DIFFERENTIAL TOPOLOGY 1 SPRING 2025
PROBLEM SHEET 10/11

PROF. DANIEL SKODLERACK

Problem 1 (10, volume form of a hypersurface)**.** Let $f : \mathbb{R}^{m+1} \to \mathbb{R}$ be a smooth map with regular value 0. We consider $N := f^{-1}(0)$ with orientation given by the gradient of f, i.e. we consider $\Sigma_{i=1}^{m+1} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$ at a point as the outwards pointing normal vector at N. Let g_N be the restriction of the standard metric of $\mathbb{R}^{m+1}$ to N. Compute the volume form of g_N with respect to the given orientation.

Problem 2 (10, volume form for a curve)**.** Let C be a one-dimensional submanifold of $\mathbb{R}^m$ $(m \geqslant 2)$ parametrised by a smooth $r : \mathbb{R} \to \mathbb{R}^m$ such that $r'(t)$ is non-zero for all $t \in \mathbb{R}$. We consider on C the orientation given in direction of r'. Let g_C be the restriction of the standard metric of $\mathbb{R}^m$ to C. Compute the volume form of g_C with respect to the given orientation.

Problem 3 (10, orientation on a product manifold)**.** Let (M, o_M), (N, o_N) be oriented manifolds. Show that there is an orientation on $M \times N$ such that the product of an oriented atlas of M with an oriented atlas of N is an oriented atlas of $M \times N$.

Problem 4 (10, volume form of a torus)**.** We consider the torus $T = S^1 \times S^1$ embedded into $\mathbb{R}^4$, via inclusion $S^1 \subseteq \mathbb{R}^2$. The orientation is given by choosing on S^1 counter-clockwise orientation. Let g be the restriction of the standard Riemannian metric of $\mathbb{R}^4$ to T. Compute the volume form of g with respect to the given orientation.

Problem 5 (10, Poincaré's lemma for compact cohomology)**.** Prove that j commutes with the exterior derivative in the proof of Lemma 169, see Step 1.

Problem 6 (10+10, Lemma 171 for manifolds with boundary)**.**
  (i) Let M be a connected manifold. Prove that $M \backslash \partial M$ is connected.
  (ii) Prove Lemma 171 for manifolds with non-empty boundary.

Date: Please hand in before the lecture on Thursday, **May 13th$^{\text{th}}$ 2025**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference.