DIFFERENTIAL TOPOLOGY 1 SPRING 2025
PROBLEM SHEET 2

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Problem 1 (10, constant maps)**.** Let $f : \mathrm{M} \to \mathrm{N}$ be a C^1-map between smooth manifolds. Suppose that M is connected and the derivative of f vanishes everywhere. Show that the image of f consists of just one element.

Problem 2 (20, manifolds as fibers)**.**
(i) Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^∞-map and $y \in \mathrm{im}(f)$ such that at every element in the pre-image M of y the derivative of f is non-zero. Show that M is a submanifold of $\mathbb{R}^{n+1}$.
(ii) Is the above statement still true if we allow M to contain one element where the derivative of f is zero?

Problem 3 (10+(5+5) points, projective n-space)**.** Let n be a positive integer.
(i) Proof that the quotient map $\pi : S^n \to \mathbb{R}P^n$ is smooth.
(ii) (a) Show that the following map is a metric on $\mathbb{R}P^n$.

$$d(l_1, l_2) := \inf\{|x - y|_2|\ x \in l_1,\ y \in l_2, \text{ such that } |x|_2 = |y|_2 = 1\},\ l_1, l_2 \in \mathbb{R}P^n$$

(b) Prove that the quotient topology on $\mathbb{R}P^n$ is equal to the topology defined by the metric d.

Problem 4 (10 points, submanifold of a torus)**.** Consider the two-dimensional torus $\mathbb{T}^2 = S^1 \times S^1$. (Convince yourself that there is a canonical way to define a differential structure on a product of two manifolds.) Find all $s \in \mathbb{R}$ such that the subset

$$S_s = \{(e^{i\alpha}, e^{i\alpha s})|\ \alpha \in \mathbb{R}\}$$

is a submanifold of $\mathbb{T}^2$.

Problem 5 (10* points, submanifolds)**.** Prove that a submanifold of a submanifold of M is a submanifold of M.

Date: Please hand in before the lecture on Thursday, **March 6$^{\text{th}}$ 2025**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference.