oplus H^k(N), & 1 \leq k \leq m-2 \end{cases}$

(ii) $H^m(M \# N) \simeq \begin{cases} \mathbb{R}, & \text{or}(M) \neq 0 \neq \text{or}(N) \\ 0, & \text{else} \end{cases}$

(iii) $H^{m-1}(M \# N)$

$$\simeq \begin{cases} H^{m-1}(M) \oplus H^{m-1}(N), & \text{if } M \text{ or } N \text{ is orientable} \\ H^{m-1}(M) \oplus H^{m-1}(N) \oplus \mathbb{R}, & \text{if } M \text{ and } N \text{ are not orientable} \end{cases}$$

Exercise 190: Suppose M is connected and dim $M \geq 2$ and

$\text{or}(M) = \emptyset$. Let (U, φ) be a chart

s.t. $\varphi(U) = B_1(\emptyset)$. Then

$$\text{or}(M \setminus \varphi^{-1}(B_{1/2}(\emptyset))) = \emptyset.$$

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Lemma 191: Let M be connected, $\partial M \neq \emptyset$, of $\dim m \geq 2$ and $p \in M$. Then

(a) $\forall 0 \leq k \leq m-2: H^k(M) \simeq H^k(M \setminus \{p\})$

(b) $H^{m-1}(M) \simeq H^{m-1}(M \setminus \{p\})$ if M is closed and $\text{or}(M) \neq \emptyset$.

(c) $H^{m-1}(M \setminus \{p\}) \simeq H^{m-1}(M) \oplus \mathbb{R}$ if M is closed and not orientable.

(d) $H^m(M \setminus \{p\}) = 0$ if M is closed.

Proof: (a) $k=0: \checkmark$

$1 \leq k \leq m-2$: We get in the dR

$M \setminus V: 0 \rightarrow H^k(M) \rightarrow H^k(M \setminus \{p\}) \oplus H^k(p) \rightarrow 0$

because $H^k(S^{m-1}) \simeq \begin{cases} 0, & 1 \leq k \leq m-2 \\ \mathbb{R}, & k=0. \end{cases}$

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(b) $0 \rightarrow H^{m-1}(M) \rightarrow H^{m-1}(M \setminus \{p\}) \rightarrow H^{m-1}(S^{m-1}) \xrightarrow{\delta} H^m(M) \rightarrow H^m(M \setminus \{p\}) \rightarrow 0$

$H^m(M \setminus \{p\}) \simeq H^m(M \setminus \mathcal{C}^{-1}(\text{open ball}))$
 $= H^m_c(M \setminus \mathcal{C}^{-1}(\text{open ball}))$
 $= 0$, because of Lem 171/Cor 170.

Thus δ is injective and
 $\text{or} \quad H^{m-1}(M) \simeq H^{m-1}(M \setminus \{p\})$.

(c) As in (b), but now $\delta = \text{zero}$.

(d) see (b) $\square$

Proof of Prop. 189: Just use

$dR - M \setminus V$ for

$M \setminus \mathcal{C}^{-1}(\bar{B}_{1/2}(0))$ and

$N \setminus \mathcal{C}^{-1}(\bar{B}_{1/2}(0))$ using Lem 191. $\square$

Example 192: $Lg := \underbrace{T^2 \# T^2 \# \dots \# T^2}_g$

$H^1_{dR}(Lg \times S^1) \xrightarrow[N=V]{\sim_{dR}} \mathbb{R} \oplus H^1_{dR}(Lg) \oplus \mathbb{R} \oplus H^1_{dR}(T^2) \xrightarrow[\sim_{189}]{} \mathbb{R} \oplus H^1_{dR}(T^2)$

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$$\simeq \mathbb{R}^{1+2g}$$

$\uparrow$

$$H_{dR}^1(T^2) = H_{dR}^1(S^1 \times S^1) \cong \mathbb{R}^2$$

$$\text{and } H_{dR}^1((L_g \times S^1) \# (S^2 \times S^1))$$

$$\underset{189}{\simeq} \mathbb{R}^{1+2g} \oplus \mathbb{R} = \mathbb{R}^{2+2g}$$

$$\text{and } H_{dR}^1(S^3) = 0.$$

Thus $\forall g \in \mathbb{N}$, $\exists$ orientable, closed connected 3 mf. M s.t. $H^1(M) \cong \mathbb{R}^{2g}$.

Example 193: $M := (S^2 \times S^1) \# (S^2 \times 1)$

is not diffeom. to a product of connected closed orientable mf..

Proof: By de Rham's Theorem; de Rham cohomology groups do not depend on the differential structure.

So take L_g and S^1 .

$$H^1(L_g \times S^1) \cong \mathbb{R}^{1+2g}$$

$$H^1(M) \cong \mathbb{R} \oplus \mathbb{R} \text{ even dim.}$$

□

VI Gauss-Bonnet for L_g

Def 194: (Euler characteristic)

Let M be a mf. s.t. all de Rham coh. groups are finite dim.

$$\chi(M) := \sum_{k=0}^m (-1)^k \dim_{\mathbb{R}} H^k(M)$$

is called the Euler characteristic of M .

Examples 195: $\chi(\mathbb{R}^m) = 1 - 0 + 0 - \dots = 1$

$$\chi(L_g) = 1 - 2g + 1 = 2 - 2g$$

$$\chi(\text{Möb.}) = \chi(S^1) = 1 - 1 = 0$$

$$\chi(\mathbb{R}P^2) = 1 - 0 + 0 = 1.$$

$$\chi(K^2) = 1 - 1 + 0 = 0$$

$$K^{\uparrow 2} = \mathbb{R}P^2 \# \mathbb{R}P^2, \text{ Prop. 189}$$

m odd $\Rightarrow \chi(\mathbb{R}P^m) = 0$ by Poincaré duality.

Exercise 196: Let M and N be closed, connected and of dim $m \geq 2$. Compute $\chi(M \# N)$.

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Def 197: (degree of a map)

Let M, N be closed, connected, oriented
mf. of dim $m \geq 1$ and $f \in C^\infty(M, N)$.
Take $[\omega_M] \in H^m(M)$, $[\omega_N] \in H^m(N)$.
s.t. $\int_M \omega_M = \int_N \omega_N = 1$.

Then $\exists D \in \mathbb{R}$: $\int_M f^* \omega_N = D$,
i.e. $[f^* \omega_N] = D [\omega_M]$.

D is called the degree of f .

Thm 198: Let $f: M \rightarrow N$ as in Def 197.
and $\mathcal{Q}$ be a regular value of f .
Let E be the number of points in
 $f^{-1}(\mathcal{Q})$ counted with a sign:

- + if f preserves the orientation
- - if not.

Then $(a) \forall \omega \in \Omega^m(N)$: $\int_M f^* \omega = E \int_N \omega$

(b) $\deg(f) = E$.

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Examples 199: (i) $f: S^1 \rightarrow S^1$ same orient.

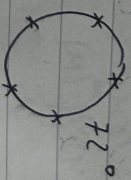
($n \in \mathbb{N}_0$)
 $\Rightarrow f^n: H^1(S^1) \rightarrow H^1(S^1)$ $f(e^{i\theta}) := e^{in\theta}$
 $\Rightarrow \deg(f) = n$.
 $f^*[d\theta] = n[d\theta]$.

$f^{-1}(1) = \{e^{2\pi i k/n} \mid k=0, 1, \dots, n-1\}$

all counted with sign +.

So $E = n$.

$n=5$



(ii) $\int_{S^3} \pi \rightarrow \mathbb{RP}^3$ $\pi(x) := [x_0, \dots, x_3]$

$Q := [1, 0, 0, 0]$.

$\pi^{-1}(Q) = \{(1, 0, 0, 0), (-1, 0, 0, 0)\}$

Orientations: On S^3 : outward pointing normal vector field.

On $\mathbb{RP}^3$: The orientation s.t. π
preserves orientations around $(1, 0, 0, 0)$.

Now: $\Phi: S^3 \rightarrow S^3$ $\Phi(x) := -x$
preserves orientations, because

$x \mapsto -x$ preserves the orientation of $\mathbb{R}^3$ and the outward pointing normal v.f. of S^3 .

Thus $T_{(-1,0,0,0)} \pi$ preserves orientation.

Thus $E = 1 + 1$. So $\deg(\pi) = 2$.

(iii) M closed, connected, oriented.
 $f \in \text{Diff}_+^{\infty}(M)$.

Then $\deg(f) = \begin{cases} +1, & \text{if } f \text{ preserves or} \\ -1, & \text{else.} \end{cases}$

(iv) $\deg(f \circ g) = \deg(f) \deg(g)$.

Proof [Thm 198]: By lemma 171 it is enough to prove (a) for $\omega \in \Omega_c^m(N)$ which are supported locally around $\emptyset$.

Case 1: $f^{-1}(\emptyset) = \emptyset$. So $E = \emptyset$.

Take a rdd V of $\emptyset$ s.t. $\text{int}(V) \cap V = \emptyset$.
 Then for $\omega \in \Omega_c^m(V)$:

$$\int_M f^* \omega = \int_M 0 = 0 = E \int_M \omega.$$

Case 2: $f^{-1}(\emptyset) = \{p_1, \dots, p_k\}$ with signs $\varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}$.

(Note that $f^{-1}(\emptyset)$ is finite, because $\emptyset$ is regular and M is compact.)

Take nbhd V of $\emptyset$ and $U_i \ni p_i$

s.t. $U_i \cap U_j = \emptyset$ $\forall i \neq j$ and

$f: U_i \xrightarrow{\sim} V$ diffeom. and

V is connected and $f^{-1}(V) = \bigcup_{i=1}^k U_i$.

$$\Rightarrow \forall \omega \in \Omega_c^m(V): \int_M f^* \omega = \sum_{i=1}^k \int_{U_i} f^* \omega$$

$$= \sum_{i=1}^k \int_{U_i} f^* \omega|_{U_i} = \sum_{i=1}^k \varepsilon_i \int_V \omega = E \int_M \omega.$$

We extend the notion of degree for proper maps.

Def 200: (proper map)

Let X, Y be locally compact T^2 spaces.

$f \in C(X, Y)$ is called proper if

$\forall K \subseteq Y$ compact: $f^{-1}(K)$ is compact.

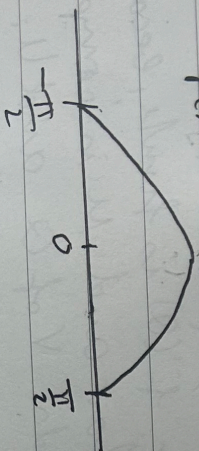
Example 201:

(i) $\text{id}: M \rightarrow M$ is proper

(ii) $f: M \rightarrow \{pt\} = \{*\}$ iff M is compact.

(iii) $f: [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \mathbb{R}$

$$f(x) := \cos(x)$$



$$M := [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{P\}, \quad P \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

f is proper.

$f|_M$ is not proper, because

compact.

(iv) $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$ is

proper, because in $\mathbb{R}$

compact = closed and bounded,

so it is enough to see that

$$f^{-1}([-R, R]) = [-\sqrt{R}, \sqrt{R}]$$

is bounded for all $R > 0$.

Def 202:

Let M, N be connected

oriented mfs of dim $m \geq 0$

a.t. $\partial M = \partial N = \emptyset$ and $f \in C^\infty(M, N)$.

be proper.

$$\text{deg}(f) := \int_M f^* \omega_N$$

where $\omega_N \in \Omega_c^m(N)$ with $\int_N \omega_N = 1$

is called the degree of f .

Rem 203: (i) Thm 198 holds for

$f: M \rightarrow N$ in Def 202 using

$\omega \in \Omega_c^m(N)$ in 198(a).

(2) In particular $\text{deg}(f) \in \mathbb{Z}$.

Example 204 (degree of the Gauss map)

Let (M, g) be a connected oriented 2-submanifold of $\mathbb{R}^3$ which is closed as a manifold. (By restriction M obtains a Riemannian metric from $\mathbb{R}^3$.)

We define $g: M \rightarrow S^2$

$$g(p) := \frac{v_1 \times v_2}{|v_1 \times v_2|},$$

$\{v_1, v_2\}$ an $\sigma(p)$ -oriented basis of $T_p M$.

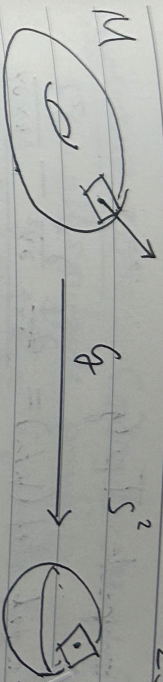
g is called the Gauss map of (M, g) .

$$W_p := T_p g: T_p M \rightarrow T_{g(p)} S^2 \stackrel{\uparrow}{\cong} T_p M$$

identification

(as $T_{g(p)} S^2$ and $T_p M$ are otherwise isomorphic to $g(p)$.)

W_p is called the Weingarten map (or shape operator)



eigenvalues of W_p = "principal curvatures of M at p "
 $\det(W_p)$ = "Gauss curvature of M at p ", denoted by $K(p)$.

Locally: let $(x, y) \mapsto (x, y, f(x, y))$ be the inverse of a chart of M adapted to σ .

$$g(p) = \frac{\begin{pmatrix} 1 \\ \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \times \begin{pmatrix} 0 \\ \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}}{\left| \begin{pmatrix} 1 \\ \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \times \begin{pmatrix} 0 \\ \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \right|}$$

$$= \frac{\begin{vmatrix} 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix}}{\left| \begin{vmatrix} 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix} \right|^2}$$

$$= \frac{\begin{pmatrix} -\frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ 1 \end{pmatrix}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

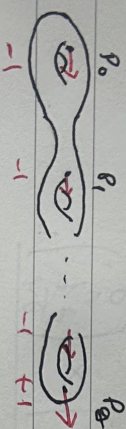
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Compute TP_g

$$\text{We get } \det(h_g) = \frac{\partial^2 g}{\partial x^2} \frac{\partial^2 g}{\partial y^2} - \left(\frac{\partial^2 g}{\partial x \partial y} \right)^2$$

Consider $M = L_g$ with outwards pointing orientation and name for S^2 .

$$\text{Then } \deg(g) = \frac{\chi(M)}{2} = 1 - g.$$

Proof:

$$a = (1, 0, 0) \in S^2$$

$$g^{-1}(a) = \{p_0, p_1, \dots, p_g\}$$

$$-1 \quad -1 \quad \dots \quad +1$$

$$\Rightarrow \deg(g) = 1 - g$$

□

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Exercise 205: Consider $M = L_g$ and S^2 with outside pointing orientation. and $g_M + g_{S^2}$ induced from $\mathbb{R}^3$.

$$\text{Then } g^* \text{vol}_{g_{S^2}} = K \text{vol}_{g_M}$$

Thm 206 (Gauss-Bonnet for L_g)

We consider $M = L_g$ and S^2 as in 205.

Then

$$\int_M K \text{vol}_{g_M} = 2\pi \chi(M)$$

$$= 2\pi(2 - 2g).$$

Proof:

$$\int_M K \text{vol}_{g_M} \stackrel{205}{=} \int_M g^* \text{vol}_{g_{S^2}}$$

$$= \deg(g) \int_{S^2} \text{vol}_{g_{S^2}}$$

$$\stackrel{204}{=} \frac{\chi(M)}{2} 4\pi = 2\pi \chi(M) \quad \square$$

VII Classification Theorem for vector bundles.

For first definitions see II.2. on Page 97.

Def 2.07: (maps between v.b.)

(i) Let $\Sigma_i = (E_i, B_i, \pi_i)$, $i=0,1$, be two vector bundles. A map $F: E_0 \rightarrow E_1$ is called a fibre map from Σ_0 to Σ_1 if $\exists f: B_0 \rightarrow B_1$ s.t.

$$\begin{array}{ccc} E_0 & \xrightarrow{F} & E_1 \\ \pi_0 \downarrow & & \downarrow \pi_1 \\ B_0 & \xrightarrow{f} & B_1 \end{array} \quad \text{commutes}$$

Write $F: \Sigma_0 \rightarrow \Sigma_1$.

We say that F covers f .

(ii) A fibre map $F: \Sigma_0 \rightarrow \Sigma_1$ is called a morphism of vector bundles if F is linear on all fibres.

$$\text{i.e. } \forall p \in E_0: E_0 \xrightarrow{\pi_0^{-1}(p)} E_{\pi_0(p)}$$

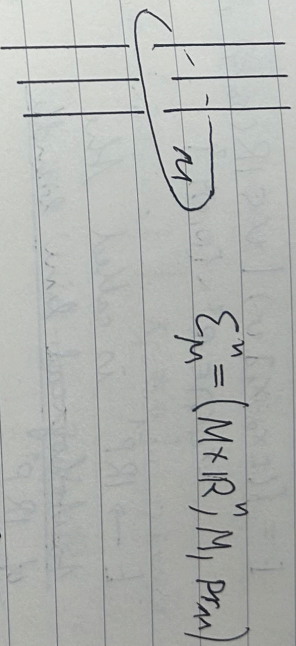
is linear. A morphism F is called a monomorphism (epimorphism / bimorphism (vector bundle map)) if $\forall p \in E_0: F_p$ is injective (surjective / bijective).

(iii) A bimorphism carrying a homeomorphism is called an equivalence.

(iv) An equivalence F s.t. $B_0 = B_1$ and $f = \text{id}_{B_0}$ is called an isomorphism. We write $\mathcal{Z}_0 \cong \mathcal{Z}_1$.

Re: In our course the base will always be a mf. vector bundles will always be smooth and every map is smooth. In (iii) f needs to be a diffeomorphism.

Examples 2.08: (a) $\mathcal{Z}_M^n$ trivial bundle over M of rank n .



$$(b) \quad p_0 \in M \quad F: \mathcal{Z}_M^n \rightarrow \mathcal{Z}_{\{p_0\}}^n$$

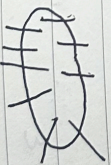
$$F: M \times \mathbb{R}^n \rightarrow \{p_0\} \times \mathbb{R}^n$$

$$F(p, v) := (p_0, v)$$

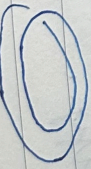
carries $M \rightarrow \{p_0\}$.

F is a bimorphism.

(c) Möb = vt. over S^1 of rk 1.



$\nexists$ smooth isomorphism to $\mathcal{Z}_{S^1}^1$

because  this section

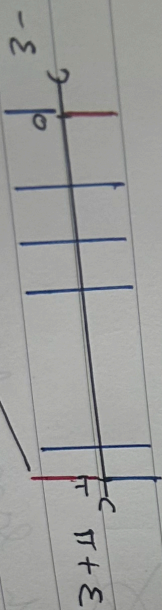
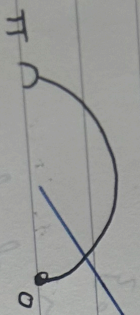
would be mapped onto a disconnected subset of $\mathbb{R}$.

(d)

$$E := \{([x_0 : x_1], v) \mid v \in \mathbb{R}(x_0, x_1), (x_0, x_1) \in \mathbb{R}^2 \setminus \{[0, 0]\}\}$$

$E \rightarrow \mathbb{RP}^1$ is called the tautological line bundle of $\mathbb{RP}^1$

$E \rightarrow \mathbb{RP}^1$ is equivalent to $Möb \rightarrow S^1$.



identify

More precisely a trivialization of $E \rightarrow \mathbb{RP}^1$ is given by

$$\{(U_i, \Phi_i) \mid i = 0, 1\}$$

$$U_i := \{[x_0 : x_1] \mid x_i \neq 0\}, \quad i = 0, 1.$$

$$\Phi_i : E|_{U_i} = \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}$$

$$\Phi_i : ([x_0 : x_1], \lambda(x_0, x_1)) \mapsto ([x_0 : x_1], \lambda x_i (1 + (x_0/x_i)^2))$$

$$(\Phi_1)_{[x_0 : x_1]}^{-1} \circ (\Phi_0)_{[x_0 : x_1]}^{-1}([x_0 : x_1], \Delta)$$

$$= \begin{cases} ([x_0 : x_1], \Delta), & \text{if } x_0/x_1 > 0 \\ ([x_0 : x_1], -\Delta), & \text{if } x_0/x_1 < 0 \end{cases}$$

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Def 2.09: (Grassmannian)

Let $1 \leq k \leq n$.

$G(k, n) := \{W \subseteq \mathbb{R}^n \mid W \text{ } k\text{-dim subspace of } \mathbb{R}^n\}$

Grammation of k -planes in $\mathbb{R}^n$

Ex: $G(1, n) = \mathbb{RP}^1$.

Prop 2.10: $G(k, n)$ has a structure of a smooth manifold.

Proof: Topology - on $G(k, n)$:

$G(k, n)$ is a quotient in the following way.

$$F(k, n) = \{A \in \mathbb{R}^{n \times k} \mid \text{rk}(A) = k\}$$

$\sim$

A, B

$A \sim B \iff \exists g \in GL_k(\mathbb{R}) : A = Bg$

$\text{span}\{e_1(A), \dots, e_k(A)\} \xrightarrow{\sim} [A]$

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Then $G(k, n) \subseteq F(k, n) / \sim$

We choose on $G(k, n)$ the quotient topology.

Then $\cdot G(k, n)$ is T^2
 $\cdot G(k, n)$ is 2^{nd} C
 $\cdot G(k, n)$ is compact } existence

Existence of the differential structure (exercise). $\square$

Example 2.11: $k=2, n=4$.

open cover: $1 \leq i < j \leq 4$

$V_{ij} := \{A \in F(2, 4) \mid \det(A_{ij}) \neq 0\}$

$$A_{ij} = \begin{pmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{pmatrix}$$

$V_{ij} := V_{ij} / \sim$ is open in $G(2, 4)$ as the quotient map is open.

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Example:

$$U_{1,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ * & * \end{bmatrix} \quad U_{2,3} = \begin{bmatrix} * & * \\ 1 & 0 \\ 0 & 1 \\ * & * \end{bmatrix}$$

An atlas is given by

$$(U_{ij}, \varphi_{ij})$$

$$\varphi_{ij}: U_{ij} \longrightarrow \mathbb{R}^{2 \times 2}$$

$$\begin{matrix} i \\ j \end{matrix} = \begin{bmatrix} b_{11} & b_{12} \\ 1 & 0 \\ 0 & 1 \\ b_{21} & b_{22} \end{bmatrix} \longmapsto \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

$$\text{e.g. } \varphi_{1,2}([A]) = A_{3,4} A_{1,2}^{-1}$$

Then φ_{ij} is a homeomorphism
and

$$\varphi_{23} \circ \varphi_{12}^{-1}(B) = \begin{pmatrix} 1 & 0 \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ b_{11} & b_{12} \end{pmatrix}^{-1}$$

$$\begin{aligned} \text{for } B \in \varphi_{12}(U_{12} \cap U_{23}) \\ = \left\{ \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2} \mid c_{11} \neq 0 \right\} \end{aligned}$$