DIFFERENTIAL TOPOLOGY 1 SPRING 2025
PROBLEM SHEET 6

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Problem 1 (10, smooth k-forms)**.** Let ω be a k-form on a manifold M. Prove that the following assertions are equivalent:

(i) ω is smooth.

(ii) For every k-tuple of vector fields $X_1, \ldots, X_k$ the map

$$P \mapsto \omega_P(X_1(P), X_2(P), \ldots, X_k(P))$$

is smooth.

(Note that if we say vector field then we include smooth.)

Problem 2 (10, $dx \wedge dy$)**.** Consider the plane E in $\mathbb{R}^3$ affinely spanned by the points

$$(1, 1, 1), (2, 2, 1), (3, 3, 0)$$

Compute $(dx \wedge dy)_P$ for $P \in E$.

Problem 3 (10, nowhere vanishing form on a hypersyrface)**.** A hypersurface in $\mathbb{R}^m$ is by definition the zero locus of a smooth map $f : \mathbb{R}^m \to \mathbb{R}$ with regular value zero. Prove that a hypersurface in $\mathbb{R}^3$ admits a nowhere vanishing smooth degree 2-form.

Problem 4 (10, exterior derivative)**.**

(i) Finish the proof of Theorem 109 in the lecture notes, i.e. in Step 3 of Case 2, show in detail that D is well-defined and that D is an exterior derivative on M.

(ii) The proof of Theorem 109 is written for manifolds without boundary. What should be changed to make the proof work for manifolds with boundary, as the Theorem is stated for all manifolds, also those with boundary?

Problem 5 (10*, Liouville form)**.** Read Example [Tu11, Ch. 5 Example 17.4] about the Liouville form.

(i) Compute the Liouville form for S^1 in terms of the angle covector field $d\theta$.

(ii) Compute the Liouville form for an arbitrary manifold in terms of local coordinates.

References

[Tu11] Loring W. Tu. *An introduction to manifolds.* Universitext. Springer, New York, second edition, 2011.

Date: Please hand in before the lecture on Thursday, **April 3rd 2025**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference.