# DIFFERENTIAL TOPOLOGY 1 SPRING 2025
# PROBLEM SHEET 1

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**Problem 1** (10 points, wedge product)**.** Prove the statement in Definition 8. For $\tau \in \Omega^q(\mathbb{R}^n)$, $\omega \in \Omega^p(\mathbb{R}^n)$, say

$$\tau = \sum_{|I|=q} f_I dx_I, \ \omega = \sum_{|J|=p} g_J dx_J$$

we have

$$\tau \wedge \omega = \sum_{|I|=q} \sum_{\substack{|J|=p \\ I \cap J = \varnothing}} f_I g_J \operatorname{sgn}(\sigma_{I,J}) dx_{I \cup J}.$$

**Problem 2** (10 points, de Rham cohomology as an algebra)**.** Let $U$ be an open subset of $\mathbb{R}^n$. Prove that $\wedge$ reduces on $\mathrm{H}^*(U)$ to a well-defined map $\mathrm{H}^*(U) \times \mathrm{H}^*(U) \to \mathrm{H}^*(U)$, given for $[\tau] \in \mathrm{H}^q(U)$, $[\omega] \in \mathrm{H}^p(U)$ via

$$[\tau] \wedge [\omega] := [\tau \wedge \omega].$$

**Problem 3** (10 points, cochain complexes)**.** Read the paragraph about differential complexes (also called cochain complexes) on Page 16 in [BT82]. Solve the exercise on the long exact sequence in cohomology on Page 17.

**Problem 4** (10 points, de Rham cohomology)**.** Compute $\mathrm{H}^q_{DR}(X)$, $X = \mathbb{R}^2 \backslash \{P, Q\}$, for two different points $P, Q \in \mathbb{R}^2$. *Hint: You can choose two open subsets of $X$ in a clever way, then use an exact sequence of de Rham complexes and apply Problem 3.*

**Problem 5** (10, maximal atlas)**.** Let M be a $\mathrm{C}^r$-manifold and $\Phi$ is a $\mathrm{C}^r$-atlas on M. Show that there is a unique maximal $\mathrm{C}^r$-atlas on M which contains $\Phi$.

**Problem 6** (10, atlas on the $n$-dimensional sphere)**.** We consider the $\mathrm{C}^1$-differential structure $\alpha_1$ on $\mathrm{S}^n$ $(n \geqslant 1)$ containing the atlas given by the projections onto the coordinate hyperplanes. Find in $\alpha_1$ an atlas of minimal cardinality.

## References

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

*Date*: Please hand in before the lecture on Thursday, February **27th 2025**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference.